On the Asymptotic Properties of Random Multidimensional Assignment Problems

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Abstract

The Multidimensional Assignment Problem (MAP) is a NP-hard combinatorial optimization problem, occurring in many applications, such as data association. In this paper, we prove two conjectures made in [1] and based on data from computational experiments on MAPs. We show that the mean optimal objective function cost of random instances of the MAP goes to zero as the problem size increases, when assignment costs are independent exponentially or uniformly distributed random variables. We also prove that the mean optimal solution goes to negative infinity when assignment costs are independent normally distributed random variables.

Keywords: Multidimensional Assignment, Combinatorial Optimization, Asymptotic Results.

1 Introduction

The multidimensional assignment problem (MAP) is a higher dimensional version of the standard (two-dimensional or linear) assignment problem. The objective of the MAP is to find tuples of elements from given sets, such that the total cost of the tuples is minimum.

A well-known instance of the MAP is the three-dimensional assignment problem (3DAP). The 3DAP consists of minimizing the total cost of assigning n_i items to n_j locations at n_k points in time. This is also referred to in the literature as the three-dimensional axial assignment problem, and may

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be extended to higher dimensions. The three-dimensional axial MAP can be formulated as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n_i}\sum_{j=1}^{n_j}\sum_{k=1}^{n_k} c_{ijk}x_{ijk} \\
\text{s.t.} \quad & \sum_{j=1}^{n_j}\sum_{k=1}^{n_k} x_{ijk} = 1 \qquad \text{for all } i = 1, 2, \ldots, n_i, \\
& \sum_{i=1}^{n_i}\sum_{k=1}^{n_k} x_{ijk} \leq 1 \qquad \text{for all } j = 1, 2, \ldots, n_j, \\
& \sum_{i=1}^{n_i}\sum_{j=1}^{n_j} x_{ijk} \leq 1 \qquad \text{for all } k = 1, 2, \ldots, n_k, \\
& x_{ijk} \in \{0, 1\} \qquad \text{for all } i, j, k \in \{1, \ldots, n\}, \\
& n_i \leq n_j \leq n_k,
\end{aligned}
$$

where c_{ijk} is the cost of assigning item i to location j at time k. In this formulation, the variable x_{ijk} is equal to 1 if and only if the i-th item is assigned to the j-th location at time k. If we consider additional dimensions for this problem, the formulation can be similarly extended in the following way:

$$
\begin{aligned}
\min \quad & \sum_{i_1=1}^{n_1}\cdots\sum_{i_d=1}^{n_d} c_{i_1\cdots i_d}x_{i_1\cdots i_d} \\
\text{s.t.} \quad & \sum_{i_2=1}^{n_2}\cdots\sum_{i_d=1}^{n_d} x_{i_1\cdots i_d} \;=\; 1 \qquad \text{for all } i_1 = 1, 2, \ldots, n_1, \\
& \sum_{i_1=1}^{n_1}\cdots\sum_{i_{k-1}=1}^{n_{k-1}}\sum_{i_{k+1}=1}^{n_{k+1}}\cdots\sum_{i_d=1}^{n_d} x_{i_1\cdots i_d} \;\leq\; 1 \\
& \quad \text{for all } k = 2, \ldots, d_1, \text{ and } i_k = 1, 2, \ldots, n_k, \\
& \sum_{i_1=1}^{n_2}\cdots\sum_{i_{d-1}=1}^{n_{d-1}} x_{i_1\cdots i_d} \;\leq\; 1 \qquad \text{for all } i_d = 1, 2, \ldots, n_d, \\
& x_{i_1\cdots i_d} \in \{0, 1\} \qquad \text{for all } i_1, i_2, \ldots, i_d \in \{1, \ldots, n\}, \\
& n_1 \leq n_2 \leq \cdots \leq n_d,
\end{aligned}
$$

where d is the dimension of the axial MAP. Also, let $z(I)$ be the value of the optimum solution for an instance I of the MAP. We denote by $\bar{z}^*$

the expected value of $z(I)$, over all instances I constructed from a random distribution, where the specific distribution will become clear by the context.

