Social Capital and Conventions: A Social Networks Perspective

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Abstract

We introduce a spatial cost topology in the network formation model analyzed by Jackson and Wolinsky, *Journal of Economic Theory* **71** (1996), 44–74. This cost topology might represent geographical, social, or individual differences. It describes variable costs of establishing social network connections. Participants form links based on a cost-benefit analysis. We examine the pairwise stable networks within this spatial environment. Incentives vary enough to show a rich pattern of emerging behavior. We also investigate the subgame perfect implementation of pairwise stable and efficient networks. We construct a multistage extensive form game that describes the formation of links in our spatial environment. Finally, we identify the conditions under which the subgame perfect Nash equilibria of these network formation games are stable.

We analyze the dynamic implications of learning in a large population coordination game where both the actions of the players and the communication network evolve over time. Cost considerations of social interaction are incorporated by considering a circular model with endogenous neighborhoods, meaning that the locations of the players are fixed but players can create their own communication network. The dynamic process describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long run, when mistake probabilities are small but nonvanishing, coexistence of conventions is no longer sustainable as the risk-dominant convention becomes the unique stochastically stable state.

We create and investigate a system that is capable of observing the accumulation of social capital and the effect of social capital accumulation on behavior of individually rational players. In the first model, we develop a restricted system to show that social capital forms and is maintained at a steady state level. The resulting network is the chain. The second model uses a congestion function in conjunction with social capital to show a network emerge that contains links that costlier than those in the chain network.

Acknowledgments

Robert Gilles was my mentor for the research that follows. I am grateful for his invaluable advice and support, his enthusiasm for economic research and his dedication to his students. He inspired me to achieve more than I thought possible. I am also indebted to him for introducing me to Edward Droste to whom I am grateful for his insight and experience in evolutionary game theory.

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Finally, I would like to thank my parents, F.A. and Sara Johnson, and my husband Gabor Csepregi. Without their unwavering support I would not have been able to return to, continue or finish my studies.

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Chapter 1

Introduction

> The social world is accumulated history, and if it is not to be reduced to a discontinuous series of instantaneous mechanical equilibria between agents who are treated as interchangeable particles, one must reintroduce into it the notion of capital and with it, accumulation and all it effects.
>
> –Pierre Bourdieu [11], page 241.

Capital can be embodied in four forms: physical, human, cultural, and social. The capacity of physical and human capital to generate profits is well understood in economics. However, the role of cultural and social capital is not yet explored in economics. Bourdieu [11] defines cultural capital in its fundamental state as the acquisition of cultural characteristics (education, culture, language, etc.) by an individual. In particular, this definition is not completely satisfactory from an economic perspective because it coincides with aspects of human capital. A better definition would be the value of investment by an individual in the ability to participate in society with regard to social or cultural institutions. Coordinating behavior falls within this category of cultural capital.

Social capital is more simply defined as the value of social obligations or contacts formed through a social network. Anecdotal evidence for the importance of social capital formation for the well-functioning of our society is provided by Jacobs [33] on page 180: "These [neighborhood] networks are a city's irreplaceable social capital. When the capital is lost, from whatever cause, the income from it disappears, never to return until and unless new capital is slowly and chancily accumulated." Knack and Keefer [38] recently explored the link between social capital and economic performance. They found that trust and civic cooperation have significant impacts on aggregate economic activity.

It would be near impossible for humans to exist without the benefits derived from social learning, participation in social institutions (family, community, norms, etc.), and information transference. Fundamentally, these activities are a function of interaction. When complemented with accumulated labor, these activities result in the accumulation of all forms of capital. This dissertation is the beginning of a research program investigating the importance of social interaction on economic outcomes.

1.1 Networks

Initially, we explore the concept of social networks at a fundamental level, progress to the application of endogenous social interaction to the coordination problem, and, finally, examine the endurance of social networks with regard to a model of social capital formation. Social networks form as individuals establish and maintain relationships. Economically and socially, being "connected" greatly benefits an individual. Yet, maintaining relationships is costly. As a consequence individuals limit the number of their active relationships. These social-relationship networks develop from the participants' comparison of costs versus benefits of connecting.

More recently the focus of network analysis in economics has turned to a full cost-benefit analysis of network formation. In 1988, Aumann and Myerson [3] presented an outline of such a research program. However, not until recently has this type of program been initiated. Within the resulting literature we can distinguish three strands: a purely cooperative approach, a purely noncooperative approach, and an approach based on both considerations, in particular the equilibrium notion of pairwise stability.

The cooperative approach was initiated by Myerson [43] and Aumann and Myerson [3]. Subsequently Qin [48] formalized a non-cooperative link formation game based on these considerations. In particular, Qin showed this link formation game to be a potential game as per Monderer and Shapley [42]. Slikker and van den Nouweland [49] have further extended this line of research. Whereas Qin only considers costless link formation, Slikker and van den Nouweland introduce positive link formation costs. They conclude that due to the complicated character of the model, results beyond the three-player case seem difficult to obtain.

Bala and Goyal [4] and [5] use a purely non-cooperative approach to network formation resulting into so-called *Nash networks*. They assume that each individual player can create a one-sided link with any other player. This concept deviates from the notion of pairwise stability at a fundamental level: a player cannot refuse a connection created by another player, while under pairwise stability both players have to consent explicitly to the creation of a link. Bala and Goyal show that the set of Nash networks is significantly different from the ones obtained by Jackson and Wolinsky [32] and Dutta and Mutuswami [19].

Jackson and Wolinsky [32] introduced the notion of a *pairwise stable network* and thereby initiated an approach based on cooperative as well as non-cooperative considerations. Pairwise stability relies on a cost-benefit analysis of network formation, allows for both link severance and link formation, and gives some striking results. Jackson and Wolinsky prominently feature two network types: the star network and the complete network. Dutta and Mutuswami [19] and Watts [51] refined the Jackson-Wolinsky framework further by introducing other stability concepts and derived implementation results for those different stability concepts.

In the second chapter we examine social networks and we use the network structure defined there as foundation for the following chapters. We extend the Jackson-Wolinsky [32] framework by introducing a *spatial* cost topology. Thus, we incorporate the main hypotheses from Debreu [16] that players located closer to one another incur less cost to establish communication. We limit our analysis to the simplest possible implementation of this spatial cost topology within the Jackson-Wolinsky framework. The consequences of this simple extension are profound. A rich structure of stable and efficient social networks emerges.

First, we identify the *pairwise stable networks* introduced by Jackson and Wolinsky [32]. We mainly distinguish two classes of such networks: If costs are *high* in relation to the potential benefits, only the empty network is stable. If costs are *low* in relation to the potential benefits, an array of stable network architectures emerges. However, we derive that *locally complete networks* are the most prominent stable network architecture in this spatial setting. In these networks, localities are completely connected. This represents a situation frequently studied and applied in spatial games, as exemplified in the literature on local interaction, e.g., Ellison

[20] and Goyal and Janssen [27]. *Non-locally complete networks* also emerge as a stable network architecture in the spatial setting. These networks are discussed extensively by Burt [12] and are credited for creating structural holes in real world social networks.

Next, we turn to the consideration of Pareto optimal and efficient spatial social networks. A network is *efficient* if the total utility generated is maximal. Pareto optimality leads to an altogether different collection of networks. We show that efficient networks exist that are not pairwise stable. This is comparable to the conflict demonstrated by Jackson and Wolinsky [32].

Finally, we present an analysis of the subgame perfect implementation of stable networks by creating an appropriate network formation game. We introduce a class of defined, multi-stage *link formation games* in which all pairs of players sequentially have the potential to form links. The order in which pairs take action is given exogenously.[1] We show that subgame perfect Nash equilibria of such link formation games may consist of pairwise-stable networks only. Chapter 2 is a slightly revised version of Johnson and Gilles [34].

1.2 Conventions

In a wide variety of economic and social environments, an agent's utility depends on successful coordination with other individuals. The following two examples illustrate this point. First, as suggested by Lewis [39], suppose that oligopolists are confronted with a change in the price of their raw material and therefore must set new prices of their product. It is to no one's advantage to set his price higher than the others set theirs, since if he does, he tends to lose his share of the market. Nor is it to anyone's advantage to set his price lower than the others set theirs, since if he does, he menaces his competitors and incurs their retaliation. Hence, each competitor must set his price close to the price he expects the others to set. Second, as described by Diamond [17], in various parts of the world in the early stages of food production hunter-gatherer societies were confronted with the introduction of cultivation of plants and the domestication of animals. It was to one's advantage to coordinate in either hunting and gathering or food production.[2]

[1] Our link formation game differs from the network formation game considered by Aumann and Myerson [3] in that each pair of players takes action only once. In the formation game considered by Aumann and Myerson, all pairs that did not form links are asked repeatedly whether they want to form a link or not. See also Slikker and van den Nouweland [49].

[2] Although, as Diamond points out on page 105 that most peasant farmers and herders were not necessarily better off than hunter-gatherers: "Archeologists have demonstrated that the first farmers in many areas were

Once coordination has been achieved on a certain behavior, then it is likely that this behavior will become the convention. For this reason Lewis [39] and Schelling [50] already stated that a convention should be considered a solution to a coordination problem. More precisely, Lewis defines a convention as a behavioral regularity such that everyone conforms to the regularity, expects others to conform, and wants to conform given that others conform.

The above examples illustrate two fundamental factors important in determining optimal behavior when agents face a coordination problem. An agent's expectation about the behavior of others plays a significant role. But underlying those expectations is an interaction structure governing communication between the players. Fundamentally, for any value to be transferred between two players from an interaction, there must be some coordination due to the fact that successful communication entails the use of some coordination device, i.e., language, gestures, email, signals, etc. Implicit in our discussion of conventions, we find ourselves talking about localities: geographic or social. Diamond [17] stresses the local, spatial interaction throughout his narrative.[3] Other examples of well-known conventions are languages, currencies, product standards, codes of dress and accounting standards.

In the third chapter we assume that the payoffs from a network connection are the outcome of a coordination game rather than an uniform value. We analyze the dynamic implications of learning in a large population coordination game where agents are distributed spatially, and both the actions of the players and the communication network between these players evolve over time. We follow the conventional evolutionary game theoretic models on coordination in assuming that players use the same pure strategy against all opponents they interact with, i.e., the players with whom they are linked directly, and we allow for this strategy to be adjusted over time. We depart from the conventional models in assuming that the interaction network itself is also subject to evolutionary pressure. Jackson and Watts [31] develop a similar setting; we depart from that model by incorporating cost considerations of social interaction. We devise a circular model with an endogenous communication network, meaning that the locations of

smaller and less well-nourished, suffered from more serious diseases, and died on the average at a younger age than the hunter-gatherers they replaced." The explanation offered for the increase in farming and herding communities is that individuals were seeking to minimize the risk of starvation.

[3] On page 103 Diamond writes,"In short, only a few areas of the world developed food production independently, and they did so at wildly differing times. From those nuclear areas, hunter-gatherers of some neighboring areas learned food production,..."

the players are fixed but players can create their own interaction neighborhood by forming and severing links with other players.

Players typically react myopically to their environment by deciding about both pure strategies as well as links based on a best-reply dynamics. Sometimes, however, players make mistakes when implementing their decisions, or alternatively players experiment with non-optimal replies. Whether or not these mistakes should be included explicitly in the model depends on the span of time over which we are interested in the players' behavior as predicted by the model. As explained by Binmore, Samuelson and Vaughan [8] the model corresponds to the players' medium-run behavior in the absence of the perturbations representing the players' mistakes. We find that in this case, the dynamic process converges to an absorbing state. As the set of absorbing states includes states in which different kinds of behavior are observed, the population's medium-run behavior is possibly characterized by coexistence of conventions. In the long run, i.e., when the perturbations representing the players' mistakes are taken into account, coexistence of conventions is no longer possible. Namely, the risk-dominant convention is the unique stochastically stable convention, meaning that it will be observed almost surely when the mistake probabilities are small but nonvanishing. Chapter 3 is a version of Droste, Gilles and Johnson [18].

1.3 Social Capital

Social Capital is an important determinant in some economic situations because it is an asset, albeit a difficult to measure and sometimes non-convertible one. In many parts of the world social capital is the only asset that some individuals have. For example, Bornstein [10] documents the Grameen Bank in Bangladesh that only lends to small groups called *lending circles*. The primary asset of these circles is social capital.

Before now, social capital has been the subject of much discussion but it has not been formally modelled using economic and game theoretic techniques. It is the objective of the fourth chapter to model social capital from such an economic perspective. Agents rationally form and sever relationships according to the cost and benefit of those relationships. The benefit of accumulated social capital may allow a relationship to persist over time that would otherwise be severed. In society this may have negative or positive consequences. One could imagine a relationship where credit is extended from one player to another, not because of the potential for future income, but because there is a history of a profitable relationship. If the loan is made because of past performance and with the full expectation that it won't be repaid, it may seem irrational from the perspective of the lending agent to make the loan. But if the lending agent receives a positive benefit from the memory of the past relationship, it is not irrational for the lending agent to continue to participate in the relationship. This phenomenon demonstrates a form of inertia in economic relationships. Once a relationship is formed, and it has benefited both parties, it is not easily severed.

In the final chapter we use the rational formation of links between individuals as the foundation for an investigative model of the accumulation of social capital. We assume that the players are coordinated in that each link offers a payoff in the current period. A network link is the current investment in a relationship. As links are formed and maintained over time they begin to accumulate social capital. Like any other asset, social capital pays a return every period and it depreciates. It is necessary for links to be present to create social capital. However, links need not be present to use social capital, but as we show, social capital diminishes when is not maintained through links. We presented two simulation models of social capital formation. Both models of social capital accumulation show that social capital can be accumulated without creating the incentive for all players to be connected. The first model is a model of social capital accumulation and the second shows an additional benefit of accumulated capital – the choice of some agents to form links that have a higher maintenance cost.

Chapter 2

Spatial Social Networks

2.1 Introduction

Increasing evidence shows that *social capital* is an important determinant in trade, crime, education, health care and rural development. Broadly defined, social capital refers to the institutions and relationships that shape a society's social interactions (see Woolcock [53]). Anecdotal evidence for the importance of social capital formation for the well-functioning of our society is provided by Jacobs [33] on page 180: "These [neighborhood] networks are a city's irreplaceable social capital. When the capital is lost, from whatever cause, the income from it disappears, never to return until and unless new capital is slowly and chancily accumulated." Knack and Keefer [38] recently explored the link between social capital and economic performance. They found that trust and civic cooperation have significant impacts on aggregate economic activity. Social networks, especially those networks that take into account the social differences among persons, are the media through which social capital is created, maintained and used. In short, spatial social networks convey social capital. It is our objective to study the formation and the structure of such spatial social networks.

Social networks form as individuals establish and maintain relationships.[1] Being "connected" greatly benefits an individual. Yet, maintaining relationships is costly. As a con-

[1] Watts and Strogetz [52] recently showed with computer simulations using deterministic as well as stochastic elements one can generate social networks that are highly efficient in establishing connections between individuals. This refers to the "six degrees of separation" property as perceived in real life networks.

sequence individuals limit the number of their active relationships. These social-relationship networks develop from the participants' comparison of costs versus benefits of connecting.

To study spatial social networks we extend the Jackson-Wolinsky [32] framework by introducing a *spatial* cost topology. Thus, we incorporate the main hypotheses from Debreu [16] that players located closer to one another incur less cost to establish communication. We limit our analysis to the simplest possible implementation of this spatial cost topology within the Jackson-Wolinsky framework. Individuals are located along the real line as in Akerlof's [1] model of social distance, and the distance between two individuals determines the cost of establishing a direct link between them. The consequences of this simple extension are profound. A rich structure of social networks emerges, showing the relative strength of the specificity of the model.

First, we identify the *pairwise stable networks* introduced by Jackson and Wolinsky [32]. We find an extensive typology of such networks. We mainly distinguish two classes: If costs are *high* in relation to the potential benefits, only the empty network is stable. If costs are *low* in relation to the potential benefits, an array of stable network architectures emerges. However, we derive that *locally complete networks* are the most prominent stable network architecture in this spatial setting. In these networks, localities are completely connected. This represents a situation frequently studied and applied in spatial games, as exemplified in the literature on local interaction, e.g., Ellison [20] and Goyal and Janssen [27]. This result also confirms the anecdotal evidence from Jacobs [33] on city life. Furthermore, we note that the networks analyzed by Watts and Strogetz [52] and the notion of the *closure* of a social network investigated by Coleman [15] also fall within this category of locally complete networks.

Next, we turn to the consideration of Pareto optimal and efficient spatial social networks. A network is *efficient* if the total utility generated is maximal. Pareto optimality leads to an altogether different collection of networks. We show that efficient networks exist that are not pairwise stable. This is comparable to the conflict demonstrated by Jackson and Wolinsky [32].

Finally, we present an analysis of the subgame perfect implementation of stable networks by creating an appropriate network formation game. We introduce a class of defined, multi-stage *link formation games* in which all pairs of players sequentially have the potential to form links.

The order in which pairs take action is given exogenously.[2] We show that subgame perfect Nash equilibria of such link formation games may consist of pairwise-stable networks only.

Related Literature

In the literature on network formation, economists have developed cost-benefit theories to study the processes of link formation and the resulting networks. One approach in the literature is the formation of social and economic relationships based on cost considerations only, thus neglecting the benefit side of such relationships. Debreu [16], Haller [28], and Gilles and Ruys [25] theorized that costs are described by a topological structure on the set of individuals, being a *cost topology*. Debreu [16] and Gilles and Ruys [25] base the cost topology explicitly on characteristics of the individual agents. Hence, the space in which the agents are located is a topological space expressing individual characteristics. We use the term "neighbors" to describe agents who have similar individual characteristics. The *more similar* the agents, with regard to their individual characteristics, the *less costly* it is for them to establish relationships with each other. Haller [28] studies more general cost topologies. The papers cited investigate the coalitional cooperation structures that are formed based on these cost topologies. Thus, cost topologies are translated into constraints on coalition formation. Neglecting the benefits from network formation prevents these theories from dealing with the hypothesis that the more *dissimilar* the agents, the more beneficial their interactions might be.

A second approach in the literature emphasizes the benefits resulting from social interaction. The cost topology is a priori given and reduced to a set of constraints on coalition formation or to a given network. Given these constraints on social interaction, the allocation problem is investigated. For an analysis of constraints on coalition formation and the core of an economy, we refer to, e.g., Kalai et al. [35] and Gilles et al. [24]. Myerson [43] initiated a cooperative game theoretic analysis of the allocation problem under such constraints. For a survey of the resulting literature, we also refer to van den Nouweland [44] and Borm, van den Nouweland and Tijs [9].

[2] Our link formation game differs from the network formation game considered by Aumann and Myerson [3] in that each pair of players takes action only once. In the formation game considered by Aumann and Myerson, all pairs that did not form links are asked repeatedly whether they want to form a link or not. See also Slikker and van den Nouweland [49].

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2.2 Social Networks

We let $N = \{1, 2, ..., n\}$ be the set of players, where $n \geqq 3$. We introduce a spatial component to our analysis. As remarked in the introduction, the spatial dispersion of the players could be interpreted to represent the social distance between the players. We require players to have a *fixed* location on the real line $\mathbb{R}$. Player $i \in N$ is located at x_i. Thus, the set $X = \{x_1, ..., x_n\} \subset [0, 1]$ with $x_1 = 0$ and $x_n = 1$ represents the spatial distribution of the players. Throughout the paper we assume that $x_i < x_j$ if $i < j$ and the players are located on the unit interval. This implies that for all $i, j \in N$ the *distance* between i and j is given by $d_{ij} := |x_i - x_j| \leqq 1$.

Network relations among players are formally represented by graphs where the nodes are identified with the players and in which the edges capture the pairwise relations between these players. These relationships are interpreted as *social* links that lead to benefits for the communicating parties, but on the other hand are costly to establish and to maintain.

We first discuss some standard definitions from graph theory. Formally, a *link* ij is the subset $\{i, j\}$ of N containing i and j. We define $g^N := \{ij \mid i, j \in N\}$ as the collection of all links on N. An arbitrary collection of links $g \subset g^N$ is called an (undirected) *network* on N. The set g^N itself is called the *complete network* on N. Obviously, the family of all possible networks on N is given by $\{g \mid g \subset g^N\}$. The number of possible networks is $\sum_{k=1}^{c(n,2)} c(c(n, 2), k) + 1$, where for every $k \leqq n$ we define $c\,(n, k) := \frac{n!}{k!(n-k)!}$.

Two networks $g, g' \subset g^N$ are said to be of the same *architecture* whenever it holds that $ij \in g$ if and only if $n - i + 1, n - j + 1 \in g'$. It is clear that this defines an equivalence relation on the family of all networks. Each equivalence class consists exactly of two mirrored networks and will be denoted as an "architecture."[3]

Let $g + ij$ denote the network obtained by adding link ij to the existing network g and $g - ij$ denote the network obtained by deleting link if from the existing network g, i.e., $g + ij = g \cup \{ij\}$ and $g - ij = g \setminus \{ij\}$.

