Radiative Transfer with Many Spectral Lines

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Abstract. The radiative transfer equation, a partial integro-differential equation, is of particular interest for astronomers since it links the spectral properties of the light received (e.g. on Earth) with the properties of the matter from the place of origin (e.g. a star) to the place of the observer.

One major difficulty in its solution is the strong variability of the extinction coefficient entering the equation (e.g. often by more than 6 dex in a small frequency interval). Furthermore, often contributions from more than 10^8 narrow spectral lines have to be included. This has essentially inhibited up to now the accurate consideration of photon fluxes and pressures in radiation-hydrodynamic modelling.

In this contribution two new algorithms developed in collaboration with B. Baschek and W. v. Waldenfels are introduced that allow the efficient calculation of radiation fields with many lines whenever the detailed spectral information is not required. In the first one the extinction coefficient is represented by a 'generalized opacity distribution function'. In a second method the line positions, strengths and profiles are described by a Poisson point process.

The resulting expressions which are valid both in static and differentially moving media become particularly convenient inside a very optically thick medium ("diffusion limit").

1 Introduction

For astronomers *photons* are by far the most important source of information about the celestial objects they are interested in. Therefore the analysis of light is of paramount importance in astronomy. Fortunately, it is usually sufficient to consider the time independent *specific intensity* $I_\nu(\mathbf{x},\mathbf{n},\nu)$, which is essentially a time average $<\mathbf{E}\mathbf{E}^*>$ of the electric field vector making up the light. Usually, the specific intensity is either introduced heuristically (cf. [13], [7]) by

$$dE = I_\nu(\mathbf{x},\mathbf{n},\nu)\,d\nu\,d\omega\,dF\cos\vartheta\,dt \tag{1}$$

(dE radiant energy of a beam of solid angle $d\omega$ in direction $\mathbf{n}$ with a frequency spread $\nu....\nu+d\nu$ passing per time interval dt through an area element dF at position $\mathbf{x}$ whose normal forms an angle ϑ with the beam direction) or by means of the photon distribution function $\phi(\mathbf{x},\mathbf{n},\nu)$ (cf. [8])

$$I_\nu(\mathbf{x},\mathbf{n},\nu) = \frac{h^4\nu^3}{c^2}\phi(\mathbf{x},\mathbf{n},\nu) \tag{2}$$

(h Planck's constant, c speed of light).

Due their huge distances from the Earth the objects are usually not spatially resolved and therefore only their flux

$$F_\nu((\mathbf{x}, \mathbf{n}, \nu) = \int_{4\pi} I_\nu(\mathbf{x}, \mathbf{n}', \nu)\mathbf{n} \cdot d\omega' \tag{3}$$

can be observed.

Even in such a case the spectral resolution may still be extremely high; as an example see Fig. 1 where the observed fluxes $F(\lambda)$ of the cool dwarf stars GL1, GL887 and GL832 are displayed for a small wavelength range (about 0.6% of the range over which the human eye is sensitive). It is seen that the number of spectral lines, i.e. narrow depressions, is extremely high in these objects; higher resolution data (as e.g. shown in Fig. 2) even indicate that many features still contain unresolved line contributions. In fact, the modeling of this small range requires the inclusion of about 10^5 spectral lines (see Fig. 3); for the whole range observable today it implies $\sim 10^8$ lines. The positions, strengths, and shapes of the lines make it possible to determine not only the abundance of the various elements present but also the pressures, temperatures and velocity fields of the outer layers of the star.

If high-resolution data as those of Fig. 2 are to be modelled it seems unavoidable to take each line individually into account. However, often just the total flux or some average value over a certain not too small wavelength interval is required. In such a case the distribution of lines in Fig. 3 suggests a statistical treatment. For media without large-scale velocity fields such a modeling has been possible for several decades by means of *opacity distribution functions* and by means of *opacity sampling methods*. However, for differentially moving media these descriptions failed.

In this paper we review –mainly from an astrophysical point of view– a recent attempt by B. Baschek, W. v. Waldenfels and the author [18],[3] to remove this problem by modelling the emergent line spectra and the radiative fluxes deep inside of media statistically by means of a Poisson point process. The aspects of probability theory that are involved are well established in mathematics for long time but –as far as we know– have never been used before in our context.

In the next section we introduce the radiative transfer equation and give some simple solutions. Subsequently (Sect. 3), we discuss the mean value of the specific intensity. In Section 4 a Poisson point process model is presented and the corresponding characteristic function are calculated. The distribution function is then derived by means of Levy's theorem, i.e. by means of a Fourier transformation (Sect. 5), and compared with one obtained by simulations. The following part (Sect. 6) is devoted to the diffusion limit, i.e. to the flux and the momentum transfer from photons to matter ('radiative acceleration') inside an optically thick medium far from the surface. We close with a discussion in which we particularly address presently open problems.