In general, the axial MAP is known to be NP-hard, a fact which follows from results in [2]. Solving even moderate sized instances of the MAP has been a difficult task, since a linear increase in the number of dimensions brings an exponential increase in the size of the problem.

1.1 Applications

There are many applications of the MAP, and these arise in different areas. A simple application concerns the assignment of employees to specific tasks, considering additional constraints, such as time of deployment, amount of resources for each task, etc. Each of these resources can be seen as a dimension of the problem, that can be assigned independently from the others. A similar application occurs when a company tries to define the optimal allocation of resources, given a number of mutually exclusive constraints. These and similar applications are discussed by Pierskalla [3].

Recently, many applications of the MAP have been found in the area of Multitarget Multisensor (MTMS) problems, which have important military applications [4]. The main challenge in MTMS problems is to match objects from a given set to observations taken from distinct points. The observations are subject to noise, and therefore some probability measure must be applied to the correct assignment of observations to objects. The resulting joint probability function, when reduced to its logarithmic form, can be formulated as a MAP with d dimensions, where d is the number of observations, and with n elements in each set, where n is the number of objects being observed. In [5] there is a detailed description of the modeling issues resulting from this application. The class of MTMS problems can be used in the context of Unmanned Air Vehicles (UAVs), as well as in related areas.

Other applications of the MAP include, for example, image recognition [6] and high energy particle research [7]. For a complete treatment and more applications of the MAP, please refer to [8].

1.2 Objectives

In order to better understand the inherent complexity of the MAP, a possible technique is to study the behavior of random generated instances. Such instances aim to be representative of some natural processes, but at the same time, have interesting stochastic properties, which can be exploited for the construction of better algorithms.

The random instances considered in this paper have cost values drawn from a random distribution. For such instances, it is interesting to determine how they behave in terms of solution value, given some distribution function and parameters from which values are taken. This question turns out to be very difficult to solve in general. As an example, for the Linear Assignment Problem (LAP), results have not been easy to prove, despite intense research in this field [9, 10, 11, 12, 13].

In recent work with the MAP [1], we proposed the following conjectures based on data derived from computational experiments with random MAPs. We assume for the rest of the paper that each dimension has the same number of elements, i.e., $n_i = n$, for $i = 1, \ldots, d$.

Conjecture 1.1 *Given a d-dimensional MAP with n elements in each dimension, if the n^d cost coefficients are independent exponentially distributed random variables with mean 1 or independent uniformly distributed random variables in [0,1], $\bar{z}^* \to 0$ as $n \to \infty$ or $d \to \infty$.*

Conjecture 1.2 *Given a d-dimensional MAP with n elements in each dimension, if the n^d cost coefficients are independent standard normal random variables, $\bar{z}^* \to -\infty$ as $n \to \infty$ or $d \to \infty$.*

Although the above conjectures were not proven, we proved the following special case when assignment costs are independent exponentially distributed random variables.

Theorem 1.3 ([1]) *For $n = 2$, $\bar{z}^* \to 0$ as $d \to \infty$.*

In this paper we provide proofs of more generalized instances of Conjecture 1.1 and prove Conjecture 1.2. The proofs are based on building an index tree [3] to represent the cost coefficients of the MAP and then selecting a minimum subset of cost coefficients such that at least one feasible solution can be expected from this subset. Then an upper bound on the cost of this feasible solution is established and used to complete the proofs.