Let $N(g) = \{i \mid ij \in g \text{ for some } j\} \subset N$ be the set of players involved in at least one link and let $n(g)$ be the cardinality of $N(g)$. A *path* in g connecting i and j is a set of distinct players $\{i_1, i_2, ..., i_k\} \subset N(g)$ such that $i_1 = i$, $i_k = j$, and $\{i_1i_2, i_2i_3, ..., i_{k-1}i_k\} \subset g$. We call a network

[3] Bala and Goyal [5] define an architecture as a set of networks that are equivalent for arbitrary permutations. We only allow for mirror permutations to preserve the cost topology.

connected if between any two nodes there is a path. A *cycle* in g is a path $\{i_1, i_2, ..., i_k\} \subset N(g)$ such that $i_1 = i_k$. We call a network *acyclic* if it does not contain any cycles. We define t_{ij} as the number of links in the shortest path between i and j. A *chain* is a connected network composed of exactly one path with a spatial requirement.

Definition 1 *A network $g \subset g^N$ is called a* ***chain*** *when (i) for every $ij \in g$ there is no h such that $i < h < j$ and (ii) g is connected.*

Since $i < j$ if and only if $x_i < x_j$, there exists exactly one chain on N and it is given by $g = \{12, 23, \ldots, (n-1)\, n\}$.

Let $i, j \in N$ with $i < j$. We define $i \leftrightarrow j := \{h \in N \mid i \leqq h \leqq j\} \subset N$ as the set of all players that are spatially located between i and j and including i and j. We let $n\,(ij)$ denote the cardinality of the set $i \leftrightarrow j$. Furthermore, we introduce $\ell\,(ij) := n\,(ij) - 1$ as the *length* of the set $i \leftrightarrow j$. The set $i \leftrightarrow j$ is a *clique* in g if $g^{i \leftrightarrow j} \subset g$ where $g^{i \leftrightarrow j}$ is the complete network on $i \leftrightarrow j$.

Definition 2 *A network g is called* ***locally complete*** *when for every $i < j : ij \in g$ implies $i \leftrightarrow j$ is a clique in g.*

Locally complete networks are networks that consist of spatially located cliques. These networks can range in complication from any subnetwork of the chain to the complete network. In a locally complete network, a connected agent will always be connected to at least one of his direct neighbors and belong to a complete subnetwork.

To illustrate the social relevance of locally complete networks we refer to Jacobs [33], who keenly observes the intricacy of social networks that turn city streets, blocks and sidewalk areas into a city neighborhood. Using the physical space of the a city street or sidewalk as an example of the space for the players, the concept of local completeness could be interpreted as each player knowing everyone on his block or section of the sidewalk.

Definition 3 *Let $i, j \in N$. The set $i \leftrightarrow j \subset N$ is called a* ***maximal clique*** *in the network $g \subset g^N$ if it is a clique in g and for every player $h < i$, $h \leftrightarrow j$ is not a clique in g and for every player $h > j$, $i \leftrightarrow h$ is not a clique in g.*

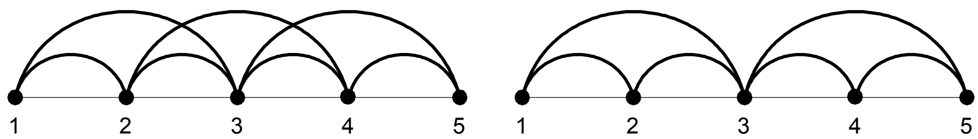

Figure 2-1: Examples of locally complete networks.

A maximal clique in a certain network is a subset of players that represent a maximal complete subnetwork of that network. For some results in this paper a particular type of locally complete network is relevant.

Definition 4 *Let $k \leqq n$. A network g is called **regular of order k** when for every i, $j \in N$ with $\ell(ij) = k$, the set $i \leftrightarrow j$ is a maximal clique.*

Examples of regular networks are the empty network and the chain; the empty network is regular of order zero, while the chain is regular of order one. The complete network is regular of order $n - 1$.

Finally, we introduce the concept of a *star* in which one player is directly connected to all other players and these connections are the only links in the network. Formally, the star with player $i \in N$ as its center is given by $g_i^s = \{ij \mid j \neq i\} \subset g^N$.

To illustrate the concepts defined we refer to Figure 2-1. The left network is the second order regular network for $n = 5$. The right network is locally complete, but not regular.

2.3 A Spatial Connections Model

A network creates benefits for the players, but also imposes costs on those players who form links. Throughout we base benefits of a player $i \in N$ on the connectedness of that player in the network: For each player $i \in N$ her individual payoffs are described by a utility function $u_i : \{g \mid g \subset g^N\} \to \mathbb{R}$ that assigns to every network a (net) benefit for that player.

Following Jackson and Wolinsky [32] and Watts [51] we model the total *value* of a certain network $g \subset g^N$ as

$$v(g) = \sum_{i \in N} u_i(g). \tag{2.1}$$

This formulation implies that we assume a transferable utility formulation.

We modify the Jackson-Wolinsky *connections model*[4] by incorporating the spatial dispersion of the players into a non-trivial cost topology. This is pursued by replacing the cost concept used by Jackson and Wolinsky with a cost function that varies with the spatial distance between the different players.

Let $c : g^N \to \mathbb{R}_+$ be a general cost function with $c(ij) \geqq 0$ being the cost to create or maintain the link $ij \in g^N$. We simplify our notation to $c_{ij} = c(ij)$. In the Jackson-Wolinsky connections model the resulting utility function of each player i from network $g \subset g^N$ is now given by

$$u_i(g) = w_{ii} + \sum_{j \neq i} w_{ij} \delta^{t_{ij}} - \sum_{j:\, ij \in g} c_{ij}, \tag{2.2}$$

where t_{ij} is the number of links in the shortest path in g between i and j, $w_{ij} \geqq 0$ denotes the intrinsic value of individual i to individual j, and $0 < \delta < 1$ is a communication depreciation rate. In this model the parameter δ is a depreciation rate based on network connectedness, not a spatial depreciation rate.

Using the Jackson-Wolinsky connections model and a linear cost topology we are now able to re-formulate the utility function for each individual player to arrive at a *spatial connections model*. We assume that the n individuals are uniformly distributed along the real line segment $[0, 1]$. We define the cost of establishing a link between individuals i and j as $c_{ij} = c \cdot \ell(ij)$ where $c \geqq 0$ is the spatial unit cost of connecting. Finally, we simplify our analysis further by setting for each $i \in N$: $w_{ii} = 0$ and $w_{ij} = 1$ if $i \neq j$. This implies that the utility function for $i \in N$ in the Jackson-Wolinsky connections model — given in (2.2) — reduces to

$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - c \sum_{j:\, ij \in g} \ell(ij). \tag{2.3}$$

The formulation of the individual benefit functions given in equation (2.3) will be used throughout the remainder of this paper. For several of our results and examples we make an additional simplifying assumption that $c = \frac{1}{n-1}$.

[4] Jackson and Wolinsky discuss two specific models, the connections model and the co-author model, and a general model. The connectons model and the co-author model are completely characterized by a specific formulation of the individual utility functions based on the assumptions underlying the sources of the benefits of a social network. Here we only consider the connections model.

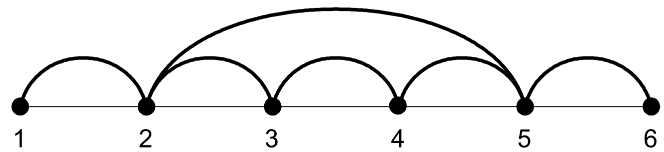

Figure 2-2: Pairwise stable network for $n = 6$, $c = \frac{1}{5}$, $\delta = \frac{7}{10}$

The concept of *pairwise stability* (Jackson and Wolinsky [32]) represents a natural state of equilibrium for certain network formation processes; The formation of a link requires the consent of both parties involved, but severance can be done unilaterally.

Definition 5 *A network $g \subset g^N$ is **pairwise stable** if*

1. *for all $ij \in g$, $u_i(g) \geqq u_i(g - ij)$ and $u_j(g) \geqq u_j(g - ij)$, and*
2. *for all $ij \notin g$, $u_i(g) < u_i(g + ij)$ implies that $u_j(g) > u_j(g + ij)$.*

Overall efficiency of a network is in the literature usually expressed by the total utility generated by that network. Consequently, a network $g \subset g^N$ is *efficient* if g maximizes the value function $v = \sum_N u_i$ over the set of all potential networks $\{g \mid g \subset g^N\}$, i.e., $v(g) \geqq v(g')$ for all $g' \subset g^N$.

2.3.1 Pairwise Stability in the Spatial Connections Model

The spatial aspect of the cost topology enables us to identify pairwise stable networks with spatially discriminating features. For example, individuals may attempt to maintain a locally complete network but refuse to connect to more distant neighbors. Conversely, it may benefit individuals who are locally connected to maintain a connection with a player who is far away and also well-connected locally. Such a link would have a large spatial cost but it could have an even larger benefit. The example depicted in Figure 2-2 illustrates a relatively simple non-locally complete network in which players 2 and 5 enjoy the benefits of close connections as well as the indirect benefits of a distant, costly connection. (Here, we call a network *non-locally complete* if it is not locally complete.) A star is a highly organized non-locally complete network.

Example 6 Let $n = 6$, $c = \frac{1}{n-1} = \frac{1}{5}$, and $\delta = \frac{7}{10}$. Consider the network depicted in Figure 2-2. This non-locally complete network is pairwise stable for the given values of c and δ. We observe that players 2 and 5 maintain a link 50% more expensive than a potential link to player 4 or 3 respectively. The pairwise stability of this network hinges on the fact that the direct and indirect benefits, δ and δ^2, are high relative to the cost of connecting. In this example $u_2(g) = 3\delta + 2\delta^2 - 5c$. If player 2 severed her long link then her utility, $u_2(g-25)$, would be $\delta + \sum_{k=1}^{4} \delta^k - 2c$. $u_2(g) - u_2(g-25) = \delta + \delta^2 - \delta^3 - \delta^4 - 3c = 0.0069 > 0$. Each players is willing to incur higher costs to maintain relationships with distant players in order to reap the high benefits from more valuable indirect connections. ♦

We investigate which networks are pairwise stable in the spatial connections model. We distinguish two major mutually exclusive cases: $\delta > c$ and $\delta \leqq c$. For $\delta > c$ there is a complex array of possibilities. We highlight the locally complete and non-locally complete insights below and leave the remaining results for the Section 2.6. For a proof to Proposition 7 we refer to Section 2.6.

For all δ and c we define

$$\hat{n}(c,\delta) := \left\lfloor \frac{\delta}{c} \right\rfloor \tag{2.4}$$

where $\left\lfloor \frac{\delta}{c} \right\rfloor$ indicates the smallest integer greater than or equal to $\frac{\delta}{c}$.

Proposition 7 *Let $\delta > c > 0$.*

(a) *For $[\hat{n}(c,\delta) - 1] \cdot c < \delta - \delta^2$ and $\hat{n}(c,\delta) \geqq 3$, there exists a pairwise stable network which is regular of order $\hat{n}(c,\delta) - 1$.*

(b) *For $c > \delta - \delta^2$, there is no pairwise stable network which contains a clique of a size of at least three players.*

(c) *For $c > \delta - \delta^2$, $\delta > \frac{1}{2}$ and $c = \frac{1}{n-1}$, for $n \leqq 5$ the chain is the only regular pairwise stable network, for $n = 6$ there are certain values of δ for which the chain is pairwise stable, and for $n \geqq 7$ the chain is never pairwise stable.*

To illustrate why the restrictions of $\hat{n}(c,\delta) \geqq 3$ and $[\hat{n}(c,\delta) - 1] \cdot c < \delta - \delta^2$ are placed in the formulation of Proposition 7(a) we refer the following example.

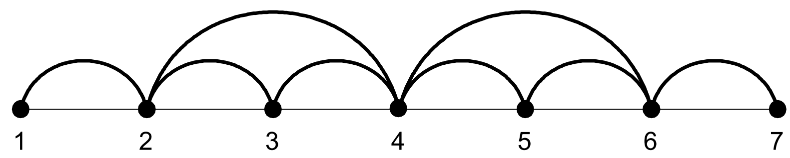

Figure 2-3: Pairwise stable network for $n = 7$, $c = \frac{1}{6}$, $\delta = \frac{1}{2}$

Example 8 Let $n = 7$, $c = \frac{1}{n-1} = \frac{1}{6}$, and $\delta = \frac{1}{2}$. Consider the network depicted in Figure 2-3. This network is pairwise stable for the given values of c and δ.
We identify two maximal cliques of size 2, $\{1, 2\}$ and $\{6, 7\}$, and two maximal cliques of the size 3, $\{2, 3, 4\}$ and $\{4, 5, 6\}$. Thus, this pairwise stable network is locally complete, but it is not regular of any order. With reference to Proposition 7(a) we note that $\hat{n}(c, \delta) = 3$. However, $[\hat{n}(c, \delta) - 1] \cdot c = \frac{1}{3} > \delta - \delta^2 = \frac{1}{4}$. ♦

For $\delta < c$ the analysis becomes involved in particular due to the possibility of cyclic pairwise stable networks. A proof of the next proposition on acyclic networks only can be found in Section 2.6.

Proposition 9 *Let* $0 < \delta \leqq c = \frac{1}{n-1}$.

(a) *For* $\delta < c$ *there exists exactly one acyclic pairwise stable network, the empty network.*

(b) *For* $\delta = c$ *there exist exactly two acyclic pairwise stable networks, the empty network and the chain.*

The following example illustrates the possibilities if we allow for cycles.

Example 10 Consider a network g_c for n even, i.e., we can write $n = 2k$. The network g_c is defined as the unique cycle given by

$$g_c = \{12, (n-1)\, n\} \cup \{i\,(i+2) \mid i = 1, \ldots, n-2\}.$$

For $k = 5$ the resulting network is depicted in Figure 2-4. This cyclic network is pairwise stable for $\delta = \sqrt[4]{\frac{1}{5}} \sim 0.66874$ and $c = \delta \frac{1-2\delta^k}{2-2\delta} \sim 0.739$

For calculations to support this example we again refer to Section 2.6.

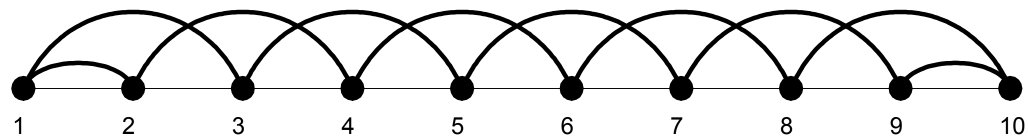

Figure 2-4: Example of a cyclic pairwise stable network.

2.3.2 Efficiency in the Spatial Connections Model

Recall that a network $g \subset g^N$ is *efficient* if g maximizes the value function $v = \sum_N u_i$ over the set of all potential networks $\{g \mid g \subset g^N\}$, i.e., $v(g) \geqq v(g')$ for all $g' \subset g^N$. We show that efficient networks exist that are not pairwise stable. This is consistent with the insight derived by Jackson and Wolinsky [32] regarding efficient networks.

Our main result shows that for $c > \delta$ any efficient network is either the chain or the empty network. This is mainly due to the fact that the chain is the least expensive connected graph.

Theorem 11 *Let* $0 < \delta < c = \frac{1}{n-1}$.

(a) *For* $c > \delta + \frac{1}{n-1} \sum_{k=2}^{n-1} (n-k)\delta^k$, *the only efficient network is the empty network.*

(b) *For* $c < \delta + \frac{1}{n-1} \sum_{k=2}^{n-1} (n-k)\delta^k$, *the only efficient network is the chain.*

For a proof of Theorem 11 we refer to Section 2.6.

Next we turn our attention numerical computations of highest valued networks. Even for relatively small numbers of players the number of possible networks can be very large, requiring us to use a computer program to calculate the value of all social networks for each n. We limit our computations to $n \leqq 7$ as the number of possible networks for $n = 8$ exceeds 250 million. Given n, $c = \frac{1}{n-1}$, and δ, Figure 2-5 summarizes our results. Figure 2-6 identifies the ranges of δ for which the social networks are both pairwise stable and efficient. Numerical values corresponding to Figures 2-5 and 2-6 can be found in Section 2.6.

We highlight some simple observations on pairwise stability and efficiency by comparing Figures 2-5 and 2-6. We focus on one non-locally complete network with 6 players and four networks with 7 players to illustrate some of the conflict and coincidence that occurs between efficiency and pairwise stability.

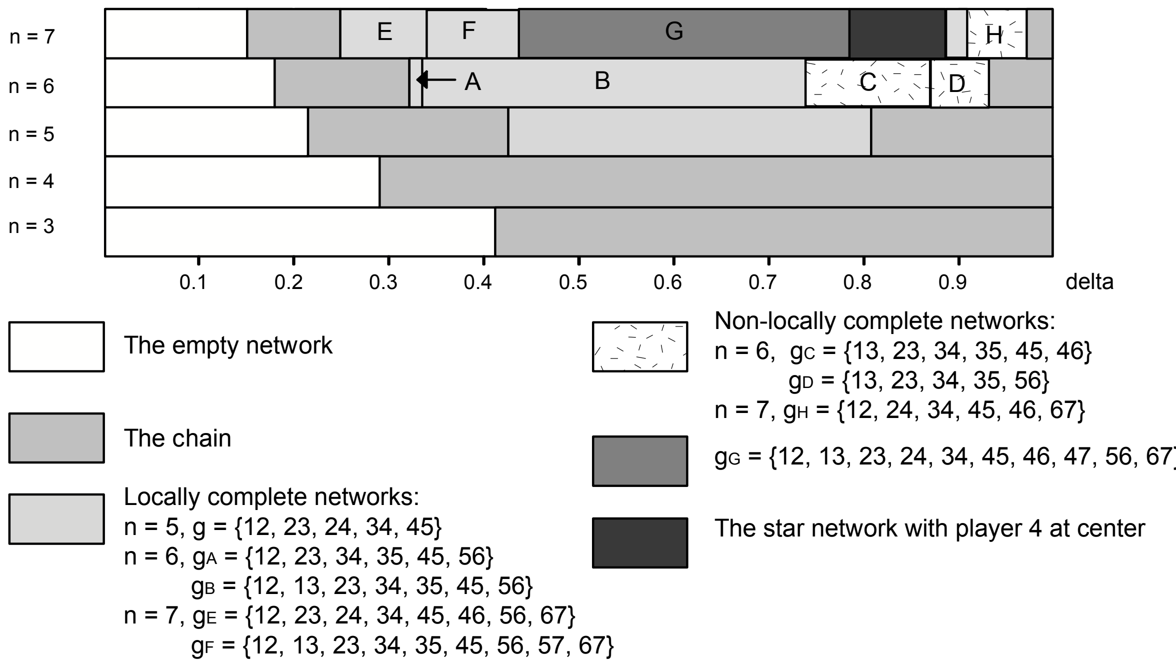

Figure 2-5: Typology of the efficient networks for $n \leqq 7$.

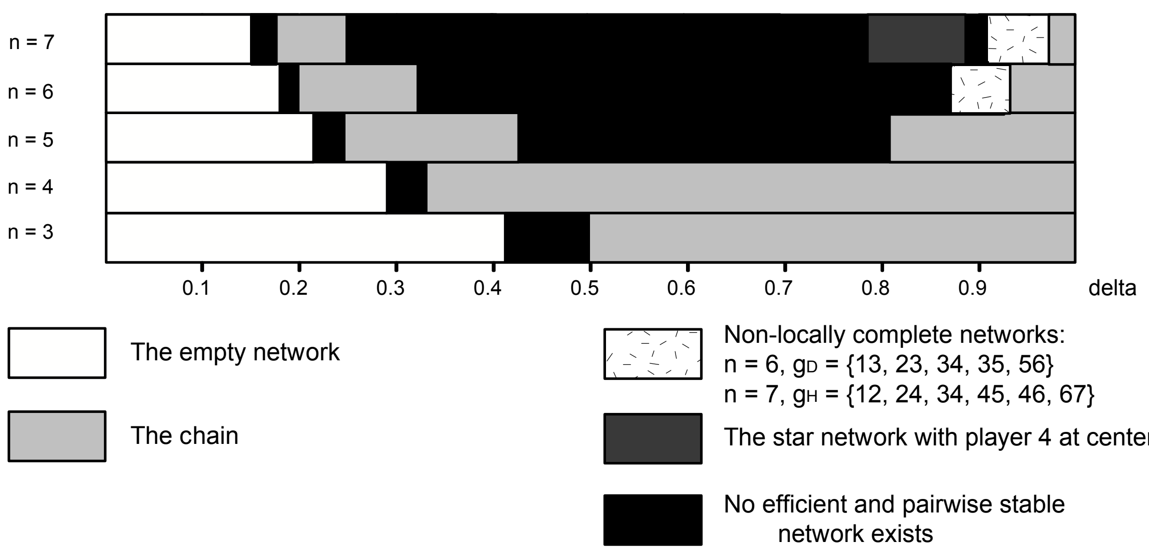

Figure 2-6: Efficient and pairwise stable networks for $n \leqq 7$.