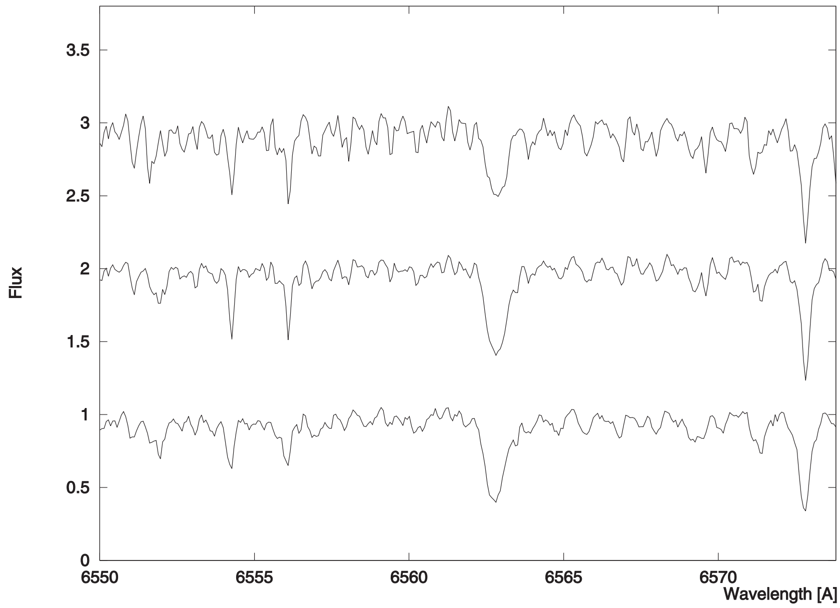

Fig. 1. High resolution spectrum for the cool dwarf stars GL1, GL887, and GL832 (from top to bottom) demonstrating the fast variation of the flux due to many spectral lines (in this case the stronger features are due to hydrogen, titanium, and calcium whereas the weaker ones are predominantly due to titanium oxide). Note that the wavelength range shown is in the red and covers only 0.6% of the sensitivity range of the human eye, which in turn is only a tiny fraction of the range accessible to astronomical instruments.

2 The radiative transfer equation and its solution

The specific intensity as measured in a frame comoving with the matter is governed by the *radiative transfer equation* which reads for unpolarized radiation in/from slowly moving stationary media (cf. [15])

$$\mathbf{n}\cdot\nabla I + w\frac{\partial I}{\partial\xi} = \frac{dI}{d\mathbf{s}} + w\frac{\partial I}{\partial\xi} = -\chi(I-S) \quad \text{with} \quad w = \mathbf{n}\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n})\,. \tag{4}$$

The ∇ operator acts on the spatial variables only. The first term lhs. is the transport term and the second describes the frequency shift due to the Doppler effect. $w = \mathbf{n}\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n})$ is the gradient of the velocity $\boldsymbol{\beta}$ (in units of the speed of light c) projected on to the ray direction. w is frequency independent since we are using logarithmic frequencies $\xi = -\log\nu + const.$ $\chi(\mathbf{x},\xi)$ is the extinction coefficient so that the term $-\chi I$ takes care of photons lost from a beam by absorption and scattering. The term χS with $S = S(\mathbf{x},\mathbf{n},\xi)$ being the source function accounts for all photons added to the beam.

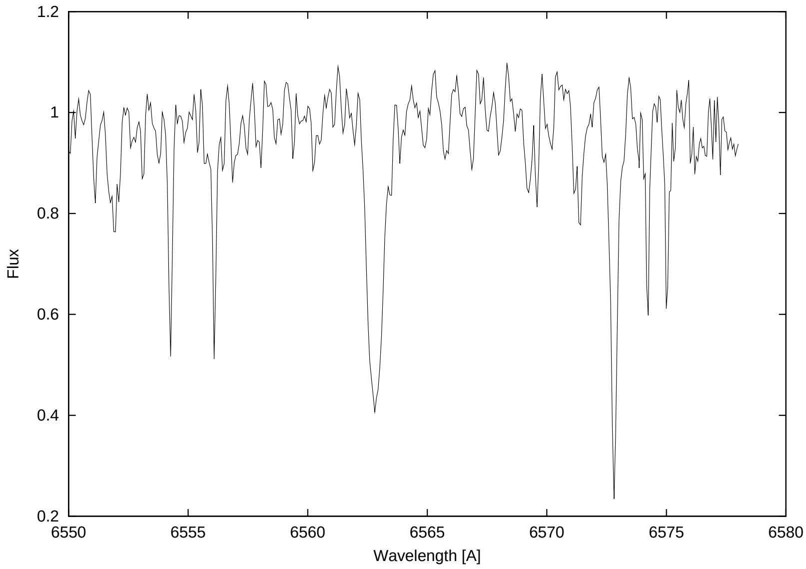

Fig. 2. Very high resolution spectrum of the cool dwarf star GL887 demonstrating that many lines still have substructures due to additional lines not seen in Fig. 1.

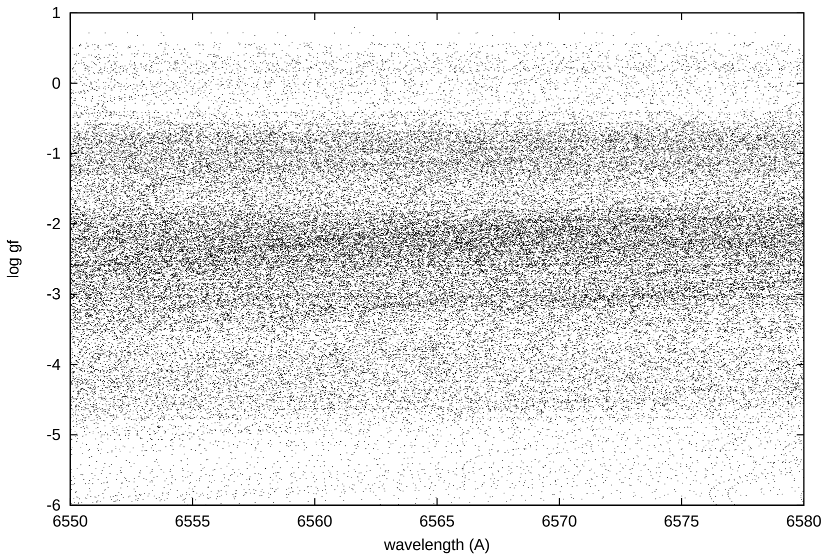

Fig. 3. Distribution of ~ 96000 titanium oxide lines [10] used in the modelling of the spectrum of Fig. 2 in the wavelength-transition probability$\times$statistical weight plane; i.e. the vertical position is a measure of the strength of a line.