The *index tree representation* for a MAP was proposed by Pierskalla [3], and is a natural way of organizing in a tree-form the possible assignments in a MAP. Each node in the index tree (with the exception of the root) represents an assignment $(i_1, \ldots, i_d)$. A path in the index tree from the root to any leaf node represents a feasible solution to the current instance. Thus, the tree is composed of n_1 levels. At level j of the tree, all assignments with $i_1 = j$ are listed. For a node v at level j, let A_i, for $i \in \{1, \ldots, d\}$, be the set of values appearing in position i on any of the assignments in the path from the root node to v. Then, v is parent of all nodes w in level $j+1$ such that $w = (j+1, k_2, \ldots, k_d)$ and $k_i \notin A_i$, for $i \in \{2, \ldots, d\}$.

This paper is organized as follows. In the next section we prove asymptotic results related to random instances, where costs are drawn from the exponential or uniform distribution. In Section 3 we prove similar results when the costs are taken from the normal distribution. Finally, we give some concluding remarks in Section 4.

2 Mean Optimal Costs of Exponentially and Uniformly Distributed Random MAPs

To find the asymptotic cost when the costs are uniformly or exponentially distributed, we use an argument based on the *probabilistic method* [14]. Basically, it is shown that, for a subset of the index tree, the expected value of the number of feasible paths in this subset is at least one. Thus, such a set must contain a feasible path and this fact can be used to give an upper bound on the cost of the optimum solution. This is explained in the next proposition.

Proposition 2.1 *Using an index tree to represent the cost coefficients of the MAP, randomly select α different nodes from each level of the tree and combine these nodes from each level into set $\mathcal{A}$. $\mathcal{A}$ is expected to produce at least one feasible solution to the MAP when*

$$\alpha = \left\lceil \frac{n^{d-1}}{(n!)^{\frac{d-1}{n}}} \right\rceil \text{ and } |\mathcal{A}| = n\alpha. \tag{1}$$

Proof: Consider there are n^{d-1} cost coefficients on each of the n levels of the index tree representation of an MAP of dimension d and with n elements in each dimension. Now consider there are $(n^{d-1})^n$ paths (not necessarily feasible to the MAP) in the index tree from the top level to the bottom level. The number of feasible paths (or feasible solutions to the MAP) in the index tree is $(n!)^{d-1}$. Therefore, the proportion ϱ of feasible paths to all paths in the entire index tree is

$$\varrho = \frac{(n!)^{d-1}}{(n^{d-1})^n}. \tag{2}$$

Create a set $\mathcal{A}$ of nodes to represent a reduced index tree by selecting α nodes randomly from each level of the overall index tree and placing on a corresponding level in the reduced index tree. The number of nodes in $\mathcal{A}$ is obviously $n\alpha$. For this reduced index tree of $\mathcal{A}$, there are α^n paths (not necessarily feasible to the MAP) from the top level to the bottom level. Since the set of nodes in $\mathcal{A}$ were selected randomly, we may now use ϱ to determine the expected number of feasible paths in $\mathcal{A}$ by simply multiplying ϱ by the number of all paths in the reduced tree of $\mathcal{A}$. That is

$$E[\text{number feasible paths in } \mathcal{A}] \quad = \quad \varrho\alpha^n$$

We wish to ensure that the expected number of feasible paths $\mathcal{A}$ is at least one. Thus, over all possible choices of the n subsets of α elements, there must be surely one choice such that there is one feasible path (in fact there is a lot of them, since the expected value gives only the average over all possible solutions). Therefore,

$$\varrho\alpha^n \quad \geq \quad 1,$$

which results

$$\alpha \quad \geq \quad \left(\frac{1}{\varrho}\right)^{\frac{1}{n}}.$$

Incorporating the value of ϱ from (2) we get

$$\alpha \quad \geq \quad \frac{n^{d-1}}{(n!)^{\frac{d-1}{n}}}.$$

Therefore, since α must be an integer, we get (1). $\square$

We now take a moment to introduce the concept of order statistics. For more complete information, please refer to statistics books, such as [15].