For $n = 6$ and the range of δ labelled C the non-locally complete network g_C is efficient. The three links $\{34, 35, 45\}$ give this network a locally complete aspect that renders it unstable. The intuition for this instability is found in the proof to Proposition 7.

For $n = 7$, the locally complete network g_E is efficient in range E. This network is described in Example 8 and depicted in Figure 2-3. Similarly the locally complete network g_F is efficient in range F. Neither network is pairwise stable. The star g_4^s is efficient for $\delta \in [0.7887, 0.8811]$ as well as pairwise stable by Lemma 20(b) found in Section 2.6. Finally, for range H, the network g_H is efficient and pairwise stable. This network architecture is discussed in the proof of Lemma 12 below.

We conclude that the empty network is always pairwise stable if it is efficient. For $n \leqq 7$ and $\delta \geqq c = \frac{1}{n-1}$, if the chain is efficient it is also pairwise stable. For relatively high δ, the chain is always efficient because the relative difference between direct and indirect connections is quite small.[5] Proposition 7(c) reminds us that for $n \geqq 7$ and $\delta > \frac{1}{2}$ the chain is never pairwise stable. Finally, for $n \leqq 7$, a locally complete network with a clique of three or more players is never efficient and pairwise stable.

Bala and Goyal [4] demonstrated that for relatively high and low connection costs, pairwise stability and efficiency coincide. We arrive at a different insight. For relatively high cost the empty network is both the unique pairwise stable and efficient network. For relatively low cost we see the chain emerge as an efficient network. However, Proposition 7(c) rules out any coincidence of stability and efficiency. In the standard non-spatial connections model with $\delta - \delta^2 < c < \delta$ a star is pairwise stable as well as the unique efficient network. (Jackson and Wolinsky [32], Proposition 1(ii) and Proposition 2(iii).) In our model with the additional assumption that $\delta > \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$ we show through Lemma 20(b) that the star is also pairwise stable. The next result confirms that the star is not efficient for relatively large values of δ in our spatial connections model.

Lemma 12 *Let* $\delta^2 - \delta^3 < \frac{c}{2}$, $c < \delta$ *with* $\delta > \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$. *Then any star is not efficient.*

Proof. Without loss of generality we may assume that n is even. We examine the value of two networks: (1) $g^s \subset g^N$ is the star with its center at $\frac{n}{2}$ and (2) $g' \subset g^N$ is the network which

[5] The chain is efficient for $n = 5$ if $\delta \in [0.215, 0.4287] \cup [0.8129, 1)$; $n = 6$ if $\delta \in [0.1727, 0.3141] \cup [0.9307, 1)$; and $n = 7$ if $\delta \in [0.1465, 0.2467] \cup [0.9695, 1)$.

is equal to g^s except that player n is linked to $n-1$ instead of the center $\frac{n}{2}$. The value of g^s is

$$v\left(g^{s}\right)=2(n-1)\delta+(n-1)(n-2)\delta^{2}-4\sum_{k=1}^{\frac{n-1}{2}}kc-2\left(\frac{n}{2}\right)c. \tag{2.5}$$

The value of the network g' is

$$v\left(g'\right)=2(n-1)\delta+(n-2)(n-3)\delta^{2}+2\delta^{2}+2\left(n-3\right)\delta^{3}-4\sum_{k=1}^{\frac{n-1}{2}}kc-2c. \tag{2.6}$$

The difference between equation (2.5) and equation (2.6) is

$$2(n-3)\delta^{2}-2(n-3)\delta^{3}-(n-2)c$$

This is negative when $\delta^2 < \delta^3 + \frac{(n-2)}{2(n-3)}c$. We conclude that the star g^s may not be the network with the highest value. ■

2.4 Implementation of Pairwise Stable Networks

Implementation of pairwise stable networks has been explored in the literature for the Jackson-Wolinsky framework with a binary cost topology. Watts [51] explicitly models the connections model of Jackson and Wolinsky [32] as an extensive form game. She bases her analysis on the myopic players playing the Grim Strategy[6] to illustrate the resulting equilibria of such a game. Dutta and Mutuswami [19] look at the relationship between stability and efficiency, but they use a static, strategic form framework. In the spatial connections model we look at a natural extensive form game in which all pairs meet exactly once to form a link. We investigate the subgame perfect Nash equilibria of this game and show that for certain orders in which the pairs meet we can implement specific pairwise stable networks. A full analysis of this game with random order of play is deferred to future research.

Initially, in our game none of the players are connected. Over multiple playing rounds,

[6] Watts [?] defines the Grim Strategy as follows: Each player agrees to link with the fist two players he meets. Secondly, each player never severs a link as long as all the other players cooperate. However, if player i deviates, then every player $j \neq i$ severs all ties with i and refuses to form any links with i for the rest of the game. Thus, if player i deviates, his payoff will be 0 in all future periods.

players make contact with the other players and determine whether to form a link with each other or not. Exactly one pair of players meets each round — or "stage." Each pair of players meets once and only once in the course of the game. The resulting extensive form game is called the *link formation game*.

We remark that our link formation game differs considerably from the one formulated in Aumann and Myerson [3]. There the pairs that did not link in previous stages of the game, meet again to reconsider their decision. The game continues until a stable state has been reached in which no remaining unlinked pairs of players are willing to reconsider. Obviously our structure implies that the "order of play" is crucial, while the Aumann-Myerson structure this is not the case. On the other hand the analysis of our game is more convenient and rather strong results can be derived.

Formally, an "order of play" in the link formation game is represented by a bijection $O : g^N \to \{1, \ldots, c(n,2)\}$ that assigns to every potential pair of players $\{i, j\} \subset N$ a unique index $O_{ij} \in \{1, \ldots, c(n,2)\}$. The set of all orders is denoted by $\mathbb{O}$.

The link formation game has therefore $c(n,2)$ stages. In stage k of the game the pair $\{i, j\} \subset N$ such that $O_{ij} = k$ play a subgame. For any two players, i and j with $i < j$, the choice set facing each player is $A_i(ij) = \{C_{ij}, R_{ij}\}$ and $A_j(ij) = \{C_{ij}, R_{ij}\}$, where C_{ij} represents the offer to establish the link ij and R_{ij} represents the refusal to establish the link ij. Players will form a link when it is mutually agreed upon, i.e., link ij is established if and only if both players i and j select action C_{ij}. No link will be formed if either player refuses to form the link, i.e., when either one of the players i or j selects R_{ij}. Link formation is permanent; no player can sever the links that were formed during earlier stages of the game. The sequence of actions, recorded as the history of the game, determines in a straightforward fashion the resulting network. We emphasize that all players have complete information in this game.

To complete the description of strategies in the link formation game with order of play $O \in \mathbb{O}$ we introduce the notion of a *(feasible) history*. A history is a listing $h \in H(O) := \cup_{k=1}^{c(n,2)} H_k(O)$, where

$$\begin{aligned} H_k(O) &= X_1(O) \times \cdots \times X_k(O) \text{ with for every } 1 \leqq p \leqq k \\ X_p(O) &= A_i(ij) \times A_j(ij) \text{ for } \{i, j\} \subset N \text{ with } O_{ij} = p. \end{aligned}$$

The history $h = (h_1, \ldots, h_k) \in H_k(O)$ is said to have a length of k, where $h_p \in X_p(O)$ for every $1 \leqq p \leqq k$. A history describes all actions undertaken by the players in the link formation game up till a certain moment in that game. The network $g(h) \in g^N$ corresponding to history $h = (h_1, \ldots, h_k) \in H_k(O)$ is defined as the network that has been formed up till stage k of the link formation game with order O, i.e., $ij \in g(h)$ if and only if $O_{ij} \leqq k$ and $x_{O_{ij}} = (C_{ij}, C_{ij})$. Now we are able to introduce for each player $i \in N$ the *strategy set*

$$S_i = \prod_{ij \in g^N} \prod_{h \in H_{O_{ij}}(O)} A_i(ij). \tag{2.7}$$

A strategy for player i assigns to every potential link ij of which i is a member, and every possible history of the link formation game up till stage O_{ij} an action. A strategy tuple in the link formation game is now given by $a \equiv (a_1, \ldots, a_n) \in S := \prod_{i \in N} S_i$. With each strategy $a \in A$ we can define the resulting network as $g_a \subset g^N$. Furthermore, player i receives a payoff $u_i(g_a)$ for every strategy tuple $a \in S$.

Formally, for any order of play $O \in \mathbb{O}$ the above describes a game tree $\mathcal{G}_O$. This implies that for order $O \in \mathbb{O}$ the *link formation game* Γ_O may be described by the $(2n+2)$-tuple

$$\Gamma_O = (N, \mathcal{G}_O, S_1, \ldots, S_n, u_1, ..., u_n). \tag{2.8}$$

Since the link formation game is a well-defined extensive form game, we can use the concept of subgame perfection to analyze the formation of networks. Next we investigate the nature of the subgame perfect Nash equilibria of the link formation game developed above.

Our analysis mainly considers the case that $c < \delta$. As shown in Proposition 7 there is a wide range of non-trivial pairwise stable networks in this situation. It can be shown that there is a set of efficient and pairwise stable networks can be implemented as subgame perfect equilibria of link formation games. First we address the conditions under which regular networks can be implemented as subgame perfect equilibria of the link formation game.

Theorem 13 *Let $m \in \{1, \ldots, n-1\}$. Then for (c, δ) satisfying*

$$\frac{1}{m+1}\delta + \left(\frac{n-1}{m+1} + 1\right)\delta^2 < c < \frac{1}{m}\delta - \frac{1}{m}\delta^2 \tag{2.9}$$

there exists an order of play $O \in \mathbb{O}$ such that the regular network of order m can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.

A proof of this theorem is given in Section 2.7.

We remark that the chain is the unique regular network of order one on N. By substituting $m = 1$ into the condition (2.9) for the implementation of the chain as a subgame perfect Nash equilibrium of a link formation game.

From this main implementation result above we are able to derive some further conclusions. Our first conclusion concerns the support of the complete network as a subgame perfect Nash equilibrium in the link formation game. Such a complete network can be supported for high enough benefits in relation to the link costs:

Corollary 14 *For $(n-1)\,c < \delta - \delta^2$ and for any order of play $O \in \mathbb{O}$, the complete network g^N can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.*

Proof. The assertion follows from a slight modification of part (1) in the proof of Theorem 13 for $m = n - 1$. (Remark that the complete network on N is the unique regular network of order $n - 1$.) Here the order of the game is irrelevant, thus showing that any order of play leads to the establishment of the strategy $\widehat{a}$ as given in the proof of Theorem 13 as a subgame perfect Nash equilibrium. ∎

Finally we consider under which conditions the identified subgame perfect Nash equilibria generate a pairwise stable network. The following corollary of Proposition 7 and Theorem 13 summarizes some insights:

Corollary 15 *The following properties hold:*

(a) *Suppose that $\frac{1}{2}\delta + \frac{n+1}{2}\delta^2 < c = \frac{1}{n-1} < \delta$. Then there exists an order of play $O \in \mathbb{O}$ such that at least one subgame perfect Nash equilibrium of the link formation game with order O is pairwise stable.*

(b) *Suppose that $\hat{n}(c,\delta) \geqq 3$. If*

$$\frac{1}{\hat{n}(c,\delta)+1}\delta + \frac{n+\hat{n}(c,\delta)}{\hat{n}(c,\delta)+1}\delta^2 < c < \frac{1}{\hat{n}(c,\delta)}\delta - \frac{1}{\hat{n}(c,\delta)}\delta^2 \tag{2.10}$$

then there exists an order of play $O \in \mathbb{O}$ such that at least one subgame perfect Nash equilibrium of the link formation game with order O is pairwise stable.

Proof.

(a) First we remark that $\delta > \frac{1}{n-1} > \frac{1}{n+3}$ implies that

$$c > \frac{1}{2}\delta + \frac{n+1}{2}\delta^2 > \delta - \delta^2. \tag{2.11}$$

Now condition (2.11) implies that Lemma 20(a) holds. Hence, the chain is pairwise stable. From (2.11) it follows that Theorem 13 holds, implying that there is an order of play O such that the chain can be supported as a SPNE of that link formation game.

(b) First we remark that from (2.10) it follows immediately that $[\hat{n}(c,\delta) - 1] \cdot c < \hat{n}(c,\delta) \cdot c < \delta - \delta^2$, and so Proposition 7(a) is satisfied. Hence, the regular network of order $\hat{n}(c,\delta) - 1$ is pairwise stable. Furthermore, from (2.10) it follows through Theorem 13 that the regular network of order $\hat{n}(c,\delta) - 1$ can be supported as a subgame perfect Nash equilibrium for some order of play $O \in \mathbb{O}$ in the link formation game. ■

We use an example to illustrate the tension between the order of play, efficiency and pairwise stability when $c < \delta$.

Example 16 Consider the case where $n = 5$, $c = \frac{1}{n-1} = \frac{1}{4}$, and $\delta = \frac{5}{8}$. The star $g_3^s = \{13, 23, 34, 35\}$ is pairwise stable but not efficient. The chain, $g^c = \{12, 23, 34, 45\}$, is also pairwise stable and has a higher total value than the star. The locally complete network $g^l = \{12, 23, 24, 34, 45\}$ is efficient but not pairwise stable; maintaining link 24 decreases utility for both player 2 and player 4.

g	$u_1(g) = u_5(g)$	$u_2(g) = u_4(g)$	$u_3(g)$	$v(g) = \sum_i u_i(g)$
g_3^s	$\delta + 3\delta^2 - 2c$	$\delta + 3\delta^2 - c$	$4\delta - 6c$	$8\delta + 12\delta^2 - 12c$
g^c	$\delta + \delta^2 + \delta^3 + \delta^4 - c$	$2\delta + \delta^2 + \delta^3 - 2c$	$2\delta + 2\delta^2 - 2c$	$8\delta + 6\delta^2 + 4\delta^3 + 2\delta^4 - 8c$
g^l	$\delta + 2\delta^2 + \delta^3 - c$	$3\delta + \delta^2 - 4c$	$2\delta + 2\delta^2 - 2c$	$10\delta + 8\delta^2 + 2\delta^3 - 12c$

Player 3 prefers the chain or the locally complete network over the star; all other players prefer the star to the chain. Players 2 and 4 also prefer the star over the locally complete network.

Depending on the order of play, we can generate the star or the chain; yet never both from the same ordering. For the star to form, we must allow pairs $\{12, 45\}$ to refuse to connect before player 3 has an opportunity to refuse any connection to the furthest star points. The order of play $\{\mathbf{12}, \mathbf{45}, 23, 34, \mathbf{15}, \mathbf{14}, \mathbf{25}, \mathbf{24}, 13, 35\}$ guarantees that the star with the center at 3 forms. The pairs bold-faced in the ordering will not form a link because both players will refuse to make the connection to guarantee the that the network that each of them prefers to form will indeed form. For the chain to form, we must allow player 3 to refuse the links $\{13, 35\}$ before other players have the opportunity to refuse the links $\{12, 45\}$. An ordering that would result in the chain is $\{\mathbf{13}, \mathbf{35}, 23, 34, \mathbf{15}, \mathbf{14}, \mathbf{25}, \mathbf{24}, 12, 45\}$.

If there was a strategy available to encourage players 2 and 4 into enduring the link 24, the players could create the efficient locally complete graph $g^l = \{12, 23, 24, 34, 45\}$. This is because there are two pairwise stable graphs with one link lower than g^l: the chain and the non-locally complete graph $\{12, 24, 34, 45\}$. ♦

Next we turn to an example for the case $c > \delta$.

Example 17 Consider $n = 5$, $c = \frac{1}{n-1} = \frac{1}{4}$, and $\delta = 0.22$. As shown in Theorem 11 the chain is the unique efficient network. However, it is not pairwise stable for these parameter values. In the link formation game the chain can be generated by an order similar to that of the order described in proof of Theorem 13. The difference is that the pairs ij where $n(ij) = 2$ must meet in a specific order: the players located at the end points must meet their direct neighbors first, before any interior pairs meet. For example, the order of play $\{12, 45, 23, 34, \mathbf{13}, \mathbf{24}, \mathbf{35}, \mathbf{14}, \mathbf{25}, \mathbf{15}\}$ guarantees that the chain forms. The first four pairs are ordered so the pairs located nearest to the endpoints have the option of linking. By backward induction, players 2 and 4 realize that if they do not offer to connect to players 1 and 5 respectively, they will not form an attractive potential link for player 3. ♦

The two examples above capture how for a given order of play players can strategically influence the creation of a network. We continue with showing that for a given order we find an outcome where players create a network that is neither efficient nor pairwise stable.

Example 18 Given $n = 4$, $c = \frac{1}{n-1}$ and $\delta = 0.7$. The order $\{\mathbf{34}, \mathbf{23}, 12, 13, 24, \mathbf{14}\}$ generates the network $g = \{12, 13, 24\}$ which is neither efficient nor pairwise stable. (See Figure 2-7.) Both

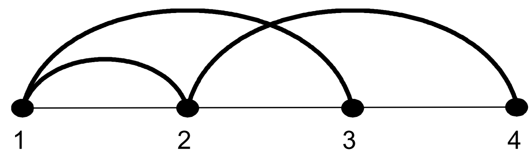

Figure 2-7: Generated network for $n = 4$, $c = \frac{1}{3}$, $\delta = \frac{7}{10}$.

the chain and non-locally complete graph $\{12, 13, 34\}$ are Pareto Superior to g. Furthermore, given the opportunity, players 3 and 4 would benefit from forming a link. This order creates this graph because players use their linking strategies as votes against the graphs that they earn the least in. The central players, 2 and 3, will refuse link 23 so as to not become the center of the star. The players located at the end points, 1 and 4, refuse link 14 so as to veto the graph $\{12, 14, 34\}$ in which they have very little positive utility. Player 3 refuses the first link merely to flip the resulting network architecture forcing player 2 to maintain two links in the network. ♦

We observe that specific structures can emerge from specific orders for certain parameter values. Our results differ from other models of sequential network formation. With myopic players as implemented by Watts [51] sequential play would result in a pairwise stable network; one would not obtain an efficient network as in Example 17. In addition, Aumann and Myerson [3] introduce a sequential game with foresight, but allow unlinked players a last chance to form a link. This would eliminate the possibility of a network as in Example 18 to form. These results suggest that future research should investigate how to model network formation.

2.5 Concluding Remarks

In this paper we introduced a spatial cost topology in a specific network formation model analyzed by Jackson and Wolinsky [32]. There are four main assumptions that determine our results: (i) links are undirected in the sense that both players incur the cost of maintaining the link, (ii) we apply a linear cost topology, (iii) we assume a polynomial benefit function, and (iv) the benefit function is founded on a uniform, constant δ. Extensions of this model can address changes to any of these four hypotheses.

With regard to (i), Bala and Goyal [4] and [5] develop a non-cooperative model of social communication where links are directional. Furthermore, concerning assumption (iii), Bala and Goyal [4] introduce the possibility that links are not fully reliable, thereby changing the benefit function. Adding both of these changes to our model may make some types of spatial social networks unstable. Since Bala and Goyal find networks in their setup to be 'super-connected', we would not expect to see the star or the chain emerge as a stable network when links that are not fully reliable, even if the links are undirected.

As for assumption (ii) we mention replacing the linear cost topology with arbitrary cost functions would unnecessarily complicate the analysis and most likely not lead to richer insights.

Benefits could also be generated by playing games on the network. The benefit from forming a direct link would then be the payoff from the game played by two linked players. Chapter 3 explores such a model. There we use an evolutionary framework inspired by Jackson and Watts [30] and Ellison [20]. Their agents are spatially located on a circle and play a coordination game in an endogenously formed network. In this framework there are no benefits from indirect connections. Their main result is that the emerging network is always locally complete. One could also include indirect benefits in the payoff structure of the game. We hypothesize in such a setting there emerge other pairwise stable networks.

2.6 Proofs from Section 2.3

We first show some intermediate results.

Lemma 19 *For $\delta > c > 0$, every pairwise stable network is connected.*

Proof. Assume there exists a pairwise stable network $g \subset g^N$ that is not connected. Since the network is not connected there will be two direct neighbors, i and j, with the following characteristics: player i is in one connected component of g and player j is in another connected component of g. Since i and j are direct neighbors, the cost to i and j to connect is equal to c. The benefit of connection to each will always be at least the direct benefit of δ. Therefore, both i and j will always want to form a connection since $\delta > c$. ■

Lemma 20 *Let $\delta > c > 0$ and $c = \frac{1}{n-1}$.*

(a) *For* $c > \delta - \delta^2$ *and* $\delta < \frac{1}{2}$*, the chain is pairwise stable.*

(b) *For* $c > \delta - \delta^2$ *and* $\delta > \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$*, there exists a star which is pairwise stable.*

Proof. Let $g \subset g^N$ be pairwise stable.