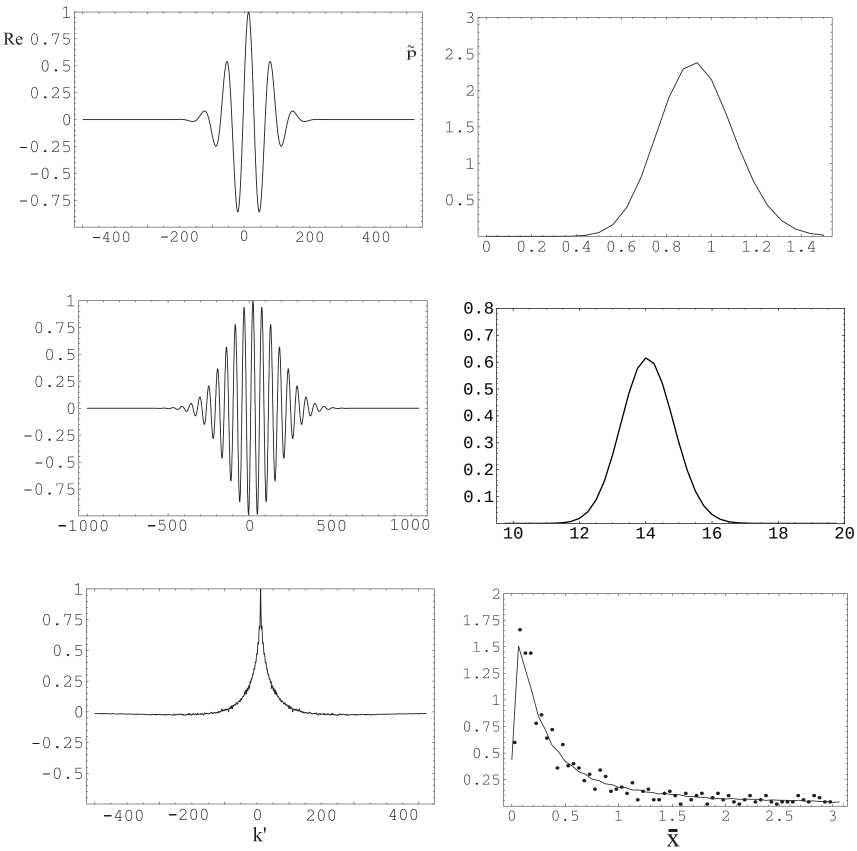

Fig. 4. Generalized opacity distribution functions $\tilde{p}(\bar{x};\xi,\Delta)$ (*right column*) and the real parts of their corresponding expectation values, $\mathrm{Re}\left(\left\langle \exp(-\bar{X}(\xi;\Delta)\cdot ik)\right\rangle\right)$ (*left column*), for Poisson distributed lines at an arbitrary (logarithmic) wavelength ξ. Shown are examples for three combinations of the line density ϱ and the averaging parameter Δ: (10,100), (100,100), and (1,0.1) (*top to bottom*), where $k = ak'$ with $a = 64$, 512, and 64 (*top to bottom*). For comparison, we have added (as *dots*) in the *lower right* panel values for the generalized opacity distribution function calculated according to the procedure described in Sect. 3. The lines have the same statistical properties in both cases. Note that the small wiggles in the *lower left* panel result from inaccuracies in the $\hat{\xi}$-integration (cf. Eq. 22) as a consequence of a highly oscillatory behavior of the integrand. From [3].

Depending on the situation the source function can have various forms (cf. [8]). An approximation often used is that of a two-level atom for which

$$S(\mathbf{x},\mathbf{n},\xi) = \epsilon(\mathbf{x},\xi)B(\mathbf{x},\xi)+\frac{1-\epsilon(\mathbf{x},\xi)}{4\pi}\int_{4\pi}\int_0^\infty R(\mathbf{x},\xi,\xi',\mathbf{n},\mathbf{n}')I(\mathbf{x},\mathbf{n}',\xi')d\xi' d\omega' \tag{5}$$

(ϵ de-excitation coefficient, B Planck function, R redistribution function, which represents the 'memory' of the scattering particle). By definition I and S are positive quantities. In this paper we assume in addition $\chi > 0$ and therefore exclude e.g. laser emission.

As appropriate boundary conditions the intensities I_{bc} impinging from the outside on to the medium are used. Note that other choices usually lead to unstable systems.

Although the transfer equation is usually derived by means of energy conservation arguments it is in fact a linearized Boltzmann equation. It has successfully been used in spectroscopy both of laboratory and celestial plasmas. However, its range of proper application is still unknown [11], [6]

Since in this contribution we want to concentrate on the consequences of extinction coefficients that are highly variable in the frequency domain, we neglect in the following the spatial dependencies of the extinction coefficient and of the velocity gradient. If a variation is to be taken into account one can use the formulae below for sufficiently short distances and then add the contributions [9]. The source function S is assumed to be given. Then the solution of Eq. 4 along a ray in direction $\mathbf{n}$ can be written for $w \neq 0$ [1], [2].

$$I(s,\xi;w) = \mathcal{I}_0(w) \tag{6}$$

$$+\frac{1}{w}\int_{\xi-ws}^{\xi}\exp\left(-\frac{1}{w}\int_{\eta}^{\xi}\chi(\zeta)\,d\zeta\right)\cdot\chi(\eta)S\left(s-\frac{\xi-\eta}{w},\eta\right)d\eta$$

$$= \mathcal{I}_0(w) + S(s,\xi)$$

$$-S(0,\xi-ws)\exp\left(-\frac{1}{w}\int_{\xi-ws}^{\xi}\chi(\zeta)d\zeta\right) \tag{7}$$

$$-\int_{\xi-ws}^{\xi}\exp\left(-\frac{1}{w}\int_{\eta}^{\xi}\chi(\zeta)d\zeta\right)\frac{dS\left(s-\frac{\xi-\eta}{w},\eta\right)}{d\eta}d\eta$$

where

$$\mathcal{I}_0(w) = I(0,\xi-ws;w)\cdot\exp\left(-\frac{1}{w}\int_{\xi-ws}^{\xi}\chi(\zeta)d\zeta\right). \tag{8}$$

The second form is derived from the first one by partial integration. Note that the limit $w \to 0$ can easily be performed and that it leads us back to the well known solution for static media.