Definition 2.2 *Suppose that* $X_1, X_2, \ldots, X_k$ *are* k *independent identically distributed variables. Let* $X_{(i)}$ *be the* i*-th smallest of these. Then* $X_{(i)}$ *is called the* i*-th order statistic for the set* $\{X_1, X_2, \ldots, X_k\}$*.*

In the rest of the paper, we will consider bounds for the value of the α-th order statistic of i.i.d. variables drawn from a random distribution. This value will be used to derive an upper bound on the cost of the optimal solution for random instances, when n or d increases. Note that, in some places (e.g., Equation (5)), we assume that $\alpha = \frac{n^{d-1}}{n!^{\frac{d-1}{n}}}$. This is a good approximation in the following formulas because

(a) if n is fixed and $d \to \infty$, then $\alpha \to \infty$, and therefore there is no difference between α and $n^{d-1}/n!^{\frac{d-1}{n}}$;

(b) if d is fixed and $n \to \infty$, then $\alpha \to e^{d-1}$. This is not difficult to derive, since

$$\frac{n}{n!^{\frac{1}{n}}} \approx \frac{n}{\left[(\frac{n}{e})^n (2\pi n)^{\frac{1}{2}}\right]^{\frac{1}{n}}} = \frac{e}{(2\pi n)^{\frac{1}{2n}}}.$$

But

$$(2\pi n)^{\frac{1}{2n}} = (2\pi e^{\log n})^{\frac{1}{2n}} = (2\pi)^{\frac{1}{2n}} \cdot e^{\frac{\log n}{2n}},$$

and both factors in the right have limit equal to 1. However, e^{d-1} is a constant value, and will not change the limit of the whole formula, as $n \to \infty$.

Proposition 2.3 *Let $\bar{z}_u^* = nE[X_{(\alpha)}]$, where $E[X_{(\alpha)}]$ is the expected value of the α^{th} order statistic for each level of the index tree representation of the MAP. Then, $\bar{z}_u^*$ is an upper bound to the mean optimal solution cost of an instance of an MAP with independent identically distributed cost coefficients.*

Proof: Consider any level j of the index tree and select the α elements with lowest cost on that level. Let A_j be the set composed by the selected elements. Since the cost coefficients are independent and identically distributed, the nodes in A_j are randomly distributed across the level j. Now, pick the maximum node $v \in A_j$, i.e., $v = \max\{w \mid w \in A_j\}$. The expected value of v is the same as the expected value of the α^{th} order statistic among n^{d-1} cost values for this level of the tree. Since each level of the index tree has the same number of independent and identically distributed cost values, we may conclude that $E[X_{(\alpha)}]$ is the same for each level in the index tree. By randomly selecting α cost values from each of the n levels of the index tree, we expect to have at least one feasible solution to the MAP by Proposition 2.1. Thus, it is clear that an upper bound cost for the expected feasible solution is $\bar{z}_u^* = nE[X_{(\alpha)}]$. □

Theorem 2.4 *Given a d-dimensional MAP with n elements in each dimension, if the n^d cost coefficients are independent exponentially distributed random variables with mean $\lambda > 0$, $\bar{z}^* \to 0$ as $n \to \infty$ or $d \to \infty$.*

Proof: We first note that for independent exponentially distributed variables the expected value of the α^{th} order statistic for k i.i.d. variables is given by

$$E[X_{(\alpha)}] = \sum_{j=0}^{\alpha-1} \frac{\lambda}{k-j}. \tag{3}$$

Note that (3) has α terms and the term of largest magnitude is the last term. Using the last term, an upper bound on (3) is developed as

$$\begin{aligned} E[X_{(\alpha)}]_u &\leq \sum_{j=0}^{\alpha-1} \frac{\lambda}{k-(\alpha-1)} \\ &= \frac{\alpha\lambda}{k-\alpha+1}. \end{aligned}$$