(a) Suppose g is the chain on N. The net benefit to any player severing a link with their nearest neighbor would be at most $c - \delta < 0$. Therefore no player will sever a link.

A player $i \in N$ will connect to a player j with $\ell(ij) = 2$ only if $2c \leqq \delta - \delta^{\tilde{n}}$. Because $\delta^2 < \frac{\delta}{2}$ and $\delta > c > \delta - \delta^2$, we know $2c > \delta \geqq \delta - \delta^{\tilde{n}}$. Thus, player i will not make such a connection.

Next consider j with $\ell(ij) \geqq 3$. Player i will make a link with j if the net benefit of such a connection is positive. Let $\ell(ij) = k$. For k odd, the net benefit for player i connecting to player j is

$$\delta + 2\sum_{l=2}^{\frac{k-1}{2}} \delta^l + \delta^{\frac{k-1}{2}+1} \quad - \sum_{m=\tilde{n}-(k-2)}^{\tilde{n}} \delta^m - kc.$$

For k even, the net benefit for player i connecting to player j is

$$\delta + 2\sum_{l=2}^{\frac{k}{2}} \delta^l \quad - \sum_{m=\tilde{n}-(k-2)}^{\tilde{n}} \delta^m - kc.$$

We proceed with a proof by induction with regard to the parameter $k \geqq 3$. When $k = 3$, the net benefit expression above simplifies to $\delta + \delta^2 - \delta^{\tilde{n}} - \delta^{\tilde{n}-1} - 3c$. If $3c + \delta^{\tilde{n}} + \delta^{\tilde{n}-1} < \delta + \delta^2$, player i would consider making a link with player j. This expression is never true for $\delta < \frac{1}{2}$ and $c > \delta - \delta^2$. For higher values of k the positive elements of the net benefit value increase by less than δ^2 and the negative elements increase by c. As $c > \delta^2$, the net benefit function decreases with respect to k. Thus for any $k \geqq 3$, player i will not consider creating a link with player j.

Thus, we have shown that no player will sever or add a link when g is chain on N and, therefore, the chain is pairwise stable.

(b) Let g be a star on N with the central player located at $\lfloor \frac{n}{2} \rfloor$. Refer to all players except the center as "points." The benefit of maintaining a connection to the center for all points is $\delta + (n-2)\delta^2$. The maximal cost of any connection in this star is $\left\lfloor \frac{n}{2} \right\rfloor \cdot c = \frac{\left\lfloor \frac{n}{2} \right\rfloor}{n-1} < \delta$. Thus, no player will sever a connection, not even the center. The net benefit of adding an additional connection for a player is $\delta - \delta^2 < c$. Thus, the star with the central player located at $\left\lfloor \frac{n}{2} \right\rfloor$ is pairwise stable.

This completes the proof of Lemma 20 ■

2.6.1 Proof of Proposition 7

Let $g \subset g^N$ be pairwise stable.

(a) Consider $g \subset g^N$ on N to be regular of order $\hat{n}(c,\delta) - 1$. Then the maximal net benefit of severing a link $ij \in g$ within a clique in g would be $c_{ij} + \delta^2$. Since $c_{ij} \leqq [\hat{n}(c,\delta) - 1] \cdot c < \delta - \delta^2$, it holds that $\delta > c_{ij} + \delta^2$ and, so, no player would be willing to sever a link. An additional link would form if $c_{ij} \leqq \delta - \delta^{\tilde{n}}$, where $\delta^{\tilde{n}}$ represents the value of an indirect connection lost due to a shorter path being created when a new link is created in a connected network. Since by Lemma 19 the network is connected, if a player were to add a link, his net benefit would be composed of three parts: The benefit of the new link and possibly higher indirect connections, the loss of indirect connections replaced by a shorter path created by the new link, and the cost of maintaining the link. We let $\delta^{\tilde{n}}$ represent the value of an indirect connection lost due to a shorter path being created when a new link is created. If more than one indirect connection is replaced by a shorter path, we use the convention of ranking the benefits $\delta^{\tilde{n}}$ by decreasing $\tilde{n}$. We know that $c_{ij} \geqq \hat{n}(c,\delta) \cdot c > [\hat{n}(c,\delta) - 1] \cdot c$ because the location for any player that i could form an additional link with would lie beyond the maximal clique. Using the definition of $\hat{n}(c,\delta)$, we know that $c_{ij} \geqq \delta$. Therefore no player will try to form an additional link outside the maximal clique. Hence, g is pairwise stable.

(b) Suppose $g^{i \leftrightarrow j} \subset g \subset g^N$ with $\ell(ij) \geqq 2$. If player i severs one of his links to a player within the clique $i \leftrightarrow j$, the resulting benefits from replacing a direct with an indirect connection are $\delta^2 + c - \delta > 0$. Therefore, player i will sever one of his connections. This shows that networks with a cliques of at least 3 members are not pairwise stable, thus showing the assertion.

(c) From assertion (b) shown above, it follows that any pairwise stable network $g \subset g^N$ does not contain a clique of at least three players. This implies that the chain is the only regular pairwise stable network to be investigated. Let g be the chain on N. First note that since $c < \delta$ no player has an incentive to sever a link in g. We will discuss three subcases, $n \geqq 7$, $n = 6$, and $n \leqq 5$.

(1) Assume $n \geqq 7$. Select two players i and j, $i < j$, who are neither located at the end locations of the network nor direct neighbors. Also assume that $\ell(ij) = 3$. If i were to connect to a player j the minimum net benefit of such a connection to either i or j would be $\delta + \delta^2 - \delta^3 - \delta^4 - 3c$. The maximal cost of connection c_{ij} when $\ell(ij) = 3$ is $\frac{1}{2}$ since $c = \frac{1}{n-1} \leqq \frac{1}{6}$. Since $\delta > \frac{1}{2}$, the minimum benefit, $\delta + \delta^2 - \delta^3 - \delta^4$, of such a connection is greater than the maximal cost. Thus, the additional connection will be made.[7] Also, note that player i is not connected to j's neighbor to the left. This player has essentially been skipped over by player i. Nor does player i have any incentive to form a link with the player that was skipped over. A connection to this player would cost $2c$, and the benefit would only be $\delta - \delta^2$. Thus, the chain is not pairwise stable.

(2) Assume $n = 6$. From assertion (b) shown above, we need only to examine two situations of link addition for two players i and j: a) $\ell(ij) = 3$ and $1 \neq i \neq n$, and b) $\ell(ij) \geqq 3$, $i = 1$ or $i = n$.

a) Select two players i and j with i not located at the end of the network, i.e., $1 \neq i \neq n$, and $\ell(ij) = 3$. If i were to connect to a player j the cost of such a connection would be $3c = \frac{3}{5}$ and the net benefit of this connection would be $\delta + \delta^2 - \delta^3 - \delta^4$. Because $c > \delta - \delta^2$, and $\delta > \frac{1}{2}$, we know that $\delta + \delta^2 - \delta^3 - \delta^4$ has a minimum value which is less than $\frac{3}{5}$.[8]

b) Select two players i and j with i, $\ell(ij) \geqq 3$, $i = 1$ or $i = n$. If player j were to connect to player i the minimum cost of a connection would be $3c$ or $\frac{3}{5}$. The minimal net benefit of such connection would be $\delta - \delta^{\tilde{n}}$ where $\tilde{n} \in \{3, 4, 5\}$. Since $c > \delta - \delta^2$, we know that $3c > \delta - \delta^{\tilde{n}}$. We can conclude that a link to an end agent will never be stable from such

[7] Because $\frac{1}{6} \geqq c > \delta - \delta^2$, and $\delta > \frac{1}{2}$, we know that $\delta + \delta^2 - \delta^3 - \delta^4$ has a minimum value of $\left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) + \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^2 - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^3 - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^4$ which is approximately equal to 0.53. Here we note that this minimum is attained in a corner solution determined by the constrained $\delta - \delta^2 < c$.

[8] Because $\frac{1}{5} \geqq c > \delta - \delta^2$, and $\delta > \frac{1}{2}$, we know that the polynomial $\delta + \delta^2 - \delta^3 - \delta^4$ has a minimum value given by $\left(\frac{1}{2} + \frac{1}{10}\sqrt{5}\right) + \left(\frac{1}{2} + \frac{1}{10}\sqrt{5}\right)^2 - \left(\frac{1}{2} + \frac{1}{10}\sqrt{5}\right)^3 - \left(\frac{1}{2} + \frac{1}{10}\sqrt{5}\right)^4$ which is approximately equal to 0.594. Again this minimum is determined by the constraint $\delta - \delta^2 < c$.

a distance.

We thus conclude that for δ such that $\delta + \delta^2 - \delta^3 - \delta^4 \leqq \frac{3}{5}$ the chain is pairwise stable and for some values of δ a non-locally complete network is stable.

(3) Assume $n \leqq 5$. Select three players i, j and k, where $i < j < k$ and j, k with $\ell(ij) \geqq 2$. We know $ij \notin g$ and $ik \notin g$. Suppose that $i = 1$. If player j were to make a new connection with player i, the maximum net benefit of such a connection to player j would be $\delta - \delta^2 - 2c < 0$. For player k we have that $\ell(ik) \geqq 3$, so, the net benefit of such a connection for player k would be at most $\delta - \delta^3 - 3c < 0$. If the player at the opposite end of the network linked with player i the net benefit would always be negative.[9] We conclude that no player would decide to connect with a player at either end points of the chain. From this it can easily be concluded that a similar argument can be applied to the other players for the case $n = 5$. (Note that the cases $n \leqq 4$ are trivially excluded.) Therefore, no player will form an additional link, and we conclude that the chain is pairwise stable.

This completes the proof of Proposition 7

2.6.2 Proof of Proposition 9

In this proof we first introduce some auxiliary notions. We define a path $\{i_1, \ldots, i_m\} \subset N(g)$ in the network $g \subset g^N$ to be *terminal* if $\#\{i_m j \in g \mid j \in N(g)\} = 1$ and for every $k = 2, \ldots, m-1$ it holds that $\#\{i_k j \in g \mid j \in N(g)\} = 2$. We also say that player i_1 *anchors* this terminal path.

(a) Let $\emptyset \subset g^N$ represent the empty network on N. For any two players the cost of connecting is at least c and the benefit of connection to each is equal toδ. Since $\delta < c$, no two players would like to add a link. So, the empty network $\emptyset$ is pairwise stable.

We now consider a network $g \subset g^N$ that is non-empty, pairwise stable, as well as acyclic. Hence, in the network $g \subset g^N$ there is at least one player $i \in N(g) \neq \emptyset$ such that $\#\{ij \in g \mid j \in N(g) \setminus \{i\}\} = 1$. Clearly since $\delta < c$, player $j \neq i$ with $ij \in g$ is better off by severing the link with i. Thus, g cannot be pairwise stable. Therefore we conclude that any acyclic pairwise stable network has to be empty.

[9]For $n = 3$, $\delta - \delta^2 - 2c < 0$. For $n = 4$, $\delta - \delta^3 - 3c < 0$. For $n = 5$, $\delta + \delta^2 - \delta^3 - \delta^4 - 4c < 0$. ($\delta + \delta^2 - \delta^3 - \delta^4$ is maximized at $\delta = \frac{1}{8} + \frac{1}{8}\sqrt{17}$ at a value of approximately 0.62 and $4c = 1$)

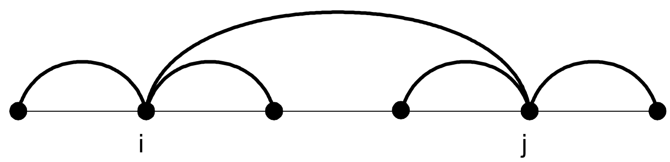

Figure 2-8: Case for $\ell(ij) = 3$.

(b) It is obvious that both the empty network and the chain on N are pairwise stable given that $\delta = c$. Next let $g \subset g^N$ be pairwise stable, non-empty, as well as acyclic. We first show that g is connected.

Suppose to the contrary that g is not connected. Then there will be two direct neighbors i and j with: player i is in a non-empty connected component of g of size at least 2 and player j is in another connected component of g. (Here we remark that $\{j\}$ is a trivially connected component of any network in which j is not connected to any other individual.) Since i and j are direct neighbors, the cost to i and j to connect is c. The net benefit for i of making a connection to j is then at least $\delta - c = 0$. The net benefit for j for making a connection to i is at least $\delta + \delta^2 - c = \delta^2 > 0$. Therefore, g is not pairwise stable. This contradicts our hypothesis and therefore g has to be connected.

Next we show that g is the chain. Suppose to the contrary that g is not the chain. From the assumptions it can easily be derived that there exists a player $i \in N$ with $\# \{ij \in g \mid j \in N(g)\} \geqq 3$.

First, we show that there is no player $j \in N$ with $ij \in g$, $\ell(ij) \geqq 2$, and the link ij is the initial link in a terminal path in g that is anchored by player i. Suppose to the contrary that such a player j exists and that the length of this terminal path is m. Then the net benefit for player i to sever ij is at least

$$2c - \sum_{k=1}^{m} \delta^k = 2\delta - \frac{\delta - \delta^{m+1}}{1-\delta} = \frac{\delta - 2\delta^2 + \delta^{m+1}}{1-\delta} > \delta \frac{1-2\delta}{1-\delta} \geqq 0$$

since $\delta = c = \frac{1}{n-1} \leqq \frac{1}{2}$. Thus, we conclude that player i is better off by severing the link ij. Hence, there is no player $j \in N$ with $ij \in g$, $\ell(ij) \geqq 2$, and the link ij is the initial link in a terminal path in g that is anchored by player i.

From this property it follows that the *only* case not covered is that $n \geqq 6$ and there exists a player j with $ij \in g$, $\ell(ij) \geqq 3$, $\#\{jh \in g \mid h \in N(g)\} = 3$, and that the two other links at j have length 1 that are connected to terminal paths, respectively of length m_1 and m_2. (The smallest network satisfying this case is depicted in Figure 2-8 and is the situation with $n = 6$ and $\ell(ij) = 3$.) The maximal net benefit of agent i to sever ij is

$$
\begin{aligned}
2c - \delta - \sum_{k=2}^{m_1} \delta^k - \sum_{k=2}^{m_2} \delta^k &= \delta - \frac{\delta^2 - \delta^{m_1+1}}{1-\delta} - \frac{\delta^2 - \delta^{m_2+1}}{1-\delta} \\
&= \delta \frac{1 - 3\delta + \delta^{m_1+1} + \delta^{m_2+1}}{1-\delta} \\
&> \delta \frac{1-3\delta}{1-\delta}
\end{aligned}
$$

Since $n \geqq 6$ it follows immediately that $\delta \leqq \frac{1}{5}$, and thus the term above is positive. This shows that g cannot be pairwise stable. Thus, every non-empty acyclic pairwise stable network has to be the chain.

This completes the proof of Proposition 9.

2.6.3 Proof of Theorem 11

(a) We partition the collection of all potential networks $\{g \mid g \subset g^N\}$ into four relevant classes: (a) $\emptyset \subset g^N$ the empty network, (b) $g^c \subset g^N$ the chain , (c) all acyclic networks, and (d) any network with a clique of at least three players. For each of these four classes we consider the value of the networks in that subset: The value of $\emptyset$ is zero. $v(g^c) = 2\sum_{k=2}^{n-1}(n-k)\delta^k - 2(n-1)c < 0$ from the condition on c and δ.

We partition acyclic networks into two groups: (i) all partial networks of the chain and (ii) all other acyclic networks.

(i) Take $\emptyset \neq g \subset g^c \subset g^N$ with $g \neq g^c$. Then g is not connected and there exists $ij \in g$ with $n(ij) = 2$. Since $c > \delta + \frac{1}{n-1}\sum_{k=2}^{n-1}(n-k)\delta^k$ deleting ij increases the total value of the network. By repeated application we conclude that $v(g) < 0$.

(ii) Assume $g \neq g^c$ is acyclic and not a subset of the chain. We define with g the partial chain $\emptyset \neq g^p \subseteq g^c \subset g^N$ given by $ij \in g$ if and only if $i \leftrightarrow j \in N(g^p)$. There are two situations: (A) the total cost of g is identical to the total cost of the corresponding g^p, (B) The total cost of g is higher than the total cost of the corresponding g^p.

Situation (A) could only occur if there is a player k with $ij \in g$ and $k \in i \leftrightarrow j$. Now $v(g^p) > v(g)$ due to more direct and possibly indirect connections.

Next consider situation (B). Assume g has one link ij with $n(ij) \geqq 3$. The cost of g is at least $2c$ higher than the cost of g^p. The gross benefit of g is at most $2\delta^2$ higher than that of g^p.[10]

Next consider $g \subset g^N$ with $K \geqq 2$ links where $n(ij) \geqq 3$. As compared to g^p, the value of g is decreased at least by $K \cdot 2c$. The maximum gross benefit of g is thus at most $2K\delta^2$ higher than the corresponding g^p.[11] In either subcase as $c > \delta > \delta^2$ we conclude that $v(g) < v(g^p)$.

Finally we consider $g \subset g^N$ containing $ij \in g$ with $n(ij) = 3$. We can quickly rule out any network with a clique greater than 3 as a candidate for higher utility than the chain. (Indeed, given the conditions for c and δ, the sum, of any extra benefits generated by forming a longer link on the chain could not compensate for the minimum additional cost of $2c$.) Next, we examine the possibility of a cycle having a higher value than the chain. Two links of length 2 must be present to have a cycle other than a trivial cycle of a neighborhood of three players or a clique of 3 players. These two links would cost at least $6c$ more than the total cost of the chain. A cycle that is nowhere locally complete has a gross maximal value of $2n\sum_{k=1}^{\frac{n}{2}-1}\delta^k + n\delta^{\frac{n}{2}}$. Recall that the chain has a gross value of $2\sum_{k=1}^{n-1}(n-k)\delta^k$. The gross value of the cycle exceeds that of the chain by $\frac{-n\delta^{\frac{1}{2}n}+n\delta^{\frac{1}{2}n+2}-2\delta^{n+1}+2\delta}{(\delta-1)^2} < 6c$. Thus, g is not efficient.

(b) The value of the chain network is $2\sum_{k=1}^{n-1}(n-k)\delta^k - 2(n-1)c$. For any value of n, given the condition $c < \delta + \frac{1}{n-1}\sum_{k=2}^{n-1}(n-k)\delta^k$, $v(g^c) > 0$ and $v(g^c) > v(g)$ for every $g \subsetneq g^c$. We refer to the preceding proof to verify that the value of any other network formation is less than the chain for $c > \delta$. The chain is the efficient network formation.

This completes the proof of Theorem 11.

[10]This value would be lessened by at least $-2\delta^{n-1}$ if g was connected.

[11]This value would be lessened by at least $-\sum_{m=1}^{k}\delta^{(n-m)}$ if g was connected.

2.6.4 Calculations for Example 10

We investigate for which values of k and (δ, c) with $0 < \delta < c < 1$ the described cyclic network g_c is pairwise stable. It is clear that there is only one condition to be considered, namely whether the severance of one of the links of length 2 in g_c is beneficial for one of the players. The net benefit of severing a link of length 2 is

$$\Delta = 2c - \sum_{m=1}^{k-1} \delta^m + \sum_{m=k+1}^{n-1} \delta^m = 2c - \frac{\delta - \delta^k - \delta^{k+1} + \delta^n}{1-\delta} \tag{2.12}$$

We analyze when $\Delta \leqq 0$. Remark that $\delta - \delta^k - \delta^{k+1} + \delta^n > \delta\left(1 - 2\delta^k\right)$. Now we consider values of (δ, c) such that

$$\delta \frac{1-2\delta^k}{1-\delta} = 2c > 2\delta \tag{2.13}$$

We note that for high enough values of k condition (2.13) is indeed feasible.
As an example we consider $k = 5$ and $\delta = \sqrt[k-1]{\frac{1}{k}} = \sqrt[4]{\frac{1}{5}} \sim 0.66874$. Then

$$\frac{1-2\delta^k}{1-\delta} = \frac{1-2\left(\sqrt[4]{\frac{1}{5}}\right)^5}{1-\sqrt[4]{\frac{1}{5}}} \sim 2.211$$

and we conclude that condition (2.13) is indeed satisfied for

$$c = \delta \frac{1-2\delta^k}{2-2\delta} = \sqrt[4]{\frac{1}{5}} \frac{1-2\left(\sqrt[4]{\frac{1}{5}}\right)^5}{2-2\sqrt[4]{\frac{1}{5}}} \sim 0.739$$

For further details we refer to Example 10 and Figure 2-4.