3 The expectation value of the specific intensity

If we assume that only the extinction coefficient varies stochastically with ξ we can simply write for the *expectation value of the specific intensity* according to Eqs. 6 to 8

$$\begin{aligned}
&\Big\langle I(s,\xi;w)\Big\rangle = \Big\langle \mathcal{I}_0(w)\Big\rangle \qquad (9)\\
&+\tfrac{1}{w}\int_{\xi-ws}^{\xi}\left\langle \exp\left(-\tfrac{1}{w}\int_{\eta}^{\xi}\chi(\zeta)\,d\zeta\right)\cdot\chi(\eta)\right\rangle S\Big(s-\tfrac{\xi-\eta}{w},\eta\Big)\,d\eta\\
&=\Big\langle \mathcal{I}_0(w)\Big\rangle + S(s,\xi)\\
&\qquad -S(0,\xi-ws)\left\langle \exp\left(-\tfrac{1}{w}\int_{\xi-ws}^{\xi}\chi(\zeta)d\zeta\right)\right\rangle \qquad (10)\\
&\qquad -\int_{\xi-ws}^{\xi}\left\langle \exp\left(-\tfrac{1}{w}\int_{\eta}^{\xi}\chi(\zeta)d\zeta\right)\right\rangle \frac{d\,S\Big(s-\frac{\xi-\eta}{w},\eta\Big)}{d\eta}\,d\eta
\end{aligned}$$

where

$$\Big\langle \mathcal{I}_0(w)\Big\rangle = \left\langle I(0,\xi-ws;w)\cdot\exp\left(-\tfrac{1}{w}\int_{\xi-ws}^{\xi}\chi(\zeta)d\zeta\right)\right\rangle . \qquad (11)$$

Eq. 10 indicates that the expectation value of the emergent specific intensity can be calculated in a straightforward way if the characteristic function

$$\varphi(s) = \left\langle \exp\left(-\tfrac{is}{\Delta}\int_{\xi-\Delta}^{\xi}\chi(\zeta)d\zeta\right)\right\rangle \qquad (12)$$

of the probability density function $\tilde{p}$ for the extinction coefficient averaged over a ξ-interval $\Delta = ws$ is known. In the next sections we show how to calculate φ and $\tilde{p}$ by means of a Poisson point process.

As an alternative one can use the ergodic hypothesis and calculate $\bar{X}_i = \int_{\xi_i-\Delta}^{\xi_i}\chi(\zeta)d\zeta/\Delta$ at many positions ξ_i in a ξ-interval of suitable length. The normalized distribution of the $\bar{X}_i$ should then be a good approximation to $\tilde{p}$. Note that this procedure in the limit $w\to 0$ leads back to the way used by astrophysicists for decades in the static case [12].

In terms of the probability density the crucial term in Eq. 10 is then evidently given by

$$\Big\langle \mathrm{e}^{-\bar{X}(\xi;\Delta)\cdot z}\Big\rangle = \int_0^{\infty}\mathrm{e}^{-\bar{x}\cdot z}\tilde{p}(\bar{x};\xi,\Delta)\,d\bar{x}\,. \qquad (13)$$

4 The Poisson point process

Although the Poisson point process (cf. [5],[4]) is one of the most simple stochastic processes it seems to represent very well spectral line distributions that are known from the laboratory and/or from quantum-mechanical calculations [18]. In addition, the process is very flexible to account for the strengths and shapes of the lines and it can easily be combined with the solution of the radiative transfer solution as shown below.

In order to apply the concept of this process to our problem we have to realize that the extinction coefficient is composed of a continuous contribution χ_c which can be considered to be frequency independent and of contributions of individual lines which we write

$$X(\xi) = \sum_{l=1}^{L} \chi_l(\xi) = \sum_{l=1}^{L} \chi(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l)\,, \tag{14}$$

Each contribution $\chi(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l)$ can be assumed to be the product of the strength A_l and the profile function Φ which depends on the frequencies ξ, the wavelength of the line center $\hat{\xi}$, and on the type of the line (as eg. Lorentzian or Gaussian). ϑ_l is used to summarize the line parameters.

$$\chi(\hat{\xi}, \vartheta, \xi - \hat{\xi}) = A \cdot \Phi(\hat{\xi}, \gamma, \xi - \hat{\xi})\,. \tag{15}$$

We now assume that within a given ξ-interval $\xi \ldots \xi + \Delta\xi$ the mean density of lines with properties ϑ (i.e. strengths, shape etc.) is given by $\rho(\hat{\xi}, \vartheta)$ and that the actual number L is given by a Poisson distribution with mean $\langle L \rangle = \rho(\xi, \vartheta)\Delta\xi\Delta\vartheta$, i.e.

$$\mathbb{P}\{L = n\} = \frac{\langle L \rangle^n}{n!}\, \mathrm{e}^{-\langle L \rangle}\,. \tag{16}$$

The centers of the lines $\hat{\xi}$ are assumed to be a sequence of independent, identically distributed random variables: the $(\hat{\xi}_i, \vartheta_i)$ $(i = 1, 2, \ldots)$ form a *Poisson point process.*

The characteristic function can now be calculated in a straightforward way. Since we later need a slightly more complicated expression we give it here for

$$h(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) = \\ \Lambda \cdot \chi(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) - \frac{1}{w} \int_{\xi - ws}^{\xi} \chi(\hat{\xi}_l, \vartheta_l, \zeta - \hat{\xi}_l) d\zeta\,. \tag{17}$$