Now, using Propositions 2.1 and 2.3, the upper bound for the mean optimal solution to the MAP with exponential costs may be developed as

$$\begin{aligned} \bar{z}_u^* &= n\frac{\alpha\lambda}{k-\alpha+1} \leq n\frac{\alpha\lambda}{k-\alpha} \\ &= \frac{n\lambda}{\frac{k}{\alpha}-1}, \end{aligned} \tag{4}$$

where $k = n^{d-1}$ is the number of cost elements on each level of the index tree. To prove $\bar{z}_u^* \to 0$, we must first substitute the values of k and α into (4), which gives

$$\bar{z}_u^* \leq \frac{n\lambda}{(n!)^{\frac{d-1}{n}}-1}. \tag{5}$$

Let $n = \gamma$ and $n! = \delta$, where γ and δ are some fixed numbers. Then (5) becomes

$$\begin{aligned} \bar{z}_u^* &\leq \frac{\gamma\lambda}{\delta^{\frac{d-1}{\gamma}}-1} \\ &\approx \frac{\gamma\lambda}{\delta^{\frac{d-1}{\gamma}}}, \quad \text{as } d \text{ gets large.} \end{aligned}$$

Therefore,

$$\lim_{d\to\infty} \bar{z}_u^* \leq \lim_{d\to\infty} \frac{\gamma\lambda}{\delta^{\frac{d-1}{\gamma}}} = 0.$$

Now, let $d-1=\gamma$, where γ is some fixed number. Then (5) becomes

$$\begin{aligned}\bar{z}_u^* &= \frac{n\lambda}{(n!)^{\frac{\gamma}{n}}-1}, \\ &\approx \frac{n\lambda}{(n!)^{\frac{\gamma}{n}}} \quad \text{as } n \text{ gets large.}\end{aligned}$$

Using the Stirling's approximation $n! \approx (n/e)^n\sqrt{2\pi n}$,

$$\begin{aligned}\frac{n\lambda}{(n!)^{\frac{\gamma}{n}}} &\approx \frac{n\lambda}{((n/e)^n\sqrt{2\pi n})^{\frac{\gamma}{n}}} \\ &= \frac{n\lambda}{((n/e)^{\gamma}(2\pi n)^{\frac{\gamma}{2n}}} \\ &= \frac{n\lambda}{n^{(\gamma+\frac{\gamma}{2n})}(\frac{1}{e})^{\gamma}(2\pi)^{\frac{\gamma}{2n}}} \\ &\leq \frac{n\lambda}{n^{(\frac{2n\gamma+\gamma}{2n})}(\frac{1}{e})^{\gamma}} &(6)\\ &= \frac{\lambda}{n^{(\frac{2n(\gamma-1)+\gamma}{2n})}(\frac{1}{e})^{\gamma}}, &(7)\end{aligned}$$

where (6) holds because $(2\pi)^{\frac{\gamma}{2n}}$ approaches one from the right as $n \to \infty$. Considering that $(\frac{1}{e})^{\gamma}$ is a constant and that the exponent to n is greater than one for any $\gamma \geq 2$, which holds because $d \geq 3$, then (7) will approach zero as $n \to \infty$. Therefore, for the exponential case

$$\lim_{n\to\infty} \bar{z}_u^* = 0 \text{ and } \lim_{d\to\infty} \bar{z}_u^* = 0 \text{ from above.}$$

Note that $\bar{z}^*$ is bounded from below by zero because the lower bound of any cost coefficient is zero (a characteristic of the exponential random variable with $\lambda > 0$). Since $0 \leq \bar{z}^* \leq \bar{z}_u^*$, the proof is complete. □

Theorem 2.5 *Given a d-dimensional MAP with n elements in each dimension and the n^d cost coefficients are independent uniformly distributed random variables in [0,1], $\bar{z}^* \to 0$ as $n \to \infty$ or $d \to \infty$.*