2.6.5 Numerical Values of δ for Figures 2-5 and 2-6

Figure 2-5

n = 7 $[0, 0.1464]$, $[0.1465, 0.2467]$, $[0.2468, 0.3480]$, $[0.3481, 0.4299]$, $[0.4300, 0.7886]$, $[0.7887, 0.8811]$, $[0.8812, 0.9030]$, $[0.9031, 0.9694]$, $[0.9695, 1)$

n = 6 $[0, 0.1726]$, $[0.1727, 0.3141]$, $[0.3142, 0.3375]$, $[0.3376, 0.7236]$, $[0.7237, 0.8788]$,

$[0.8789, 0.9306]$, $[0.9307, 1)$

n = 5 $[0, 0.2149]$, $[0.2150, 0.4287]$, $[0.4288, 0.8128]$, $[0.8129, 1)$

n = 4 $[0, 0.2799]$, $[0.2800, 1)$

n = 3 $[0, 0.4142]$, $[0.4143, 1)$

Figure 2-6

n = 7 $[0, 0.1464]$, $[0.1465, 0.1666]$, $[0.1667, 0.2467]$, $[0.2468, 0.3480]$, $[0.3481, 0.4299]$,

$[0.4300, 0.7886]$, $[0.7887, 0.8811]$, $[0.8812, 0.9030]$, $[0.9031, 0.9694]$, $[0.9695, 1)$

n = 6 $[0, 0.1726]$, $[0.1727, 0.1999]$, $[0.2, 0.3141]$, $[0.3142, 0.3375]$, $[.3376, 0.7236]$,

$[0.7237, 0.8788]$, $[0.8789, 0.9306]$, $[0.9307, 1)$

n = 5 $[0, 0.2149]$, $[0.2150, 0.2499]$, $[0.2500, 0.4287]$, $[0.4288, 0.8128]$, $[0.8129, 1)$

n = 4 $[0, 0.2799]$, $[0.2800, 0.3333]$, $[0.3334, 1)$

n = 3 $[0, 0.4142]$, $[0.4143, 0.4999]$, $[0.5000, 1)$;

2.7 Proofs From Section 2.4

2.7.1 Proof of Theorem 13

Let $m \in \{1, \ldots, n-1\}$. First we remark that (2.9) stated in Theorem 13 is indeed a feasible condition on the parameters c and δ. Namely, this holds for low enough values of δ; to be exact $\delta < (m(n+m) + m + 1)^{-1}$.

Now we partition the set of potential links g^N into $n - m$ subsets $\{G_0, G_{m+1}, \ldots, G_n\}$ where we define

$$\begin{aligned} G_0 &= \left\{ij \in g^N \mid n(ij) \leqq m+1\right\} \\ G_k &= \left\{ij \in g^N \mid n(ij) = k\right\} \text{ where } k \in \{m+2, \ldots, n\} \end{aligned}$$

We now consider the order $\widetilde{O} := \left(\widetilde{G}_0, \widetilde{G}_n, \widetilde{G}_{n-1}, \ldots, \widetilde{G}_{m+2}\right) \in \mathbb{O}$, where $\widetilde{G}_k$ is an enumeration of G_k, $k = 0, m+2, \ldots, n$. We now show that the regular network of order m is a subgame perfect Nash equilibrium of the link formation game corresponding to the order $\widetilde{O}$. For that purpose we apply backward induction to this link formation game.

We define the strategy tuple $\widehat{a}$ by $\widehat{a}_i(ij, h) = C_{ij}$ $\left(\text{where } h \in H\left(\widetilde{O}\right)\right)$ if and only if $ij \in G_0$.[12] From this definition it is clear that the resulting network $g_{\widehat{a}}$ is the unique network on N that is regular of order m. We proceed to show that the strategy described by $\widehat{a}$ is indeed a best response to any history in the link formation game, following the backward induction method.

(1) When any player i is paired with a player j where $ij \in G_0$, i.e., $n(ij) \leqq m+1$, both players will choose to make a connection because those connections will always have a positive net benefit because a lower bound for the net benefit of such a link is given by $\delta - \delta^2 - [n(ij) - 1] \cdot c \geqq \delta - \delta^2 - m \cdot c > 0$ from the right-hand side of condition (2.9). This is independent of the number of links made in the previous or later stages of the game. Hence, we conclude that if $n(ij) \leqq m+1$, the history in the link formation game with order $\widetilde{O}$ does not affect the willingness to make the connection ij.

Next, we proceed by checking the remaining pairs:

(2) Let $k \in \{m+1, \ldots, n-1\}$ and $i, j \in N$ with $i < j$ be such that $n(ij) = k+1 \geqq m+2$ and let $h \in H_{\widetilde{O}_{ij}}\left(\widetilde{O}\right)$ be an arbitrary history of the link formation game up till stage $\widetilde{O}_{ij}$. Then given the backward induction hypothesis that in later stages no links will be formed, the network $g(h)$ only consists of links of length less than $m+1$ and links of lengths k and higher. This implies that player j can be connected to at most $2m$ players with links of length m or less and to at most with $(n-k+1)$ players with links of length k and higher. So, an upper bound for the net benefits $U_i(ij)$ for player i of creating a direct link with player j can be constructed to be

$$\begin{aligned} U_i(ij) &\leqq \delta + (n - k + 2m + 1)\,\delta^2 - kc \\ &\leqq \delta + (n+m)\,\delta^2 - (m+1)c \end{aligned}$$

[12] Hence, this strategy prescribes that all links are formed in the first $|G_0|$ stages of the game corresponding to all pairs in G_0. Furthermore, irrespective of the history in the link formation game up till that moment there are no links formed in the final $C(n,2) - |G_0|$ stages of the link formation game corresponding to the pairs in $G_{m+1}, \ldots, G_n$. Obviously the outcome of this strategy is that $ij \in g_{\widehat{a}}$ if and only if $n(ij) \leqq m$.

$$
\begin{aligned}
&< \delta + (n+m)\,\delta^2 - (m+1)\left(\frac{1}{m+1}\delta + \left(\frac{n-1}{m+1}+1\right)\delta^2\right) \\
&= 0
\end{aligned}
$$

We conclude that player i will not have any incentives to create a link with player j in the link formation game with order $\widetilde{O}$.

Thus we conclude from (1) and (2) above that the strategy $\widehat{a}$ is indeed a subgame perfect Nash equilibrium of the link formation game with order $\widetilde{O}$. This shows that the regular network of order m can be supported as such for the parameter values described in the assertion.

Chapter 3

Evolution of Conventions in Endogenous Social Networks

3.1 Introduction

In a wide variety of economic and social environments, an agent's utility depends on successful coordination with other individuals. The following two examples illustrate this point. First, as suggested by Lewis [39], suppose that oligopolists are confronted with a change in the price of their raw material and therefore must set new prices of their product. It is to no one's advantage to set his price higher than the others set theirs, since if he does, he tends to lose his share of the market. Nor is it to anyone's advantage to set his price lower than the others set theirs, since if he does, he menaces his competitors and incurs their retaliation. Hence, each competitor must set his price close to the price he expects the others to set. Second, as described by Diamond [17], in various parts of the world in the early stages of food production hunter-gatherer societies were confronted with the introduction of cultivation of plants and the domestication of animals. It was to one's advantage to coordinate in either hunting and gathering or food production.[1] Once coordination has been achieved on a certain behavior, then it is likely that this behavior

[1] Although, as Diamond points out on page 105 that most peasant farmers and herders weren't necessarily better off than hunter-gatherers: "Archeologists have demonstrated that the first farmers in many areas were smaller and less well-nourished, suffered from more serious diseases, and died on the average at a younger age than the hunter-gatherers they replaced." The explanation offered for the increase in farming and herding communities is that individuals were seeking to minimize the risk of starvation.

will become the convention. For this reason Lewis [39] and Schelling [50] already stated that a convention should be considered a solution to a coordination problem. More precisely, Lewis defines a convention as a behavioral regularity such that everyone conforms to the regularity, expects others to conform, and wants to conform given that others conform.

The above examples illustrate two fundamental factors important in determining optimal behavior when agents face a coordination problem. An agent's expectation about the behavior of others plays a significant role. But underlying those expectations is an interaction structure governing communication between the players. Implicit in our discussion of conventions, we find ourselves talking about localities: geographic or social. Diamond [17] stresses the local, spatial interaction throughout his thesis.[2] Other examples of well-known conventions are languages, currencies, product standards, codes of dress and accounting standards.

Overview of the Model

We analyze the dynamic implications of learning in a large population coordination game where agents are distributed spatially, and both the actions of the players and the communication network between these players evolve over time. We follow the conventional evolutionary game theoretic models on coordination problems in assuming that players use the same pure strategy against all opponents they interact with, i.e., the players with whom they communicate, and we allow for this strategy to be adjusted over time. We depart from the conventional models in assuming that the interaction network itself is also subject to evolutionary pressure. Jackson and Watts [31] develop a similar setting; we depart from that model by incorporating cost considerations of social interaction. Instead we devise a circular model with an endogenous communication network, meaning that the locations of the players are fixed but players can create their own interaction neighborhood by forming and severing links with other players. We assume that the larger the distance between two players on the circle, the larger the maintenance costs of the mutual link will be. As maintenance costs include invested time and effort, distance should not only be interpreted as physical distance but may also represent social distance.

Players typically react myopically to their environment by deciding about both pure strate-

[2] On page 103 Diamond writes,"In short, only a few areas of the world developed food production independently, and they did so at wildly differing times. From those nuclear areas, hunter-gatherers of some neighboring areas learned food production,..."

gies and links based on a best-reply dynamics. Sometimes, however, players make mistakes when implementing their decisions, or alternatively players experiment with nonoptimal replies. Whether or not these mistakes should be included explicitly in the model depends on the span of time over which we are interested in the players' behavior as predicted by the model. As explained by Binmore, Samuelson and Vaughan [8] the model corresponds to the players' medium-run behavior in the absence of the perturbations representing the players' mistakes. We find that in this case, the dynamic process converges to an absorbing state. As the set of absorbing states includes states in which different kinds of behavior are observed, the population's medium-run behavior is possibly characterized by coexistence of conventions. In the long run, i.e., when the perturbations representing the players' mistakes are taken into account, coexistence of conventions is no longer possible. Namely, the risk-dominant convention is the unique stochastically stable convention, meaning that it will be observed almost surely when the mistake probabilities are small but nonvanishing.

Related Literature

As stated earlier, Jackson and Watts study an endogenous model of network formation. Although our models differ in cost considerations we obtain similar insights. Theorem 24 in Section 3.4 is equivalent to their first main result within a spatial setting.

Related to the present paper is also the literature on network formation. Dynamic models of network formation are considered by e.g. Bala and Goyal [**?**], Jackson and Watts [30], and Watts [51]. In these models the presence of a link however does not indicate that the involved players interact by means of playing a game. Instead there is a deterministic benefit from an emerging network. The best-known and most intuitive example is the symmetric connections model as introduced by Jackson and Wolinsky [32], which represents social communication between agents. In fact, agents communicate with all other agents they are directly or indirectly connected with. The value of the communication depends on the number of links involved in the shortest path that connects a pair of agents. Watts [51] shows convergence to an efficient network in case costs are small and closer connections are valued more than distant connections. Note that the model is deterministic and therefore involves initial state dependence. Bala and Goyal [**?**] also find convergence to efficient networks in their deterministic setting. Their model differs from Watts [51] as it focuses on directed networks and players receive the same benefit from direct and indirect connections. Jackson and Watts [30] consider a stochastic model similar to Kandori, Mailath and Rob [36] and Young [54]. They find that the networks that occur with positive probability in the stationary distribution when mistake probabilities go to zero are either stable or a cycle. In their model efficiency can not be ensured.

The evolution and stability of conventions is analyzed through the population adjustment models with persistent randomness in Kandori, Mailath and Rob [36], Young [54], and Ellison [20]. With respect to coordination games these learning models identify the risk-dominant equilibrium as the unique long-run convention. Goyal and Janssen [27] focus on nonexclusive conventions in a deterministic framework to model the idea that, at the expense of some additional costs, players can remain flexible and therefore coordinate their actions more successfully. They find that the Pareto-efficient or risk-dominant equilibrium prevails depending on whether these costs are low or high, respectively. Furthermore, at intermediate cost levels, the two conventions coexist. Coexistence of exclusive conventions is also a feature of the model with noise on the margin analyzed by Anderlini and Ianni [2].

The models mentioned above deal with exogenously given patterns of interaction. In particular, Kandori, Mailath and Rob [36] and Young [54] use uniform matching rules, meaning that every player possibly interacts with all other players, while Anderlini and Ianni [2], Ellison [20], and Goyal and Janssen [27] use local matching rules, expressing that every player can only interact with a small subset of the population. However, by fixing the pattern of interaction exogenously these models ignore that players may have the desire and, at least to some extent, the ability to affect the set of players with whom they interact.

Endogenous patterns of interaction in population adjustment models can also be realized by allowing players to migrate, i.e., choose their location. Bhaskar and Vega-Redondo [7], Ely [21], Mailath, Samuelson and Shaked [41], and Oechssler [45] show that migration implies, or at least allows for, the population to coordinate on the Pareto-efficient equilibrium. In particular, Oechssler [45] shows that in his deterministic framework, given that all conventions are initially present, the efficient one will eventually prevail throughout. Ely [21] considers a stochastic model and shows by considering the stationary distribution when mistakes probabilities become small that the efficient convention will occur independent of the initial conditions. In both models coexistence of conventions is not possible. Bhaskar and Vega-Redondo [7], in a model with asynchronous strategy and location revision opportunities, indicate the possibility of coexistence of conventions in the medium run when the game is a 'pure' coordination game. Also in the long run both conventions are possible, i.e., both appear with positive probability in the stationary distribution of the stochastic model, in the case of frequent play. Otherwise, only the efficient convention is stochastically stable. Finally, Mailath, Samuelson and Shaked [41] look at a quite different context in continuous time. Players of two continuum populations have to decide which location to visit. They show that if the evolutionary process is monotonic and players can avoid undesirable matching, then every locally stable configuration is efficient.

The paper is organized as follows. In Section **??** we introduce the model. Section 3.3 deals with the deterministic model describing the medium-run behavior of the dynamic process. Long-run behavior is analyzed in Section 3.4 by means of the stationary distribution of the stochastic model when mistakes are small but nonvanishing. The robustness of the obtained results with respect to the exact specification of the model is discussed in Section 3.5. Finally, Section 3.6 concludes and Section 3.7 contains the proofs.

3.2 The Model

Consider a large but finite population of players $N = \{1, ..., n\}$ who are spatially distributed around a circle. Players are distributed around the circle in an equidistant fashion. Each discrete-time period $k \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ the players' interaction consists of two stages. In the first stage players form or sever links connecting themselves to other players. In the second stage pairs of players who are linked play a coordination game and adjust their actions.

The state s_k of the dynamic process in each period k is given by a graph g_k, with its nodes representing the players and its edges capturing the established communication links, and an action profile a_k, specifying the action being played by every player $i \in N$. Let $\mathcal{G}$ and $\mathcal{A}$ denote the set of possible graphs and the set of possible action profiles, respectively, then $s_k = (g_k, a_k) \in \mathcal{G} \times \mathcal{A} = \mathcal{S}$.

Each period, the presence of a link between players i and j, with $i, j \in N$ and $i \neq j$, results in a maintenance cost c_{ij} to both players. The costs c_{ij} are determined by the distance between the two players on the circle. Formally, we assume that

$$c_{ij} = \min \{\gamma |j-i|, \gamma(n-|j-i|)\},$$

where $\gamma \geqq 0$ are the so-called unit costs.

Let $\mathcal{L}_{i,k} \subset N$ denote the set of players that player $i \in N$ is linked with after the first stage in period $k \in \mathbb{N}_0$. We refer to $\mathcal{L}_{i,k}$ as the *interaction neighborhood* of player i at time k. In addition, with slight abuse of notation, we write $a_k = \left(a_{i,k}, a_{\mathcal{L}_{i,k},k}, a_{-\mathcal{L}_{i,k},k}\right)$. Note that when not causing any confusion $\mathcal{L}_{i,k}$ and $a_{i,k}$ may also be denoted by $\mathcal{L}_i$ and a_i, respectively. In the second stage of each period k, a player i plays a coordination game with all players $j \in \mathcal{L}_{i,k}$. The gross benefits are determined by the utilities u in the 2×2 coordination game given below.

	A	**B**
A	a, a	c, d
B	d, c	b, b

(3.1)

The payoff to player i in period k is given by

$$\pi_i\left(a_{i,k}, a_{\mathcal{L}_{i,k},k}\right) = \begin{cases} \sum_{j \in \mathcal{L}_{i,k}} u\left(a_{i,k}, a_{j,k}\right) & \text{if } \mathcal{L}_{i,k} \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{i,k} \in \{A, B\}$, for all $i \in N$ and $\mathbb{N}_0$. The net benefits of a player i can be found by subtracting all maintenance costs, $\sum_{j \in \mathcal{L}_{i,k}} c_{ij}$, from his gross benefits. For notational convenience only we assume that $a, b, c, d \in \mathbb{N} \cup \{0\}$. Furthermore, it is required that $a > d \geqq 0$ and $b > c \geqq 0$, implying that both (A, A) and (B, B) are strict Nash equilibria. Furthermore, the payoffs a and b are taken to be larger than or equal to γ for the model to be nontrivial, i.e., for the players to have an incentive to interact. Finally, we assume that $a + c > b + d$, implying that (A, A) is the risk-dominant equilibrium as defined by Harsanyi and Selten [29]. Note that when the two actions have equal security levels, i.e., $c = d$, (A, A) is also the Pareto-efficient equilibrium.

Now, we describe how players establish and sever links, and how they adjust their pure strategies. The process of link formation and link severance is based on the process described in Jackson and Watts [30] and similar to Jackson and Watts [31]. In the first stage of each period k two players i and j, $j \neq i$, are randomly selected with probability $\rho_{ij} > 0$. Hence, we consider a matching process with players randomly meeting each other in pairs, and time periods being identified with the encounters. Only the two players constituting that pair can alter their (potential) mutual link in period k. If their mutual link already exists, they decide whether to sever it, and otherwise they decide whether to create the link.

The part of the dynamic process that describes how the two selected players establish or sever a mutual link is modelled as follows. First, suppose player i and j are not linked and therefore have to decide whether to establish the link ij. Player i compares the payoff he would have received from playing the coordination game with player j in the previous period with the maintenance costs c_{ij} of the link, i.e.,

$$u\left(a_{i,k-1}, a_{j,k-1}\right) \geqq c_{ij}.$$

Of course, player j follows exactly the same procedure. Only in case both payoffs are greater than or equal to the costs c_{ij}, the players decide to establish the mutual link. Notice that according to the deterministic part of the link formation process, two players will never establish a link if their mutual distance on the circle exceeds $\max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Namely, in that case the maintenance costs are always larger than the payoff obtained in the coordination game. Second, suppose that at the moment players i and j are linked, and that they therefore have to decide whether to sever their mutual link. Either player can sever the link ij unilaterally if $u\left(a_{i,k-1}, a_{j,k-1}\right) \leqq c_{ij}$. We assume that every linking decision made by a pair of players is implemented correctly with probability $1 - 2\tau$, where $\tau \geqq 0$. With probability 2τ, the two potential linking decisions are implemented with equal probability.

To describe the adjustment of the players' pure strategies, we follow Ellison [20] by assuming that players typically react myopically to their environment. In fact, we assume that in the second stage of period t, player i chooses

$$a_{i,k} \in \arg\max_{a_i} \pi_i\left(a_i, a_{\mathcal{L}_{i,k},k-1}\right) \tag{3.2}$$

with probability $1 - 2\varepsilon$, where $\varepsilon \geqq 0$. With probability 2ε player i chooses between the two pure strategies at random with equal probability. Player j adjusts his pure strategy in the same way.

We refer to Section 3.5 for a discussion of alternative specifications of the dynamic process.

For a given initial state $s_0 \in \mathcal{S}$, the dynamic process introduced above defines a Markov process $\{s_k\}_{k \in \mathbb{N}_0}$ in discrete time with the finite state space $\mathcal{S} = \mathcal{G} \times \mathcal{A}$. Let $P := P(\tau, \varepsilon)$ denote the (one-step) transition matrix of the Markov process. Then, if $s = (g, a)$, $\tilde{s} = (\tilde{g}, \tilde{a}) \in \mathcal{S}$, the entry $P_{s\tilde{s}} := P_{s\tilde{s}}(\tau, \varepsilon)$ of the transition matrix is the probability that the state is $\tilde{s}$ at time $k+1$ conditional on the state being s at time k, i.e.,

$$P_{s\tilde{s}} = \Pr(s_{k+1} = \tilde{s} \mid s_k = s).$$

Let $P^l := P^l(\tau, \varepsilon)$ denote the l-step transition matrix of the Markov process, then the entry $P^l_{s\tilde{s}} := P^l_{s\tilde{s}}(\tau, \varepsilon)$ of the l-step transition matrix is equal to

$$P^l_{s\tilde{s}} = \Pr(s_{k+l} = \tilde{s} \mid s_k = s).$$

We refer to the Markov process $\{s_k\}_{k \in \mathbb{N}_0}$ as *adaptive play*.