Since

$$\left\langle \exp\left(\sum_{l=1}^{L} h(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) \right) \right\rangle$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\varrho(\mathcal{S})}}{n!} \int_{\mathcal{S}} \ldots \int \Big(\varrho(\hat{\xi}_1, \vartheta_1) \ldots \varrho(\hat{\xi}_n, \vartheta_n) \Big) \\
&\quad \times \exp\left(\sum_{l=1}^{n} h(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) \right) d\hat{\xi}_1 d\vartheta_1 \ldots d\hat{\xi}_n d\vartheta_n \\
&= \mathrm{e}^{-\varrho(\mathcal{S})} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_{\mathcal{S}} \varrho(\hat{\xi}, \vartheta) \mathrm{e}^{h(\hat{\xi}, \vartheta, \xi - \hat{\xi})} d\hat{\xi} d\vartheta \right]^n \\
&= \exp\left(\int_{\mathcal{S}} \varrho(\hat{\xi}, \vartheta) \Big\{ \mathrm{e}^{h(\hat{\xi}, \vartheta, \xi - \hat{\xi})} - 1 \Big\} \, d\hat{\xi} d\vartheta \right) \qquad (18)
\end{aligned}
$$

with $\mathcal{S}$ being the set of possible ϑ-values, the characteristic function reads for the moving case

$$
\begin{aligned}
\varphi(k) = & \left\langle \mathrm{e}^{-\bar{X}(\xi;\Delta)\cdot ik} \right\rangle \qquad (19) \\
& = \exp\left(\int_{\Theta} \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \left\{ \mathrm{e}^{-ikU} - 1 \right\} d\hat{\xi} \, d\vartheta \right)
\end{aligned}
$$

with

$$
U \equiv U(\hat{\xi}, \xi, \Delta, \gamma, A) = \frac{1}{\Delta} \int_{\xi-\Delta}^{\xi} A\Phi(\hat{\xi}, \gamma, \zeta - \hat{\xi}) \, d\zeta \, . \qquad (20)
$$

and for static media ($w = 0$)

$$
\begin{aligned}
& \left\langle \mathrm{e}^{-X(\xi)\cdot ik} \right\rangle = \Omega(\xi, \xi; 0) \\
&= \left\langle \exp\left(- \sum_{l=1}^{L} \chi(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) \cdot ik \right) \right\rangle \qquad (21) \\
&= \exp\left[\int_{\Theta} \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \left(\mathrm{e}^{-\chi(\hat{\xi}, \vartheta, \xi - \hat{\xi})\cdot ik} - 1 \right) d\hat{\xi} d\vartheta \right] .
\end{aligned}
$$

5 The opacity distribution function

Using Levy's theorem (see e.g. [5]) and setting $\Lambda = 0$ we can immediately derive the distribution function $\tilde{p}$ of $1/\Delta \int_{\xi-\Delta}^{\xi} \chi(\zeta) d\zeta$ from Eq. 19

$$
\begin{aligned}
\tilde{p}(\bar{x}; \xi, \Delta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{ik\bar{x}} \cdot \left\langle \mathrm{e}^{-\bar{X}(\xi;\Delta)\cdot ik} \right\rangle dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\Bigg[\; ik \, \bar{x}
\end{aligned}
$$

$$+ \int_{A_1}^{A_2}\int_{-\infty}^{\infty} \varrho(\hat{\xi}, A)\left\{\mathrm{e}^{-ik\,A\,U} - 1\right\} d\hat{\xi}\, dA\Bigg]\, dk \tag{22}$$

$$= \frac{1}{\pi}\int_0^{\infty} \exp\left[\int_{A_1}^{A_2}\int_{-\infty}^{\infty} \varrho(\hat{\xi}, A)\left\{\cos\left(k\,A\,U\right) - 1\right\} d\hat{\xi}\, dA\right]$$
$$\times \cos\left\{k\bar{x} - \int_{A_1}^{A_2}\int_{-\infty}^{\infty} \varrho(\hat{\xi}, A)\sin\left(k\,A\,U\right) d\hat{\xi}\, dA\right\} dk\,. \tag{23}$$

Three examples for the real part of the characteristic function and the resulting distribution function are shown in Fig. 4.

6 The diffusion limit

In celestial objects one often has to calculate the radiative flux very deep inside a medium where the extinction coefficient does not vary much over a photon free mean path $1/\chi_c$ and the Planck function can be approximated by a linear function. Then the radiation field is essentially local and one refers to the *diffusion limit.* Whereas for static media already in 1924 Rosseland has derived very convenient expressions for both the monochromatic and the frequency integrated case, no corresponding formulae have been available until very recently [15],[16],[17] for the differentially moving case.

In order to derive these expressions we introduce the *spectral thickness*

$$\psi(\xi) = \int_{\xi_1}^{\xi} \chi(\zeta)d\zeta. \tag{24}$$

Since for the conditions stated the boundary conditions do not contribute we find from Eq. 7 for the flux $\mathcal{F}(s_0, \xi; w) = I(s_0, \mathbf{n}, \xi) - I(s_0, -\mathbf{n}, \xi)$ in direction $\mathbf{n}$ at position s_0 and at (logarithmic) frequency ξ in a medium extending from $s = 0$ to $s = \tilde{s}$ and expanding/collapsing with velocity gradient w

$$\begin{aligned}
&\mathcal{F}(s_0, \xi; w) = \\
&\Big(p(s_0,\xi) - q(s_0,\xi,\mathbf{n})\Big) \exp\left[\frac{-\psi(\xi) + \psi(\xi - w s_0)}{w}\right] \\
&-p(s_0,\xi)\exp\left[\frac{-\psi(\xi) + \psi(\xi - w(\tilde{s} - s_0))}{w}\right] \\
&+q(s_0,\xi,\mathbf{n})\int_{s_0}^{\tilde{s}} \exp\left[\frac{-\psi(\xi) + \psi(\xi + w(s_0 - \ell))}{w}\right] d\ell \\
&+q(s_0,\xi,\mathbf{n})\int_{0}^{s_0} \exp\left[\frac{-\psi(\xi) + \psi(\xi - w(s_0 - \ell))}{w}\right] d\ell\,.
\end{aligned} \tag{25}$$