Proof: For the case of the uniform variable in $[0,1]$, the expected value of the α^{th} order statistic for k i.i.d. variables is given by

$$E[X_{(\alpha)}] = \frac{\alpha}{k+1}.$$

Therefore, using Propositions 2.1 and 2.3, the upper bound on the mean optimal solution for an MAP with uniform costs in $[0,1]$ is

$$\bar{z}_u^* = \frac{n\alpha}{k+1} \leq \frac{n\alpha}{k}, \tag{8}$$

where $k = n^{d-1}$ is the number of cost elements on each level of the index tree. We must now substitute the values of k and α into (8), which becomes

$$\bar{z}_u^* \leq \frac{n}{(n!)^{\frac{d-1}{n}}}. \tag{9}$$

Applying to (9) the Stirling approximation, in the same way as used in Theorem 2.4, we see that $\bar{z}_u^* \to 0$ as $n \to \infty$ or $d \to \infty$. Note again that $\bar{z}^*$ is bounded from below by zero because the lower bound of any cost coefficient is zero (a characteristic of the uniform random variable in $[0,1]$). Since $0 \leq \bar{z}^* \leq \bar{z}_u^*$, this completes the proof. □

Theorem 2.6 *Given a d-dimensional MAP with n elements in each dimension, for some fixed n, if the n^d cost coefficients are independent, uniformly distributed random variables in $[a,b]$, then $\bar{z}^* \to na$ as $d \to \infty$.*

Proof: For the case of the uniform variable in $[a,b]$, the expected value of the α^{th} order statistic for k i.i.d. variables is given by (see [15])

$$E[X_{(\alpha)}] = a + \frac{(b-a)\alpha}{k+1}.$$

Therefore, using Propositions 2.1 and 2.3, the upper bound on the mean optimal solution for an MAP with uniform costs in $[a,b]$ is

$$\begin{aligned} \bar{z}_u^* &= n\left(a + \frac{(b-a)\alpha}{k+1}\right) \\ &\leq n\left(a + \frac{(b-a)\alpha}{k}\right) \\ &= na + \frac{(b-a)n\alpha}{k}, \end{aligned} \tag{10}$$

where $k = n^{d-1}$ is the number of cost elements on each level of the index tree. We must now substitute values of k and α into (10), which results

$$\bar{z}_u^* \leq na + \frac{(b-a)n}{(n!)^{\frac{d-1}{n}}}. \tag{11}$$

It becomes immediately obvious from (11) that for a fixed n and as $d \to \infty$, $\bar{z}_u^* \to n\alpha$. As $\bar{z}^* \leq \bar{z}_u^*$ and na is an obvious lower bound for this instance of the MAP we conclude that, for fixed n, $\bar{z}^* \to na$ as $d \to \infty$. □

3 Mean Optimal Costs of Normal-Distributed Random MAPs

We want now to prove results similar to the theorems above, for the case where cost values are taken from a normal distribution. This will allow us to solve the Conjecture 1.2. A bound on the cost of the optimal solution for normal distributed random MAPs can be found, using a technique similar to the one used in the previous section. However, in this case a reasonable bound is given by the median order statistics, as described in the proof of the following theorem.

Theorem 3.1 *Given a d-dimensional MAP, for a fixed d, with n elements in each dimension, if the n^d cost coefficients are independent standard normal random variables, then $\bar{z}^* \to -\infty$ as $n \to \infty$.*

Proof: First note that for odd $k = 2r+1$, $X_{(r+1)}$ is the median order statistic and for even $k = 2r$, we define the median as $\frac{1}{2}(X_{(r)} + X_{(r+1)})$. Obviously, the expected value of the median in both cases is zero. Let $k = n^{d-1}$ and note that, as n or d get large, $\alpha \ll r$ for either odd or even case. Therefore we may immediately conclude $E[X_{(\alpha)}] < 0$. Using Propositions 2.1 and 2.3, we see that $\bar{z}^* \le \bar{z}_u^* = nE[X_{(\alpha)}]$ and $\bar{z}^* \to -\infty$ as $n \to \infty$. □