3.3 Adaptive Play without Mistakes

This section deals with the model introduced in section **??** when both mistake probabilities, τ and ε, are equal to zero. We refer to this specific Markov process as *adaptive play without mistakes*. As discussed extensively by Binmore, Samuelson and Vaughan [8] adaptive behavior without mistakes describes the medium-run behavior of the dynamic process, where the medium-run refers to the time span needed for the dynamic process to reach the neighborhood of the first equilibrium near to which it will stay for a significant period of time.

Analyzing the dynamic process in the medium run boils down to identifying the absorbing states of the Markov process representing adaptive play without mistakes. An absorbing state is a state that once entered cannot be left. With respect to the present model this implies that in an absorbing state no single player wants to sever any of the links he is involved in, no pair of players wants to establish a mutual link or adapt his pure strategy. Below, we discuss what the absorbing states of adaptive play without mistakes look like, and we explore conditions which guarantee convergence to one of these absorbing states.

A convention is defined by Lewis [39] as a pattern of behavior that is customary, expected, and self-enforcing, meaning that everyone conforms, expects others to conform, and wants to conform given that everyone else conforms, respectively. Since a convention is self-enforcing, it has to be an absorbing state of the Markov process describing adaptive play without mistakes. It can easily be verified that the present model has two absorbing states which are conventions. Namely, the state such that

$$\begin{cases} a_i = A, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{a}{\gamma}, \end{cases} \tag{3.3}$$

for all $i \in N$ and the state such that

$$\begin{cases} a_i = B, \\ j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{b}{\gamma}, \end{cases} \tag{3.4}$$

for all $i \in N$. We refer to (3.3) as the $\left(A, \frac{a}{\gamma}\right)$-convention or the risk-dominant convention and to (3.4) as the $\left(B, \frac{b}{\gamma}\right)$-convention.

The $\left(A, \frac{a}{\gamma}\right)$-convention and $\left(B, \frac{b}{\gamma}\right)$-convention are, however, not the only absorbing states of adaptive behavior without mistakes. The other absorbing states indicate that the players' medium-run behavior may be characterized by coexistence of conventions. The following example shows existence of an absorbing state in which players act differently. This absorbing state therefore indicates the possibility of coexistence of conventions in the medium-run, see also Bhaskar and Vega-Redondo [7] for a similar observation.

Example 21 Consider a population consisting of 12 players. Assume $\gamma = 1$ and let the payoffs of the coordination game be given by $a = 2$, $b = 1$, and $c = d = 0$. It can easily be verified that the state represented in Figure 3-1 is an absorbing state. Since players do not conform to the same pattern of behavior, the absorbing state is not a convention. The absorbing state does, however, show the presence of local clusters of players who conform, expect others to conform, and want to conform (given that others conform) to the same pattern of behavior. We refer to

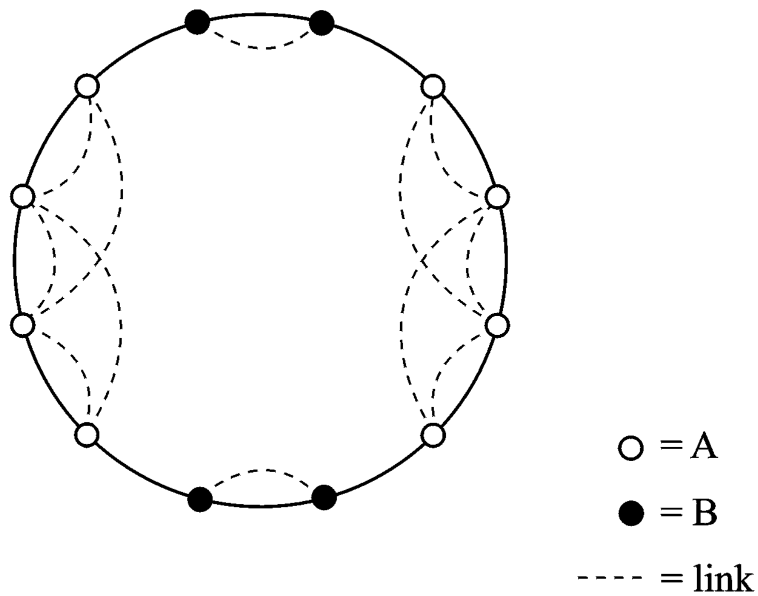

Figure 3-1: An absorbing state reflecting co-existence of conventions.

the presence of such clusters in an absorbing state as coexistence of conventions. It is not hard to see that there are typically many absorbing states which exhibit such clusters. Note that in order for such a state to be an absorbing state the clusters of A-players and B-players should be at least of size $\frac{b}{\gamma}$ and $\frac{a}{\gamma}$, respectively, the off-diagonal payoffs must be less than γ and finally, the population must be large enough so that the population could not become the interaction neighborhood for a player, i.e., $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. ♦

The following theorem specifies the condition on the size of the population under which convergence of adaptive play without mistakes can be guaranteed almost surely. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x.

Theorem 22 *Assume that* $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$. *Then for any initial state* $s_0 \in \mathcal{S}$ *adaptive behavior without mistakes converges almost surely to an absorbing state.*

The proof of Theorem 22 is contained in Section 3.7.1.

The condition $\lfloor \frac{n}{2} \rfloor > \max \left\{ \frac{a}{\gamma}, \frac{b}{\gamma} \right\}$ requires that the population is sufficiently large to ensure that for every player $i \in N$ there exist a player $j \neq i$ such that, independent of the outcome of the

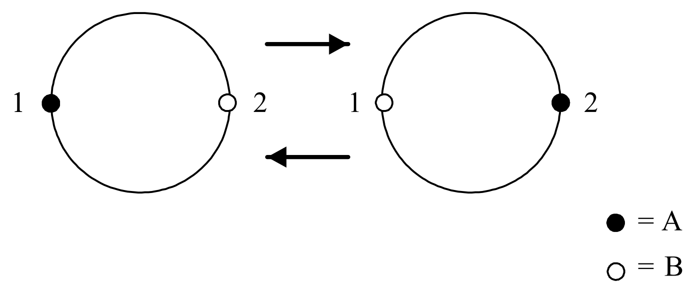

Figure 3-2: A cycle

game, it is too costly for them to be linked. According to Theorem 22 this condition excludes the possibility that the dynamic process ends up in a cycle. The necessity of this assumption is illustrated with Example 23.

Example 23 Consider a population consisting of two players who are situated on a circle and assume that $\gamma = 1$ and $\min\{c, d\} \geqq 1$, i.e., player 1 and player 2 want to be linked independent of the outcome of the game. The two players will therefore form a link (if not already present) in the first stage of period 1 and sustain it in all subsequent periods. Since the players adapt their actions myopically as specified by (3.2), it can easily be verified that any initial state such that the actions of player 1 and player 2 are different, results in a cycle as represented in Figure 3-2, where both players will adapt their pure strategies continuously. Consequently, the players' pure strategies will be different every period and adaptive behavior without mistakes will never settle down. ⧫

3.4 Adaptive Play with Mistakes

In this section we consider *adaptive play with mistakes.* With both kinds of mistakes as part of the model, i.e., $\tau > 0$ and $\varepsilon > 0$, the transition matrix of the Markov process is *irreducible* and *aperiodic.* Irreducibility means that for every pair of states s, $s' \in \mathcal{S}$, there exists a time $l := l(s, \tilde{s})$ such that $\left(P^l\right)_{s\tilde{s}} > 0$, i.e., every state in the state space can be reached from every other state with positive probability. A Markov process is aperiodic if for every state s in the state space it holds that $P_{ss} > 0$, i.e., for every state there is a positive probability of remaining there in the next period.

A *stationary distribution* of the Markov process is a row vector $\phi := \phi(\tau, \varepsilon) \in \Delta_{|\mathcal{S}|-1}$ such that $\phi P = \phi$, where

$$\Delta_{|\mathcal{S}|-1} = \left\{ \nu \in \mathbb{R}^{|\mathcal{S}|} \mid \nu_s \geqq 0 \text{ for all } s \in \mathcal{S} \text{ and } \textstyle\sum_{s \in \mathcal{S}} \nu_s = 1 \right\}.$$

In Lemma 27 found in Section 3.7.2 we summarize the standard results in the literature that an irreducible and aperiodic Markov process is ergodic, meaning that the Markov process has a unique stationary distribution, and that the process converges to this stationary distribution from any initial state. Furthermore, along any sample path the distribution of realized states approaches the stationary distribution almost surely.

As mentioned earlier, the stationary distribution is a good description of the behavior of the adaptive process in the long run. Namely, the adaptive process jumps from an absorbing state to the basin of attraction of another absorbing state due to the occurrence of very unlikely realizations of the perturbations of the dynamic process, i.e., of the players' mistakes. Only given a very long period of time, these jumps will occur often enough to produce a well-defined stationary distribution, which represents an asymptotic probability distribution over the states.

To explore what the stationary distribution of adaptive play with mutations looks like, we introduce the following notation. An x-tree t on $\mathcal{S} = \mathcal{G} \times \mathcal{A}$ is a function $t : \mathcal{S} \to \mathcal{S}$ such that $t(x) = x$ and for all $s \neq x$ there exists an m with $t^m(s) = x$. The stationary distribution $\phi(\tau, \varepsilon)$ is characterized by

$$\phi_x(\tau, \varepsilon) = c(\tau, \varepsilon) \sum_{t \in H_x} \prod_{s \neq x} P_{st(s)}(\tau, \varepsilon), \tag{3.5}$$

where H_x is the set of x-trees on $\mathcal{S}$ and $c(\tau, \varepsilon)$ is a continuous function in τ and ε. Notice that Foster and Young [22], Kandori, Mailath and Rob [36], and Ellison [20] use the same characterization of the stationary distribution. We refer to those papers and to Freidlin and Wentzell [23] for background material on the characterization.

Now consider the transition matrix $P(\tau, \varepsilon)$ describing adaptive behavior with mistakes. The transition probabilities $P_{s\tilde{s}}(\tau, \varepsilon)$, i.e., the entries of the transition matrix, are continuous in τ and ε. In fact, the transition probabilities are either zero or given by a polynomial in τ and ε. The constant term of such a polynomial is nonzero if and only if the transition $s \rightarrow \tilde{s}$ occurs with positive probability in the Markov process describing adaptive behavior without mistakes ($\tau = \varepsilon = 0$). Furthermore, since each time period only two players can alter their (potential) mutual link and adapt their actions, the polynomial is of τ-order and ε-order at most 1 and 2, respectively.

Because we are interested in small probabilities of τ-mistakes and ε-mistakes, we consider the asymptotic behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \rightarrow 0$ and $\varepsilon \rightarrow 0$. Henceforth, we denote the behavior of the stationary distribution $\phi(\tau, \varepsilon)$ as $\tau \rightarrow 0$ and $\varepsilon \rightarrow 0$ by $\lim_{\tau,\varepsilon \rightarrow 0} \phi(\tau, \varepsilon)$. In the analysis as stated in this section, we assume that whenever these limits are taken, τ and ε go to zero at the same rate, i.e.,

$$0 < \lim_{\tau,\varepsilon \rightarrow 0} \frac{\tau}{\varepsilon} < \infty. \tag{3.6}$$

The importance of this assumption will be discussed in detail in Section 3.5.

The asymptotic stationary distribution gives the percentage of the time that the Markov process will spend in any state in the long run. The following theorem states that the $\left(A, \frac{a}{\gamma}\right)$-convention is the unique stochastically stable state, which means that when the two mistake probabilities go to zero at the same rate, adaptive play with mistakes will be in the $\left(A, \frac{a}{\gamma}\right)$-convention almost all the time, provided that the population is sufficiently large. Notice that the structure of the proof of Theorem 24 is identical to the structure of the proof of part (a) of Theorem 1 in Ellison [20].

Theorem 24 *Let* $n \geqq 3$. *If* $0 < \lim_{\tau,\varepsilon \rightarrow 0} \frac{\tau}{\varepsilon} < \infty$, *then* $\lim_{\tau,\varepsilon \rightarrow 0} \phi_{\left(A, \frac{a}{\gamma}\right)}(\tau, \varepsilon) = 1$.

The proof of Theorem 24 is contained in Section 3.7.2.

The intuition behind the theorem above is similar to the corresponding results in Kandori, Mailath and Rob [36], Young [54], and Jackson and Watts [31]. Namely, the $\left(A, \frac{a}{\gamma}\right)$-convention or, equivalently, the risk-dominant convention is the state with the largest basin of attraction, meaning that the minimal numbers of τ-mistakes and ϵ-mistakes needed to escape from its basin of attraction are larger than the minimal numbers of mistakes needed to leave the basin of attraction of any other state.

3.5 Discussion and Simulations

We now discuss the robustness of the model presented in Section **??**. In particular, we focus on alternative specifications of the stochastic and deterministic part of the dynamic process and discuss how this would influence the results.

The stochastic part of the dynamic process not only consists of the mistake probabilities, but also of the random selection of a pair of players. First, consider the selection of a pair of players. Notice that this is the only stochastic element of adaptive play without mistakes, as analyzed in Section 3.3. The presence of this random element drives the convergence result as stated in Theorem 22. Without this random part of the model, i.e., with all players myopically updating links and pure strategies each period, cycles can not be excluded as shown by Ellison [20].

Random selection of a pair of players is not the only way to obtain a convergence result in a model without mistake probabilities. An alternative way to establish such a result is analyzed by Young [54] and applied by Jackson and Watts [31], where the random element is originated in a sampling procedure by the players. Young [54] assumes that players base their decisions on limited information about the actions of other players in the recent past. More precisely, every player inspects k plays drawn without replacement from the most recent $m \geqq k$ periods. A convergence result in the absence of mistake probabilities can be obtained in case the fraction $\frac{k}{m}$, which measures the completeness of the players' information, is small enough. In other words, convergence can only be ensured when the degree of randomness is sufficiently high. Results similar to those formulated in Theorem 22 and Theorem 24 could therefore still be obtained when we rephrase the present model such that it fits in the framework as analyzed by Young [54] and Jackson and Watts [31]. Alternatively, we could allow for the identified pair of players not only to alter their (potential) mutual link but also to possibly sever all other links they are involved in, without changing the results.

Second, consider the stochastic part of the dynamic process reflecting that players occasionally make mistakes when implementing their decisions. In Theorem 24, we assumed that τ and ε go to zero at the same rate. The specification of the model states that we assume the two kinds of mistakes to be independent of the state of the process. Bergin and Lipman [6] show that this assumption is of 'crucial' importance, meaning that allowing for τ-mistake probabilities or ε-mistake probabilities to differ across states and in particular to go to zero at different rates would change the results significantly. Indeed, by choosing the mistake probabilities appropriately, the $\left(B, \frac{b}{\gamma}\right)$-convention may even become the unique stochastically stable state. Van Damme and Weibull [14], however, restore the result that in 2×2 coordination games the risk-dominant equilibrium will be selected in the long run by considering endogenously determined mistake probabilities. More precisely, they consider a model where players can control the probability of making a mistake at some costs. As players will try harder to avoid mistakes leading to larger payoff losses, the mistake probabilities depend on the state of the process. They show that mistake probabilities nevertheless go to zero at the same rate when the costs become negligible.

Finally, we make two remarks concerning the deterministic part of the dynamic process. First, instead of assuming that a player can observe the pure strategies of all potential links independently of whether or not he is linked with them, we could alternatively consider the case that a player is only able to observe the pure strategies of the players he is currently linked with. The decision to establish a new link could then be based on comparing the maintenance costs with the expected payoff in the coordination game, where the expected payoff is given by a player's average gross benefit in the last period. We conjecture that coexistence of conventions in the long run will no longer be possible when this specification of the deterministic part of the dynamic process is used. To motivate this point, notice that the state illustrated in Figure 3-1 will no longer be a steady state of adaptive play without mistakes. Namely, adjacent A-players and B-players will establish a link if given the opportunity. As can be concluded from the proof of Theorem 24, the result that the $\left(A, \frac{a}{\gamma}\right)$-convention is the unique stochastically stable state of adaptive play with mistakes remains true for this alternative specification.

Second, the deterministic part of the process is characterized by sequential decision making of the players. In the first stage of a period players are concerned with the network formation process and in the second stage players decide upon their pure strategies in the coordination game. Considering players who decide on links and pure strategies simultaneously would, however, not significantly change any of the results. Convergence to an absorbing state of adaptive play without mistakes can still be obtained because simultaneous decision making does not decrease the degree of randomness in the model. Furthermore, adaptive play with mistakes will still converge to the risk-dominant convention as the size of the basins of attraction does not change significantly.

We adapt a measure introduced by Ellison [20] to illustrate the extent to which play resembles its long run limit. The long run limit with mistakes is characterized by the $\left(A, \frac{a}{\gamma}\right)$-convention. For the simulations reported below, we only examine play as it resembles its long run limit in the action of players and leave the graphical examination for future research. We denote the number of players playing A at time k as $A(a_{i,k})$ and note that this does not indicate the extent to how nearly the interaction network resembles its long run limit. $W(N, \tau, \varepsilon, \alpha)$ is the expected waiting time until at least $1 - \alpha$ of the players simultaneously play A given that everyone starts off playing B, i.e.,

$$W(N, \tau, \varepsilon, \alpha) = E(\min\{k | A(a_{i,k}) \geqq (1 - \alpha)N\} | a_{i,0} = B).$$

Tables I lists the observed values of $W(N, \tau, \varepsilon, \alpha)$ indicated by $W^R(N, \tau, \varepsilon, \alpha)$ for the endogenous interaction model such that all players initially play B and that the initial network is randomized, i.e., the starting condition is determined by a random network. In Table I-A we consider the coordination game described by $a = 2$, $b = 1$, and $c = d = 0$. In Table I-B we consider the coordination game given by $a = 4$, $b = 5$, $c = 3$, and $d = 0$. (We refer to the first set of payoffs as Game 1 and the second set of payoffs as Game 2.) Both sets of payoffs induce a player to play strategy A if at least $\frac{1}{3}$ of his interaction neighborhood play A. In both games A is the risk dominant equilibrium but in Game 2 B is Pareto superior to A. This is not the case in Game 1.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	71.262 [47.5] (71.31)	43.853 [33.5] (33.467)	30.591 [28] (15.705)
$N = 20$	527.999 [510] (304.428)	272.438 [233.5] (195.144)	105.405 [73] (90.489)
$N = 50$	3762.491 [3713] (699.923)	3340.933 [3253.5] (704.659)	1181.522 [1129] (563.383)
$N = 100$	19600.22 [19553] (1973.643)	17876.0 [17826] (2481.601)	11132.06 [11201.5] (2387.776)

Table I-A: Observed $W^R(N, \tau, \varepsilon, 0.25)$ for game 1

In Table I-A we denote by N the population size, ε the probability of a mistake in the selection of actions, and τ the probability of a mistake in link formation. All waiting times reported are until 75% of the players play A simultaneously. The mean waiting times are reported first, the median waiting times are given in square brackets, while the standard deviation is given in parentheses.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	3545.521 [80] (6654.326)	221.108 [35] (442.426)	39.926 [30] (48.626)
$N = 20$	6967.823 [4536] (7601.932)	947.117 [676] (975.146)	140.79 [78] (145.058)
$N = 50$	9704.727 [8210] (5648.254)	5066.152 [4861.5] (1739.067)	1637.692 [1598] (819.478)
$N = 100$	45612.240 [41847] (18231.810)	34602.40 [32802] (10575.47)	16792.38 [16494.5] (4798.067)

Table I-B: Observed $W^R(N, \tau, \varepsilon, 0.25)$ for game 2.

The waiting times reported in Table I-B are generated under similar conditions as those reported in Table I-A. In Table I-B the indication * refers to waiting times that average more than 30,000 iterations. These long waiting times are not feasible within the software application.

Tables II-A and II-B list the observed values of $W^E(N, \tau, \varepsilon, \alpha)$ for the endogenous interaction model such that initially all players play the risk dominated convention B and the initial interaction network is the empty one.

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	8.69 [7] (11.302)	9.904 [8] (9.596)	13.058 [10] (11.05)
$N = 20$	14.127 [13] (3.608)	15.902 [15] (5.385)	23.464 [19] (18.104)
$N = 50$	35.769 [35] (5.611)	40.611 [39] (8.656)	58.183 [50] (27.342)
$N = 100$	68.087 [68] (7.222)	77.774 [76] (12.09)	106.75 [101] (29.77)

Table II-A: Observed $W^E(N, \tau, \varepsilon, 0.25)$ for game 1

Again the reported waiting times are until 75% of the players play A. The mean number of iterations is reported first, followed by the median number of iterations in square brackets, and the standard deviation in parentheses.