Here $p(s,\xi)$ and $q(s,\xi)$ are values of the Planck function (at s and ξ) and of its gradient in direction $\mathbf{n}$, resp.. For a fixed value of ξ we then obtain in the limit s_0, $(\tilde{s}-s_0)\to\infty$

$$\begin{aligned}\mathcal{F}(s_0,\xi;w) &= 2q(s_0,\xi,\mathbf{n})\int_0^\infty \exp\left(-\frac{\psi(\xi)-\psi(\xi-w\ell)}{w}\right)d\ell \\ &= 2q(s_0,\xi,\mathbf{n})\int_0^\infty \exp\left(-\frac{1}{w}\int_{\xi-w\ell}^{\xi}\chi(\zeta)d\zeta\right)d\ell\,.\end{aligned} \tag{26}$$

Equation 26 is basic expression for the flux in the diffusion approximation.

Since in most actual situations the velocity gradients are small and in order to get some more insight into the consequences of spectral lines, edges etc. we expand the exponential term around $w=0$ up to second order

$$\begin{aligned}&\exp\left(-\frac{\psi(\xi)-\psi(\xi-ws)}{w}\right)\\ &\simeq \mathrm{e}^{-\psi'(\xi)\cdot s}\left[1+\frac{1}{2}s^2\psi''(\xi)\,w\right.\\ &\qquad\left.+\left(\frac{1}{8}s^4\psi''(\xi)^2-\frac{1}{6}s^3\psi'''(\xi)\right)w^2\right]+O(w^3).\end{aligned} \tag{27}$$

Then we can perform the depth integration analytically and obtain for the "w correction factor" θ to second order in w

$$\theta(s_0,\xi;w) = 1+\frac{\psi''(\xi)}{\psi'(\xi)^2}\,w+\frac{1}{\psi'(\xi)^3}\left(3\frac{\psi''(\xi)^2}{\psi'(\xi)}-\psi'''(\xi)\right)w^2 \tag{28}$$

Replacing $\psi'(\xi)$ by $\chi(\xi)$ etc., the flux in the diffusion limit becomes

$$\begin{aligned}&\mathcal{F}(s_0,\xi;w)\\ &=\mathcal{F}(s_0,\xi)\cdot\left[1-\frac{\partial}{\partial\xi}\frac{1}{\chi(\xi)}\cdot w+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\frac{1}{\chi(\xi)^2}\cdot w^2\right].\end{aligned} \tag{29}$$

Since in most cases only the total flux is needed we integrate over frequency to obtain

$$\mathcal{F}_{\mathrm{tot}}(s_0;w)=\mathcal{F}_{\mathrm{tot}}(s_0)\cdot\left[1+\eta_1(s_0)\cdot w+\eta_2(s_0)\cdot w^2\right]\,, \tag{30}$$

with

$$\begin{aligned}\eta_1(s_0) &= -\bar{\chi}_{\mathrm{R}}(s_0)\int_{-\infty}^{\infty}\frac{1}{\chi(\xi)}\frac{\partial}{\partial\xi}\left(\frac{1}{\chi(\xi)}\right)G(s_0,\xi)\,d\xi\\ &= -\frac{1}{2}\,\bar{\chi}_{\mathrm{R}}(s_0)\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{1}{\chi(\xi)}\right)^2 G(s_0,\xi)\,d\xi\,,\end{aligned} \tag{31}$$

$$\eta_2(s_0) = +\frac{1}{2}\,\bar\chi_{\rm R}(s_0)\int_{-\infty}^{\infty}\frac{1}{\chi(\xi)}\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\chi(\xi)}\right)^2 G(s_0,\xi)\,d\xi\,. \tag{32}$$

Here $G(s_0,\xi)$ is the weighting function

$$G(s_0,\xi) = \frac{\partial B(T,\xi)}{\partial T}\Big/\frac{\partial B(T)}{\partial T} \tag{33}$$

with

$$B(T) = \int_{-\infty}^{\infty} B(T,\xi)e^{\xi}\,d\xi. \tag{34}$$

The 'Rosseland mean opacity' $\bar\chi_{\rm R}(s_0)$ defined by

$$\bar\chi_{\rm R}(s_0) = \int_{-\infty}^{\infty}\frac{1}{\chi(s_0,\xi)}G(s_0,\xi)d\xi \tag{35}$$

and the flux

$$\mathcal{F}_{\rm tot}(s_0) = 2\mathbf{n}\cdot\nabla T\int_{-\infty}^{\infty}\frac{\partial B(T,\xi)}{\partial T}\frac{1}{\chi(s_0,\xi)}e^{\xi}\,d\xi = \frac{2\mathbf{n}\cdot\nabla T}{\bar\chi_{\rm R}(s_0)}\frac{\partial B(T)}{\partial T} \tag{36}$$

refer to the static case.

Equally, we can define an effective opacity

$$\frac{1}{\bar\chi_\beta(s_0;w)} = \frac{1}{\bar\chi_{\rm R}(s_0)}\cdot\left[1+\eta_1(s_0)\cdot w+\eta_2(s_0)\cdot w^2\right]\,, \tag{37}$$

which is of interest since up to now in all radiation-hydrodynamical models precalculated values of $\bar\chi_{\rm R}$ are used without any correction terms and this expression gives the opportunity to include the effects of motion with a minimum of code changes.