Theorem 3.2 *Given a d-dimensional MAP with n elements in each dimension, for a fixed n, if the n^d cost coefficients are independent standard normal random variables, then $\bar{z}^* \to -\infty$ as $d \to \infty$.*

Proof: We use the results from Cramér [16] (p. 376) to establish the expected value of the i^{th} order statistic of k independent standard normal variables. With $i \le \frac{k}{2}$ we have

$$E[X_{(i)}] = -\sqrt{2\ln k} + \frac{\ln(\ln k) + \ln(4\pi) + 2(S_1 - C)}{2\sqrt{2\ln k}} - O(\frac{1}{\ln k}), \qquad (12)$$

where $S_1 = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{i-1}$ and C denotes Euler's constant, $C \sim 0.57722$. As $d \to \infty, k \to \infty$ and the last term of (12) may be dropped. In addition, a slight rearrangement of (12) is useful:

$$E[X_{(i)}] \sim -\sqrt{2\ln k} + \frac{\ln(\ln k)}{2\sqrt{2\ln k}} + \frac{\ln(4\pi)}{2\sqrt{2\ln k}} + \frac{(S_1 - C)}{\sqrt{2\ln k}}. \qquad (13)$$

It is not difficult to see that as $k \to \infty$, the sum of the first three terms of (13) goes to $-\infty$. Therefore, we consider the last term of (13) as $k \to \infty$.

$$\begin{aligned} \frac{(S_1 - C)}{\sqrt{2\ln k}} &= \frac{-C + \sum_{j=1}^{i-1} \frac{1}{j}}{\sqrt{2\ln k}} \\ &\approx \frac{-C + \int_1^{i-1} \frac{1}{j}}{\sqrt{2\ln k}} \\ &= \frac{\ln(i-1) - C}{\sqrt{2\ln k}} \\ &= \frac{\ln(i-1)}{\sqrt{2\ln k}} - \frac{C}{\sqrt{2\ln k}}. \end{aligned} \tag{14}$$

Noting that the second term of (14) goes to zero as $k \to \infty$, and also making the substitutions $i = \alpha = \frac{n^{d-1}}{(n!)^{\frac{d-1}{n}}}$ and $k = n^{d-1}$, we have

$$\begin{aligned} \frac{(S_1 - C)}{\sqrt{2\ln k}} &\leq \frac{\ln\left(\frac{n^{d-1}}{(n!)^{\frac{d-1}{n}}} - 1\right)}{\sqrt{2\ln n^{d-1}}} \\ &\leq \frac{\ln\left(\frac{n^{d-1}}{(n!)^{\frac{d-1}{n}}}\right)}{\sqrt{2\ln n^{d-1}}} \\ &= \frac{\ln(n^{d-1}) - \ln((n!)^{\frac{d-1}{n}})}{\sqrt{2\ln n^{d-1}}} \\ &= \frac{(d-1)\ln(n) - (d-1)\ln(n!^{\frac{1}{n}})}{\sqrt{2\ln n^{d-1}}}. \end{aligned} \tag{15}$$

It is clear that for a fixed n, and as $d \to \infty$, the right hand side of (15) approaches zero. Therefore, using Propositions 2.1 and 2.3 we have $\bar{z}^* \leq \bar{z}_u^* = nE[X_{(\alpha)}]$ and $E[X_{(\alpha)}] \to -\infty$ for a fixed n and $d \to \infty$. The proof is complete. □

4 Conclusion

In this paper, the convergence of random MAPs with costs drawn from the exponential, uniform and normal distributions is studied. We proved that, in these cases, the expected value of the optimal objective cost approaches a limit when the number of dimensions or the size of each dimension increases. These facts have been first observed empirically in [1].

The results in this paper can be useful when, in the process of solving large MAPs, there is information available about the distribution of observed costs. In this case, one can use the limit values to speed up practical algorithms for such instances. It would be interesting to extend the theorems given here to similar problems, such as the planar MAP [17].

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