For the observed number of iterations with initially the empty network are reported in Table

II-B:

	$\tau = \varepsilon = 0.025$	$\tau = \varepsilon = 0.05$	$\tau = \varepsilon = 0.1$
$N = 10$	39.783 [7] (445.963)	11.791 [8] (30.827)	14.071 [10] (11.554)
$N = 20$	14.95 [13] (5.465)	17.074 [15] (9.054)	22.259 [19] (11.345)
$N = 50$	35.733 [35] (5.63)	41.459 [39.5] (13.231)	58.44 [51] (28.348)
$N = 100$	68.31 [67] (7.244)	76.695 [75] (11.597)	108.957 [101] (33.203)

Table II-B: Observed $W^E(N, \tau, \varepsilon, 0.25)$ for game 2.

The difference between the number of observed iterations with a random network — reported in Tables I-A and I-B — and an empty network — reported in Tables II-A and II-B — is caused by the inertia in the network formation process. Because the deterministic part of the process is characterized by a sequential decision making process, on average the random network will have too many links that are relatively costly and, therefore, have to be severed. However, until those links are severed, they are part of the interaction neighborhoods of the players involved. This is causes a higher number of iterations necessary to converge the evolutionary process.

In general, as the mistake probabilities increase, the dispersion of the expected waiting time decreases, as does the expected waiting time. Both of these observations are consistent with what we expect. The more likely a mistake occurs, the more likely the process will enter an absorbing state for the $\left(A, \frac{a}{\gamma}\right)$-convention. Considering that the deterministic part of the process is characterized by sequential decision making of the players, it is not surprising that convergence seems dependent on network size. Consider that at a minimum $\frac{n(n-1)}{2}$ periods must pass for each pair of players to meet. As a rough proxy we divide the observed average number of iterations by the number of possible links. For example, from the reported values in

Table II-A for Game 1 initiated with an empty network and mistake probabilities $\tau = \varepsilon = 0.025$ we compute

$$\begin{aligned}
\frac{W^E(10, \tau, \varepsilon, 0.25)}{45} &= 1.58, \\
\frac{W^E(20, \tau, \varepsilon, 0.25)}{190} &= 2.78, \\
\frac{W^E(50, \tau, \varepsilon, 0.25)}{1225} &= 3.07, \\
\frac{W^E(100, \tau, \varepsilon, 0.25)}{4950} &= 3.96.
\end{aligned}$$

These values indicate the average number of times all links are queried for at least $\frac{3}{4}$ of all players to play the risk dominant convention A given that everyone initially plays B. It is not surprising that the average number of times each pair needs to be queried increases with the number of players. This is caused by the assumption that all relevant pairs are queried in a random order. Another reason for the increase in the waiting times is that perhaps a pocket of the risk dominated equilibrium persists and coexists for a substantial amount of time. The probability of such an event would increase with the number of players.

Finally, we tested whether the threshold of a convergence rate of 75% is a reasonable proxy for complete convergence. We performed a set of simulations where $\alpha = 0$, meaning that the measured number of iterations were for the stopping rule that 100% of the players play A when initially all players play B. The observed number of iterations on average only exceeded the reported values in Tables I-A, I-B, II-A and II-B by approximately half the number of potential links $\frac{n(n-1)}{2}$.

3.6 Concluding Remarks

We have analyzed a large population coordination game where not only the actions of the players but also the communication network is subject to evolutionary pressure. Cost considerations of social interaction are incorporated by exploiting the spatial structure of the model, i.e., the costs of interacting increase in the distance between two players on the circle.

The Markov process without mistakes describing medium-run behavior is shown to converge to an absorbing state, which may be characterized by coexistence of conventions. In the long-run, when mistakes are possible with small but nonvanishing probabilities, coexistence of conventions is no longer possible as the risk-dominant convention is the unique stochastically stable state. These results require that $c, d \geqq \gamma$. If $c, d < \gamma$, i.e., mismatched pairs obtain net losses when making links with their closest neighbors, we conjecture that we would obtain a similar insight as the second main theorem (Proposition 2) of Jackson and Watts [31]. That is, we would expect to see both coordination equilibria as stochastically stable states.

Interesting topics for future research are as follows. First, considering a larger class of games, see e.g. Kandori, Mailath and Rob [36] and Young [54], in the framework of the present paper. Second, extending the model to also allow for players who are indirectly connected to play a game as is common practice in the literature on network formation. When value is given to indirect connections, other types of networks could be generated with perhaps other implications.

3.7 Proofs

3.7.1 Proof of Theorem 22

To explore the conditions under which adaptive play without mistakes converges to an absorbing state, we develop the two lemmas. We define the set $\tilde{S} \subset \mathcal{S}$ by

$$\tilde{S} := \left\{ s \in \mathcal{S} \;\middle|\; a_i \in \arg\max_{\tilde{a}_i} \pi_i \left(\tilde{a}_i, a_{\mathcal{L}_i}\right) \text{ for all } i \in N \right\},$$

i.e., the set $\tilde{S}$ contains all states $s \in \mathcal{S}$ such that all players $i \in N$ choose a pure strategy that results in the highest possible gross benefit, given the current pure-strategy profile of the players in their neighborhood. Lemma 25 specifies a condition on the size of the population under which adaptive play without mistakes becomes contained in $\tilde{S}$ with positive probability in a finite period of time. Let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x.

Lemma 25 *Assume that $\left\lfloor \frac{n}{2} \right\rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Then for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes reaches a state $\tilde{s} \in \tilde{S}$ in $K_s < \infty$ periods.*

Proof. Consider a state $s \in \mathcal{S}$ and a player $i \in N$. With positive probability player i is matched with a player $j \in N$ such that $i \neq j$ and $c_{ij} > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Notice that for every player

$i \in N$, existence of such a player j is guaranteed because we assume that $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Obviously, players i and j will never establish a mutual link. Furthermore, if a mutual link was already present, then it will definitely be severed. After that, both players will adapt their pure strategy and, as they no longer belong to each other's neighborhood, end up choosing a pure strategy that results in the highest possible benefit given the current pure strategies of the players in their neighborhood.

Consider the positive probability event of successively matching all players $i \in N$ in the way specified above. Notice that player i changing his pure strategy may imply that players in his neighborhood, who have already undergone the above procedure, no longer choose a pure strategy that results in the highest possible benefit given the current pure-strategy profile in their neighborhood. We restore this characteristic for the involved players by means of the positive probability event of once again matching all of them in the same pairs as before. Obviously, this may in turn result in other players, who have already undergone the above procedure, no longer playing optimally given the current pure-strategy profile in their neighborhood, and so on. It therefore remains to be shown that such a restoration process will terminate in finite time.

Suppose player i changes his pure strategy and becomes an A-player. As the restoration process does not include any link formation or link severance and $a - c > d - b$, the change by player i makes pure strategy A more attractive for all players involved in the restoration process. Consequently, all involved A-players will definitely remain A-players, while involved B-players possibly become A-players. Since we have a finite number of players such a restoration process terminates in finite time. A similar argument holds when player i becomes a B-player. ■

Lemma 26 in turn states that, starting from a state which is contained in $\tilde{S}$, adaptive play without mistakes converges to an absorbing state with positive probability in a finite period of time.

Lemma 26 *For any state $s \in \tilde{S}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.*

Proof. Consider a state $s \in \tilde{S}$ and a player $i \in N$. Suppose player i is an A-player, i.e., $a_i = A$. With positive probability player i is successively matched in pairs with all players

$j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} > \frac{c}{\gamma}$. Since the costs c_{ij} are strictly larger than the payoff $\frac{c}{\gamma}$ to player i, all links between player i and these players j will be severed. Because $b > c$ this dynamic process of link severance implies that $\pi_i \left(A, a_{\mathcal{L}_{i,k},k-1}\right)$ decreases less than $\pi_i \left(B, a_{\mathcal{L}_{i,k},k-1}\right)$, causing player i to remain an A-player. Furthermore, all players j remain B-players (when their link with player i is severed) due to the fact that $a > d$ causes $\pi_j \left(B, a_{\mathcal{L}_{j,k},k-1}\right)$ to decrease less than $\pi_j \left(A, a_{\mathcal{L}_{j,k},k-1}\right)$.

We continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. Since costs c_{ij} exceed payoffs $\frac{a}{\gamma}$ for these pairs of players, all links between player i and players j will be severed. Notice that at some point during this dynamic process of link severance, player i may become a B-player because $a > d$, i.e., $\pi_i \left(A, a_{\mathcal{L}_{i,k},k-1}\right)$ decreases more than $\pi_i \left(B, a_{\mathcal{L}_{i,k},k-1}\right)$. First, consider the case that player i is an A-player when he severs the link with a player j. Since $a > d$, the severance of this link may cause player j to become a B-player. This in turn may cause that A-players in the neighborhood of player j also want to become B-players, and so on. It can easily be verified that successively matching all the involved players in pairs, which is a positive probability event, results in a dynamic process only consisting of players changing their pure strategy from A to B and possibly severing their mutual link. Obviously, such a dynamic process will terminate in a finite number of steps and result in a state contained in $\tilde{S}$. Notice that player i may also be one of the players who changes his pure strategy from A to B in the above dynamic process.

Second, consider the case that player i, because of one of the two reasons mentioned above, becomes a B-player while he is severing links with the players j. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{d}{\gamma}$. Since $a > d$, this includes all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} > \frac{a}{\gamma}$. All these links will be severed because the costs c_{ij} are strictly larger than the payoff $\frac{d}{\gamma}$ to player i. Furthermore, player i and players j who are involved in this dynamic process will not change their pure strategy because $a > d$ and $b > c$, respectively.

The above argument shows that a player i, who is initially an A-player, will become an A-player such that

$$\begin{cases} \left[a_j = A \text{ and } c_{ij} > \frac{a}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \\ \\ \left[a_j = B \text{ and } c_{ij} > \frac{c}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \end{cases} \tag{3.7}$$

for all $j \in N$ with $i \neq j$, or a B-player such that

$$\begin{cases} \left[a_j = A \text{ and } c_{ij} > \frac{d}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \\ \\ \left[a_j = B \text{ and } c_{ij} > \frac{b}{\gamma}\right] \Rightarrow j \notin \mathcal{L}_i, \end{cases} \tag{3.8}$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is initially a B-player. In fact, starting from any state $s \in \tilde{S}$, we attain a state $\tilde{s} \in \tilde{S}$ such that all players $i \in N$ simultaneously satisfy either (3.7) or (3.8) with positive probability and in finite time. Such a state can actually be attained by successively applying the above dynamic process to all players $i \in N$.

Notice that the interactive effects may cause that players at some point in time no longer satisfy (3.7) or (3.8) even though the dynamic process has already been applied to them. For these players the dynamic process has to be repeated. However, due to the fact that the dynamic process does not include any link formation and, in case no links are severed, only consists of players changing their pure strategy in the same way, this can only happen a finite number of times.

Now consider a state $\tilde{s} \in \tilde{S}$ such that all players satisfy either (3.7) or (3.8) and there exists a player $i \in N$ such that $a_i = A$. With positive probability player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = A$, and $c_{ij} \leqq \frac{a}{\gamma}$. Obviously, not only will all these pairs of players establish a link, but also will none of the involved players change his pure strategy. We continue with the positive probability event that player i is successively matched in pairs with all players $j \in N$ such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leqq \frac{c}{\gamma}$. Again, all pairs of players will establish a link, which may possibly cause players to change their pure strategy. In fact, players i and players j may become B-players and A-players, respectively. As explained before, however, every time that link severance, or link formation for that matter, causes a player to change his pure strategy, the subsequent dynamic process of changing pure strategies by other players will terminate in finite time with positive probability. In case player i actually becomes a B-player, we continue with the positive probability event that player i is successively matched in pairs with all players j such that $j \notin \mathcal{L}_{i,k}$, $a_j = B$, and $c_{ij} \leqq \frac{b}{\gamma}$, and all players j such that $j \in \mathcal{L}_{i,k}$, $a_j = A$, and $d < \gamma \cdot c_{ij} \leqq a$, which obviously only results in link formation and link severance, respectively.

The above argument shows that a player i, who is an A-player in a state $\tilde{s} \in \tilde{S}$, will become an A-player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{a}{\gamma} \right], \\ \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{c}{\gamma} \right], \end{cases} \tag{3.9}$$

for all $j \in N$ with $i \neq j$, or a B-player such that

$$\begin{cases} a_j = A \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{d}{\gamma} \right], \\ \\ a_j = B \Rightarrow \left[j \in \mathcal{L}_i \Leftrightarrow c_{ij} \leqq \frac{b}{\gamma} \right], \end{cases} \tag{3.10}$$

for all $j \in N$ with $i \neq j$, with positive probability and in finite time. A similar argument can be used to show that this also holds for a player i who is a B-player in a state $\tilde{s} \in \tilde{S}$. Furthermore, starting from a state $\tilde{s} \in \tilde{S}$, we attain a state such that all players $i \in N$ simultaneously satisfy (3.9) or (3.10) with positive probability and in finite time. In that case, no single players wants to adapt his pure strategy or sever any of the links he is involved in, and no pair of players wants to establish a mutual link, which implies that we have attained an absorbing state of the Markov process.

Like before it may happen that in order to attain an absorbing state from a state $\tilde{s} \in \tilde{S}$, the dynamic process described above has to be applied repeatedly to some players. However, because of the following two reasons this can only happen a finite number of times. First, the part of the dynamic process dealing with link severance does not result in players changing their pure strategies. Second, parts of the dynamic process that do consist of players changing their pure strategies will always terminate in finite time with positive probability as they only involve changes in the same direction. ■

Now, we are able to prove Theorem 22 which specifies the condition on the size of the population under which convergence of adaptive play without mistakes can be guaranteed almost surely.

Theorem 22 *Assume that* $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. *Then for any initial state* $s_0 \in S$ *adaptive behavior without mistakes converges almost surely to an absorbing state.*

Proof. Lemma 25 states that for any state $s \in \mathcal{S}$ there is a probability $p_s > 0$ that adaptive play without mistakes will be given by a state $\tilde{s} \in \tilde{S}$ in $K_s < \infty$ periods, provided that $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. According to Lemma 26, for any state $s \in \tilde{S}$ there is a probability $q_s > 0$ that adaptive play without mistakes converges to an absorbing state in $L_s < \infty$ periods.

Note that $\mathcal{S}$ is finite. Let $p = \min_{s \in \mathcal{S}} p_s > 0$, $q = \min_{s \in \tilde{S}} q_s > 0$, $K = \max_{s \in \mathcal{S}} K_s < \infty$, and $L = \max_{s \in \tilde{S}} L_s < \infty$, it follows that there exists a positive integer $M = K + L < \infty$, and a positive probability $r = pq > 0$, such that from any initial state $s_0 \in \mathcal{S}$, the probability is at least r that adaptive play without mistakes converges to an absorbing state within M periods, provided that $\lfloor \frac{n}{2} \rfloor > \max\left\{\frac{a}{\gamma}, \frac{b}{\gamma}\right\}$. Obviously, both M and r are time-independent and state-independent. Let m be an arbitrary integer. Hence, the probability of not reaching an absorbing state after at least mM periods is at most $(1-r)^m$. This implies that this probability goes to zero as $m \to \infty$. ■

3.7.2 Proof of Theorem 24

Lemma 27 *Assume that the transition matrix* P *is irreducible and aperiodic. Then the stationary distribution* ϕ *is unique. Furthermore, for any* $\nu \in \Delta_{|\mathcal{S}|-1}$,

$$\nu P^k \to \phi \quad \text{as } k \to \infty.$$

Also, for all initial states $s_0 \in S$,

$$\frac{1}{K} \sum_{k=1}^{K} X_s(s_k) \to \phi_s \quad \text{almost surely as } K \to \infty,$$

where

$$X_s(s_k) = \begin{cases} 1 & \text{if } s_k = s, \\ 0 & \text{otherwise.} \end{cases}$$

(See Theorem 1.2 *and Theorem* 1.3 *in Chapter* 3 *of Karlin and Taylor [37])*

We write $r(z)$ for a successor of state z in the Markov process describing adaptive play without mistakes. Obviously, $r(z)$ is not uniquely determined since it depends on which pair of players can alter their (potential) mutual link and adapt their pure strategies. For this reason we write

$$D\left(A, \frac{a}{\gamma}\right) = \left\{ z \,\middle|\, \lim_{m \to \infty} \Pr\left(r^m(z) = \left(A, \frac{a}{\gamma}\right)\right) = 1 \right\}$$

for the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention. Hence, the basin of attraction $D\left(A, \frac{a}{\gamma}\right)$ consists of the states that will eventually be taken to the $\left(A, \frac{a}{\gamma}\right)$-convention by adaptive play without mistakes.

To determine what the stationary distribution looks like when mistake probabilities go to zero, we need the following two lemmas. Lemma 28 gives upper bounds for the number of τ-mistakes and ε-mistakes that are needed to enter the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention with positive probability.

Lemma 28 *Let $x \notin D\left(A, \frac{a}{\gamma}\right)$, $y_1, ..., y_H \notin D\left(A, \frac{a}{\gamma}\right)$, and $y \in D\left(A, \frac{a}{\gamma}\right)$. There exists a transition from x to y via $y_1, ..., y_H$ such that*

$$P_{xy_1}(\tau, \varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau, \varepsilon) \cdot P_{y_H y}(\tau, \varepsilon) > 0, \tag{3.11}$$

where (3.11) is of τ-order and ε-order at most 0 *and n, respectively.*

Proof. To prove Lemma 28 it suffices to show that $D\left(A, \frac{a}{\gamma}\right)$ can be reached from any state with positive probability using at most n ε-mistakes. First, let the population size n be even. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j - 1$ and $j = 1, ..., \frac{n}{2}$. Every time a player has the opportunity to update his pure strategy, he chooses A. Obviously, this requires at most n ε-mistakes and leaves us with a state $y \in D\left(A, \frac{a}{\gamma}\right)$.

Second, let the population size n be odd. Consider the positive probability event of successively matching players i and $i+1$ in pairs, where $i = 2j-1$ and $j = 1, ..., \frac{n-1}{2}$. Again, all players involved in this positive probability event choose pure strategy A whenever they have the opportunity to update their strategy. This requires at most $n-1$ ε-mistakes and leaves us with a state such that $a_i = A$ for all $i = 1, ..., n-1$. Now, we continue with the positive probability event of matching player n with an A-player $i^* \in \{1, ..., n-1\}$, where player i^* is an A-player who is linked to at least one other A-player. Notice that existence of such an A-player can always be established without the need for any additional mistakes. Then, at most one more ε-mistake is needed for player n to become an A-player, while player i will remain an A-player. Again, we have reached a state $y \in D\left(A, \frac{a}{\gamma}\right)$ using at most n ε-mistakes. ■

Lemma 29 gives a lower bound for the total number of mistakes in a certain product of transition probabilities. All transition probabilities contained in the product refer to transitions which are specified by an x-tree t, with $x \notin D\left(A, \frac{a}{\gamma}\right)$, and which start from a state in the basin of attraction of the $\left(A, \frac{a}{\gamma}\right)$-convention.

Lemma 29 *Let $n \geqq 3$. The sum of the τ-order and ε-order of*

$$\prod_{s \in D\left(A, \frac{a}{\gamma}\right)} P_{st(s)}(\tau, \varepsilon) > 0,$$

with t an x-tree such that $x \notin D\left(A, \frac{a}{\gamma}\right)$, is at least $n+1$.

Proof. First, consider the state $s = \left(A, \frac{a}{\gamma}\right) \in D\left(A, \frac{a}{\gamma}\right)$. Obviously, any path starting in the $\left(A, \frac{a}{\gamma}\right)$-convention and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$ contains at least one ε-mistake.

Second, consider a state s such that (i) every player $i \in N$ is linked with his two adjacent neighbors, i.e., every player $i \in N$ is linked with all players $j \neq i$ such that

$$\min\{|j-i|, n-|j-i|\} = \gamma,$$

and (ii) there is exactly one player i such that $a_i = B$. Obviously, such a state s is contained in $D\left(A, \frac{a}{\gamma}\right)$ when $n \geqq 3$ and there exist exactly n different states that satisfy the conditions (i) and (ii). Furthermore, any path starting in such a state s and eventually leaving $D\left(A, \frac{a}{\gamma}\right)$

(or reaching another state satisfying conditions (i) and (ii)) contains at least 1 mistake (either a τ-mistake or an ε-mistake). Hence, we know that the sum of the number of τ-mistakes and ε-mistakes in $\prod_{s\in D\left(A,\frac{a}{\gamma}\right)} P_{st(s)}(\tau,\varepsilon)$ is at least $n+1$. ■

Theorem 24 *Let* $n \geqq 3$. *If* $0 < \lim_{\tau,\varepsilon\to 0} \frac{\tau}{\varepsilon} < \infty$, *then* $\lim_{\tau,\varepsilon\to 0} \phi_{\left(A,\frac{a}{\gamma}\right)}(\tau,\varepsilon) = 1$.