In a similar way we find for the radiative acceleration

$$\begin{aligned} a_{\rm rad,tot}(s_0;w) &= \frac{1}{c}\int_{-\infty}^{\infty}\chi(s_0,\xi)\mathcal{F}(s_0,\xi;w)\exp(\xi)d\xi \\ &= a_{\rm rad,tot}(s_0)\cdot\left[1+\tilde\eta_1(s_0)\cdot w+\tilde\eta_2(s_0)\cdot w^2\right] \end{aligned} \tag{38}$$

with

$$a_{\rm rad,tot}(s_0) = \frac{2}{c}\frac{\partial B}{\partial T}\mathbf{n}\cdot\nabla T \tag{39}$$

being the static value and

$$\tilde\eta_1(s_0) = -\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}\left(\frac{1}{\chi(\xi)}\right)G(s_0,\xi)\,d\xi\,, \tag{40}$$

$$\tilde\eta_2(s_0) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{\chi(\xi)}\right)^2 G(s_0,\xi)\,d\xi\,. \tag{41}$$

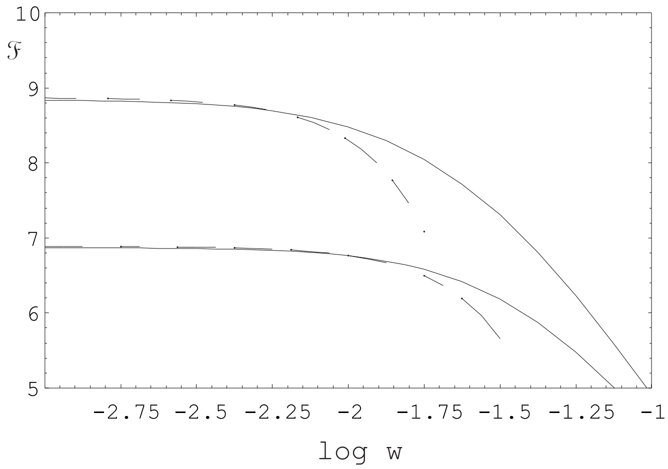

Fig. 5. Variation of the flux inside an optically very thick medium with velocity gradient w. according to Eq. 42 with $2g = 1$ (full curves) and according to the approximation Eq. 50 (broken curves). The upper curves refer to lines of Lorentzian shape with parameters strength $A = 1$, halfwidth $\gamma = 10^{-3}$, density $\rho = 1$ on a continuum $\chi_c = 0.1$. The lower curves have identical parameters except for $\gamma = 10^{-2}$. From [17]

Since the flux in the deep interior of a medium can hardly be measured, it is here even more advantageous than for the intensity at the surface to have expressions for the expectation value. We find

$$\Big\langle F(s_0, \xi; w) \Big\rangle = 2g(s_0, \xi) \int_0^\infty \mathrm{e}^{-\chi_c(\xi)\cdot s}\, \Omega(\xi, \xi - ws; w)\, ds \tag{42}$$

with

$$\Omega(\xi, \xi - ws; w) = \exp\left(\int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \times \left\{ \exp\left(-\frac{1}{w} \int_{\xi - ws}^{\xi} \chi(\hat{\xi}, \vartheta, \zeta - \hat{\xi}) d\zeta \right) - 1 \right\} d\hat{\xi} d\vartheta \right). \tag{43}$$

Fig. 5 shows the dependence of the expectation value of the flux for the diffusion approximation on the velocity gradient w. It is seen that the flux is a decreasing function of w, i.e. that lines get increasingly more important for larger velocity gradient. This implies e.g. that in order to transport a given flux the temperature gradient has to be larger in a moving medium than in a static one.

For the radiative acceleration the identity

$$\langle U \cdot V \rangle \equiv \frac{\partial}{\partial \Lambda} \left\langle \mathrm{e}^{\Lambda U} \cdot V \right\rangle \Big|_{\Lambda=0} , \tag{44}$$

which holds for any two functions U and V, and Eq. 18 have to be used so that

$$\begin{aligned} \Big\langle a_{\mathrm{rad}}(s_0, \xi; w) \Big\rangle &= \frac{1}{c} \Big\langle \chi(s_0, \xi) \cdot F(s_0, \xi; w) \Big\rangle \\ &= \frac{1}{c} \Big[\chi_{\mathrm{c}}(\xi) \cdot \Big\langle F(s_0, \xi; w) \Big\rangle \\ &\quad + \Big\langle \sum_{l=1}^{L} \chi(\hat{\xi}_l, \vartheta_l, \xi - \hat{\xi}_l) \cdot F(s_0, \xi; w) \Big\rangle \Big] \\ &= \frac{2g(s_0, \xi)}{c} \int_0^\infty \mathrm{e}^{-\chi_{\mathrm{c}}(\xi) \cdot s} \, \Omega_{\mathrm{rad}}(\xi, \xi - ws; w) \, ds \end{aligned} \tag{45}$$

where

$$\begin{aligned} \Omega_{\mathrm{rad}}(\xi, \xi - ws; w) &= \Omega(\xi, \xi - ws; w) \\ &\times \left(\chi_{\mathrm{c}}(\xi) + \int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \chi(\hat{\xi}, \vartheta, \xi - \hat{\xi}) \right. \\ &\left. \times \exp\left(-\frac{1}{w} \int_{\xi - ws}^{\xi} \chi(\hat{\xi}, \vartheta, \zeta - \hat{\xi}) d\zeta \right) d\hat{\xi} d\vartheta \right) . \end{aligned} \tag{46}$$

An approximate expression for the flux in the diffusion approximation at small w requires the expansion of Ω around $w = 0$ up to second order

$$\Omega(\xi, \xi - ws; w) = \Omega(\xi, \xi; 0) \cdot \left[1 + \omega_1 \cdot w + \frac{1}{2} \omega_2 \cdot w^2 \right] \tag{47}$$

with $\Omega(\xi, \xi; 0)$ given by Eq. (21) and

$$\begin{aligned} \omega_1 &= \frac{1}{\Omega(\xi, \xi; 0)} \cdot \frac{\partial \Omega(\xi, \xi - ws; w)}{\partial w} \Big|_{(\xi, \xi; 0)} \\ &= \frac{s^2}{2} \int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \, \chi'(\hat{\xi}, \vartheta, \xi - \hat{\xi}) \, \mathrm{e}^{-\chi(\hat{\xi}, \vartheta, \xi - \hat{\xi}) s} d\hat{\xi} d\vartheta , \end{aligned} \tag{48}$$

$$\omega_2 = \frac{1}{\Omega(\xi, \xi; 0)} \cdot \frac{\partial^2 \Omega(\xi, \xi - ws; w)}{\partial w^2} \Big|_{(\xi, \xi; 0)}$$

$$
\begin{aligned}
&= \int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi},\vartheta) \left[\frac{s^4}{4} [\chi'(\hat{\xi},\vartheta,\xi-\hat{\xi})]^2 \right. \\
&\qquad \left. - \frac{s^3}{3} \chi''(\hat{\xi},\vartheta,\xi-\hat{\xi}) \right] \mathrm{e}^{-\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})s} d\hat{\xi} d\vartheta \\
&+ \frac{s^4}{4} \left(\int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi},\vartheta)\, \chi'(\hat{\xi},\vartheta,\xi-\hat{\xi}) \right. \\
&\qquad\qquad \left. \times\ \mathrm{e}^{-\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})s} d\hat{\xi} d\vartheta \right)^2 .
\end{aligned}
\tag{49}
$$

Here χ' and χ'' denote the first and second derivative of $\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})$ with respect to $\hat{\xi}$. We stress that $\Omega(\xi,\xi;0)$ as well as ω_1 and ω_2 depend not only on ξ but also on s and hence cannot be taken out of the respective integrals in the expression for the monochromatic flux so that

$$
\begin{aligned}
\Big\langle F(s_0,\xi;w) \Big\rangle &= 2g(s_0,\xi) \int_0^{\infty} \mathrm{e}^{-\chi_c(\xi)s} \times \\
&\Omega(\xi,\xi;0) \cdot \left[1 + \omega_1 \cdot w + \frac{1}{2} \omega_2 \cdot w^2 \right] ds
\end{aligned}
\tag{50}
$$

cannot be simplified further. In a completely analogous way one finds for the radiative acceleration

$$
\begin{aligned}
\Big\langle a_{\mathrm{rad}}(s_0,\xi;w) \Big\rangle &= \frac{2g(s_0,\xi)}{c} \int_0^{\infty} \mathrm{e}^{-\chi_c(\xi)\cdot s} \times \\
&\Omega_{\mathrm{rad}}(\xi,\xi;0) \cdot \left[1 + \tilde{\omega}_1 \cdot w + \frac{1}{2} \tilde{\omega}_2 \cdot w^2 \right] ds \,,
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
\Omega_{\mathrm{rad}}(\xi,\xi;0) &= \Omega(\xi,\xi;0) \cdot \Bigg(\chi_c(\xi) + \\
&\int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi},\vartheta) \chi(\hat{\xi},\vartheta,\xi-\hat{\xi}) \cdot \mathrm{e}^{-\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})\cdot s} d\hat{\xi} d\vartheta \Bigg) ,
\end{aligned}
\tag{52}
$$

$$
\tilde{\omega}_1 = \omega_1 + \frac{\Omega(\xi,\xi;0)}{\Omega_{\mathrm{rad}}(\xi,\xi;0)} \cdot \hat{\omega}_1 \,,
\tag{53}
$$

and

$$
\tilde{\omega}_2 = \omega_2 + \frac{\Omega(\xi,\xi;0)}{\Omega_{\mathrm{rad}}(\xi,\xi;0)} \cdot (2\omega_1\hat{\omega}_1 + \hat{\omega}_2)
\tag{54}
$$

with

$$
\begin{aligned}
\hat{\omega}_1 &= \frac{s^2}{2} \int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi},\vartheta) \times \\
&\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})\, \chi'(\hat{\xi},\vartheta,\xi-\hat{\xi})\, \mathrm{e}^{-\chi(\hat{\xi},\vartheta,\xi-\hat{\xi})s} d\hat{\xi} d\vartheta
\end{aligned}
\tag{55}
$$

and

$$\hat{\omega}_2 = \int_\Theta \int_{-\infty}^{\infty} \varrho(\hat{\xi}, \vartheta) \chi(\hat{\xi}, \vartheta, \xi - \hat{\xi}) \times \left[\frac{s^4}{4} [\chi'(\hat{\xi}, \vartheta, \xi - \hat{\xi})]^2 - \frac{s^3}{3} \chi''(\hat{\xi}, \vartheta, \xi - \hat{\xi}) \right] \times \mathrm{e}^{-\chi(\hat{\xi}, \vartheta, \xi - \hat{\xi}) s} d\hat{\xi} d\vartheta \, . \tag{56}$$

Analogously to the expression for the flux, also the quantities $\Omega_{\rm rad}$, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in the monochromatic acceleration, depend on ξ *and* on s.

7 Discussion

In this paper we have reviewed the statistical treatment of radiation fields with many spectral lines by means of a generalized opacity distribution function and by means of a Poisson point process with special emphasis on the diffusion limit. The expressions derived have all been based on an analytical solution of the transfer equation that has been found recently. The density distribution (and therefore the geometry) has been assumed to be smooth and simple. In addition, we have assumed that the source function is given, i.e. we have essentially neglected scattering.

In spite of these simplifications our expressions are suitable for many applications (as e.g. for Mira stars, novae and accretion disks in astrophysics) and can reduce by orders of magnitude the computation times and memory requirements of radiation-hydrodynamical modeling; in fact, they make for the first time a modeling of such objects possible with realistic radiation fields.

In view of the progress in observational accuracy and sensitivity it is evident from this review that it also is highly desirable to develop in addition algorithms that allow to take spatial inhomogeneities and varying scattering fractions ϵ (see Sect. 2) statistically into account. It would in particular be advantageous if a differential equation for the mean values of the radiation field could be derived.

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