Proof. The characterization as specified by (3.5) allows us to express the quantity $\frac{\phi_x(\tau,\varepsilon)}{\phi_{\left(A,\frac{a}{\gamma}\right)}(\tau,\varepsilon)}$ as a ratio of polynomials in τ and ε for any state x. To prove Theorem 24 it is sufficient to show that for n sufficiently large, i.e., $n \geqq 3$, it holds that

$$\lim_{\tau,\varepsilon\to 0} \frac{\phi_x(\tau,\varepsilon)}{\phi_{\left(A,\frac{a}{\gamma}\right)}(\tau,\varepsilon)} = 0$$

for all $x \neq \left(A,\frac{a}{\gamma}\right)$. This will follow if we demonstrate that for any x-tree t $\left(x \neq \left(A,\frac{a}{\gamma}\right)\right)$ such that $\prod_{s\neq x} P_{st(s)}(\tau,\varepsilon) > 0$, we have

$$\lim_{\tau,\varepsilon\to 0} \frac{\prod_{s\neq x} P_{st(s)}(\tau,\varepsilon)}{\sum_{t'\in H_{\left(A,\frac{a}{\gamma}\right)}} \prod_{s\neq\left(A,\frac{a}{\gamma}\right)} P_{st'(s)}(\tau,\varepsilon)} = 0.$$

This in turn follows if we show that there exists an (A,a)-tree t' such that

$$\prod_{s\neq\left(A,\frac{a}{\gamma}\right)} P_{st'(s)}(\tau,\varepsilon) > 0$$

and

$$\lim_{\tau,\varepsilon\to 0} \frac{\prod_{s\neq x} P_{st(s)}(\tau,\varepsilon)}{\prod_{s\neq\left(A,\frac{a}{\gamma}\right)} P_{st'(s)}(\tau,\varepsilon)} = 0. \tag{3.12}$$

We show that (3.12) holds by distinguishing two cases.

First, assume that $x \in D\left(A,\frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A,\frac{a}{\gamma}\right), \\ t(z) & \text{otherwise.} \end{cases}$$

Notice that t' is an $\left(A, \frac{a}{\gamma}\right)$-tree because for any state z the path described by t' initially coincides with t and hence eventually enters $D\left(A, \frac{a}{\gamma}\right)$. From the first point at which $t'^{m}(z) \in D\left(A, \frac{a}{\gamma}\right)$, the tree maps every point to a successor according to adaptive play without mistakes and hence reaches $\left(A, \frac{a}{\gamma}\right)$. In this case the ratio specified in (3.12) equals

$$\frac{P_{\left(A,\frac{a}{\gamma}\right)t\left(\left(A,\frac{a}{\gamma}\right)\right)}(\tau,\varepsilon) \prod_{s \in D\left(A,\frac{a}{\gamma}\right)-\left\{\left(A,\frac{a}{\gamma}\right),x\right\}} P_{st(s)}(\tau,\varepsilon)}{P_{xr(x)}(\tau,\varepsilon) \prod_{s \in D\left(A,\frac{a}{\gamma}\right)-\left\{\left(A,\frac{a}{\gamma}\right),x\right\}} P_{sr(s)}(\tau,\varepsilon)}.$$

The above expression converges to 0 as $\tau, \varepsilon \to 0$ because $P_{st(s)}(\tau,\varepsilon)/P_{sr(s)}(\tau,\varepsilon)$ is bounded, $P_{\left(A,\frac{a}{\gamma}\right)t\left(\left(A,\frac{a}{\gamma}\right)\right)}(\tau,\varepsilon) \to 0$, and $P_{xr(x)}(\tau,\varepsilon) \to \xi > 0$.

Second, assume that $x \notin D\left(A, \frac{a}{\gamma}\right)$. Define t' by

$$t'(z) = \begin{cases} r(z) & \text{if } z \in D\left(A, \frac{a}{\gamma}\right), \\ t(z) & \text{if } z \notin D\left(A, \frac{a}{\gamma}\right),\ z \neq x, \text{ and } z \neq y_1, ..., y_H, \\ y_1 & \text{if } z = x, \\ y_2 & \text{if } z = y_1, \\ ... & ... \\ y_H & \text{if } z = y_{H-1}, \\ y & \text{if } z = y_H, \end{cases}$$

where $y_1, ..., y_H \notin D\left(A, \frac{a}{\gamma}\right)$ and $y \in D\left(A, \frac{a}{\gamma}\right)$ are as specified in Lemma 5.5. Obviously, t' is again an $\left(A, \frac{a}{\gamma}\right)$-tree. In this case the ratio specified in (3.12) equals

$$\frac{\prod_{s \in D\left(A,\frac{a}{\gamma}\right)} P_{st(s)}(\tau,\varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h t(y_h)}(\tau,\varepsilon)}{P_{xy_1}(\tau,\varepsilon) \cdot \prod_{h=1}^{H-1} P_{y_h y_{h+1}}(\tau,\varepsilon) \cdot P_{y_H y}(\tau,\varepsilon) \cdot \prod_{s \in D\left(A,\frac{a}{\gamma}\right)-\left\{\left(A,\frac{a}{\gamma}\right)\right\}} P_{sr(s)}(\tau,\varepsilon)}.$$

Because of Lemma 28 and the fact that $P_{sr(s)}(\tau,\varepsilon) \to \mu > 0$, the denominator is of τ-order 0 and of ε-order at most n. According to Lemma 29, the numerator is of a higher total order if

$n \geqq 3$. Consequently, the above expression converges to 0 as long as $0 < \lim_{\tau,\varepsilon \to 0} \frac{\tau}{\varepsilon} < \infty$. ■

Chapter 4

A Model of Social Capital Formation

4.1 Introduction

During recent years, *social capital* has become a topic of study in sociology and economics. The history of social capital in the academic circles, primarily sociological, is long but much of the recent interest can be contributed to work by Bourdieu [11], Coleman [15], Portes [46] and Putnam [47]. Loury [40], an economist, is also acknowledged for his early work in this area. Jacobs [33] used the term in her 1961 book which predates all the work of the above writers. Increasing evidence shows that social capital is an important determinant in trade, crime, education, health care, and rural development. It is an important determinant in some economic situations because it is an asset, albeit a difficult to measure and sometimes non-convertible one. In many parts of the world social capital is the only asset that some individuals have. For example, Bornstein [10] documents the Grameen Bank in Bangladesh that only lends to small groups called lending circles. The primary asset of these circles is social capital.

Bourdieu [11] on page 243 aptly defines social capital as “made up of social obligations (“connections”), which is convertible, in certain conditions, into economic capital...”. Social networks are the media through which social capital is created, maintained and used. In short, social networks convey social capital. It is our objective to study the formation and maintenance of social capital through a social network.

Social networks form as individuals establish and maintain relationships. Being “connected” greatly benefits an individual. Yet, maintaining relationships is costly. As a consequence individuals limit the number of their active relationships.

Our model uses the rational formation of links between individuals as the foundation for the formation and maintenance of social capital. A network link is the current investment in a relationship. As links are formed and maintained over time they begin to accumulate social capital. Like any other asset, social capital pays a return every period and it depreciates. In our model a small depreciation rate δ means that more social capital is accumulated each period. Social capital depreciates at different rates depending on the substitutes developed for the uses of social capital. For example, many social relationships are replaced by market activities: child care, home meal production, and elderly care. We conclude that it is necessary for links to be present to create social capital. However, links need not be present to use social capital, but as we show, social capital diminishes when is not maintained through links.

We present two simulation models of social capital formation. Both models of social capital accumulation show that social capital can be accumulated without creating the incentive for all players to be connected. The first model is a model of social capital accumulation and the second shows an additional benefit of accumulated capital – the choice of some agents to form links that have a higher maintenance cost.

Social Networks

Network relations among players are formally represented by graphs where the nodes are identified with the players and in which the edges capture the pairwise relations between these players. These relationships are interpreted as *social* links that lead to benefits for the communicating parties.

We first discuss some standard definitions from graph theory. Formally, a *link* ij is the subset $\{i, j\}$ of N containing i and j. We define $g^N := \{ij \mid i, j \in N\}$ as the collection of all links on N. An arbitrary collection of links $g \subset g^N$ is called an (undirected) *network* on N.

A *chain* is a connected network composed of exactly one path with a spatial requirement.

Definition 30 *A network $g \subset g^N$ is called a **chain** when (i) for every $ij \in g$ there is no h such that $i < h < j$ and (ii) g is connected.*

There exists exactly one chain on N and it is given by $g = \{12, 23, \ldots, (n-1)\, n, n1\}$.

4.2 A Model of Social Capital Formation

Consider a large but finite population of players $N = \{1, ..., n\}$ who are spatially distributed around a circle. Players are distributed around the circle in an equidistant fashion. Each discrete-time period $t \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ the players form or sever links connecting themselves to other players. Each player's utility depends on net benefits derived from relationships in each period. Net benefits depend on the history of links with their associated maintenance costs and benefits. We again defer to Bourdieu [11] who describes the dynamics of social capital formation on page 243:

> In other words, the network of relationships is the product of investment strategies, individual or collective, consciously or unconsciously aimed at establishing or reproducing social relationships that are directly usable in the short or long term, i.e., at transforming contingent relations, such as those of neighborhood, the workplace, or even kinship, into relationships that are at once necessary and elective, implying durable obligations subjectively felt (feelings of gratitude, respect, friendship, etc.) or institutionally guaranteed (rights). This is done through the alchemy of consecration, the symbolic constitution produced by social institution (institution as a relative–brother, sister, cousin, etc.–or as a knight, an heir, and elder, etc.) and endlessly reproduced in and through the exchange (of gifts, words, women, etc.) which it encourages and which presupposes and produces mutual knowledge and recognition.

In the context of our model relationships that are maintained over time pay b_{ij}, the net utility or net benefit that agent i receives from agent j. The benefit to player i of maintaining a relationship over time with player j is a function of six factors:

$$b_{ij,t} = f(I_{ij,t}, v_{ij,t}, S_{ij,t}, R_{j,t}, h(d_{j,t}), c_{ij})$$

They are with $i, j \in N$ and $i \neq j$,

$I_{ij,t}$ indicates if link ij is present at time t i.e., $I_{ij,t} = 1$ if $ij \in g_t$ and $I_{ij,t} = 0$ if $ij \notin g$.

$v_{ij,t}$ is the value of the link in the current period for player i linking to player j attributable to current interaction. $v_{ij,t}$ is a "gift" or "words." It is the acknowledgment and comparable to the "δ" in the Spatial Social Networks chapter. It occurs in the current period only and can only be gained if the link is present. This value is exogenous to the model and given to guarantee that at least one link will rationally form. Additionally, we make the simplifying assumption that $v_{ij,t} = v_{ji,t}$.

$S_{ij,t}$ is the social capital for player i based on the relationship between i and j. It may have a positive value if players i and j have been connected at least on period in the past. Social capital is depreciated at the rate δ or accumulated at the rate $(1-\delta)$.

$$S_{ij,t} = \sum_{s=1}^{t-1} I_{ij,s} (1-\delta)^{t-s} b_{ij,s} = (1-\delta) I_{ij,(t-1)} b_{ij,(t-1)} + (1-\delta) S_{ij,(t-1)}.$$

$R_{j,t}$ is the measure of how well agent i interacts with others. It can be interpreted as a reputation variable. If agent i does well linking with others, then the intuition is that they are a reliable agent.

$$R_{i,t} = \sum_{s=0}^{t-1} (1-\delta)^{t-s} b_{i,s} = (1-\delta) I_{ij,(t-1)} b_{i,(t-1)} + (1-\delta) R_{i,(t-1)},$$

where $b_{i,t} = \sum_j b_{ij,t}$. Reputations also depreciate over time at the rate δ.

d_j^t is a congestion factor. d_j^t is the degree of j in graph g_t at time t for all $ij \in g$.

$$d_{j,t} = \sum_{i=1}^{n} I_{ij,t}.$$

c_{ij} Each period, the presence of a link between players i and j, with $i, j \in N$ and $i \neq j$, results in a maintenance cost c_{ij} to both players. The costs c_{ij} are determined by the distance between the two players on the circle. Formally, we assume that

$$c_{ij} = \min \left\{ \gamma \left| j-i \right|, \gamma(n - |j-i|) \right\},$$

where $\gamma \geqq 0$ are the so-called unit link costs.

We further simplify our model by assuming a functional form for the net benefit of a link to a player as

$$b_{ij,t} = \kappa_1 I_{ij,t} v_{ij,t} + \kappa_2 S_{ij,t} + \kappa_3 R_{j,t} - I_{ij,t} h\left(d_{j,t}\right) - I_{ij,t} c_{ij}, \tag{4.1}$$

where $\kappa_1, \kappa_2, \kappa_3$ are constants. For every period t a link is maintained it accumulates an amount of social capital and reputation for each participating player. Social capital and reputation may pay a positive amount to players after link severance. A link can be severed by either party.

At this point it is useful to discuss the difference between depreciation of social capital and the maintenance cost of a link. They both have a negative impact net benefit. Social capital is a form of capital, an asset. Social assets lose value over time as substitutes for them are developed as do physical assets. Maintenance costs are the explicit costs of forming and maintaining a relationship. The depreciation rate δ is uniform over all links and the maintenance cost c_{ij} is specific to a particular link.

4.2.1 Deterministic Models

We begin our simulations with the empty network. Initially, $S_{ij,t=0} = 0$ and $R_{i,t=0} = 0$ for all $i, j \in N$. Players are distributed around the circle in an equidistant fashion. Each discrete-time period $t \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ one pair is chosen at random and asked if they would like to form a link, all other linked players are asked if they would like to sever. Players choose to add a link if $b_{ij,t} \geqq 0$ and will sever if a link if $b_{ij,t} < 0$. Players that continue to maintain links for more than one period accumulate social capital and reputation.

We assume that for $i, j \in N$ $v_{ij,t} = v$ is exogenously given and constant. Other exogenously given parameters are $\kappa_1, \kappa_2, \kappa_3$, the weights of current benefit, social capital and reputation, respectively, γ, the per unit cost of connection, and δ, the depreciation rate. $h(d_{j,t})$ will have different functional forms. In the first example $h(d_{j,t})$ is linear and in the second example we allow it to be quadratic for stabilizing purposes. The rest of the variables are as described before.

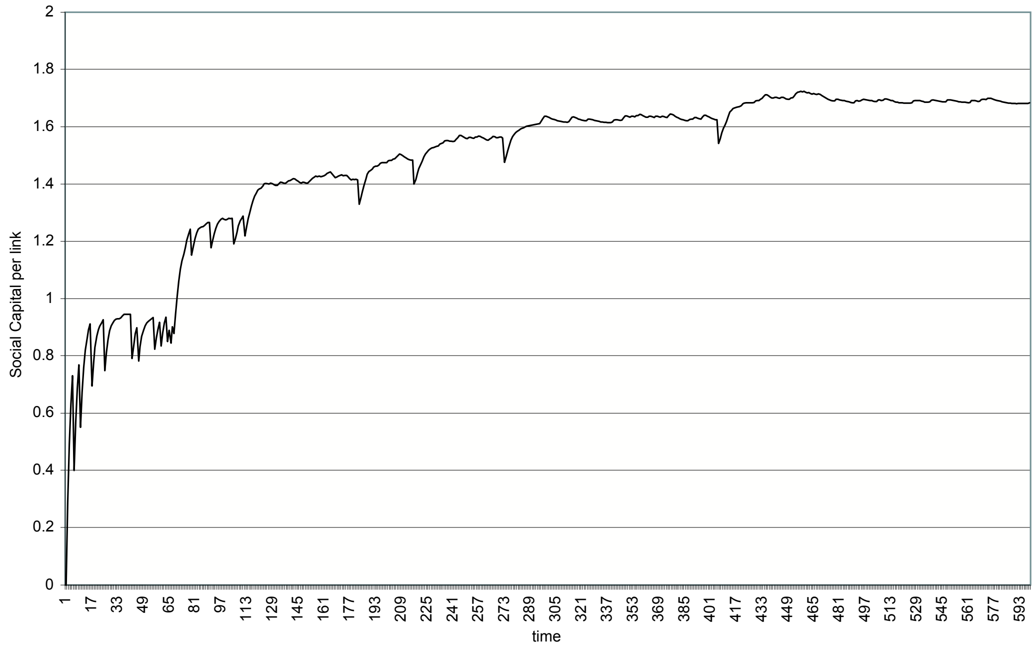

Figure 4-1: Social capital per link; Linear model

Linear Model

In the first model, we develop a system to show that social capital forms and is maintained at a certain level. However, the system we describe below is constructed so that social capital would form and continue to be maintained, but not to the level that would encourage players to maintain links that are farther away than their direct neighbors.

We let $\delta = 0.6, \gamma = 0.59, v = 1,\ \kappa_1 = 1, \kappa_2 = 0.35, \kappa_3 = 0.2$ and $h(d_{j,t}) = 0.05d_{j,t}$. For all $i, j \in N$, $t \in \mathbb{N}_0$ and $v_{ij,t} = v$ Equation (4.1) simplifies to

$$b_{ij,t} = I_{ij,t} + 0.35S_{ij,t} + 0.2R_{j,t} - I_{ij,t} \cdot 0.05d_{j,t} - I_{ij,t}c_{ij} \tag{4.2}$$

We started the system with 20 players and an empty network and let it run for 10,000 periods. The resulting network is the chain. We are able to show almost all the variability in the system through Figures 4-1, 4-2, and 4-3 represent the first 600 periods of the interaction. The size of the network formed is depicted by the bold straight lines of Figure 4-2. The size of the graph steadily increases from 1 player to 20. The size of the network is graphed in conjunction with

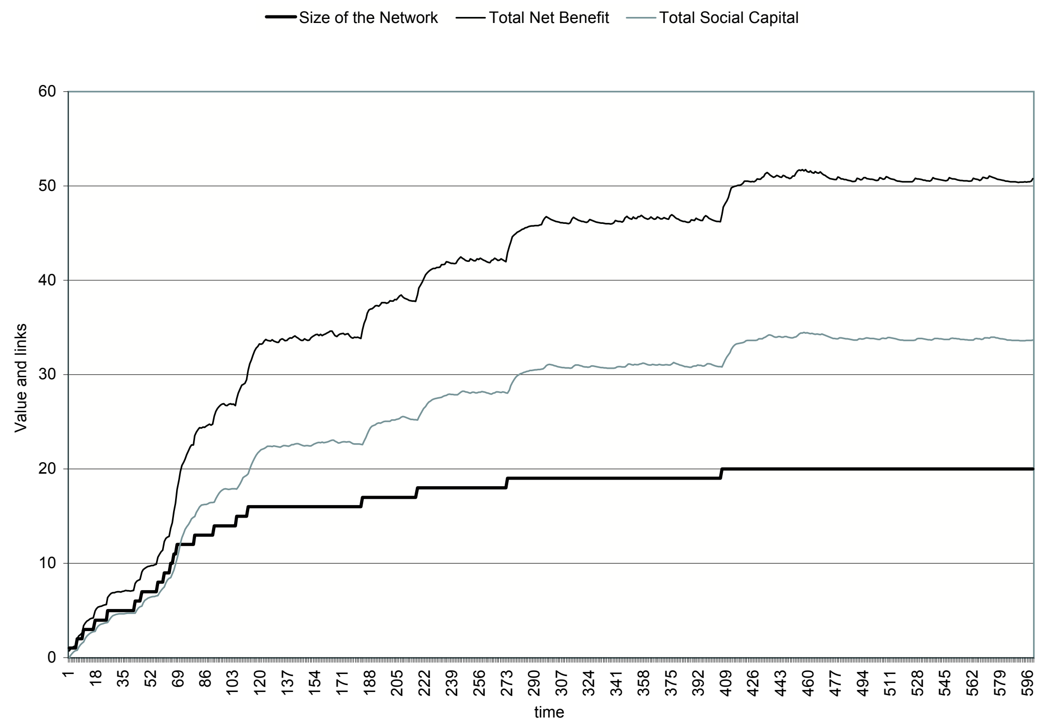

Figure 4-2: Comparison of size of network, total social capital, and total net benefits; Linear model.

the total social capital and total net benefit to verify that both of these sums generally increase with the size of the network.

We use Equation (4.1) to verify that the only benefit received from the current period for $ij \in g$ that is not a part of the history of the link, meaning $S_{ij,t}$ or $R_{j,t}$, is $\kappa_1 I_{ij,t} v_{ij,t} - I_{ij,t} h\left(d_{j,t}\right) - I_{ij,t} c_{ij}$. If we assume that each player is linked to his nearest neighbor and every player has $d_{i,t} = 2$ we can conclude from Equation (4.2) that the current value from the connection is $1 - 0.59 - 0.05(2) = 0.31$. Figure 4-1 shows that the social capital generated and maintained per link is much higher than twice $0.3\dot{1}$. Figure 4-3 verifies that the chain is the architecture of the network formed. After period 406 in the simulation results pictured, all players are members of the chain network and no player is ever linked to someone more than one unit away. Figure 4-4 illustrates the steady level of social capital for several hundred periods after the chain has formed. The remaining 8000 periods (not pictured) look very much like a continuation of Figure 4-4.

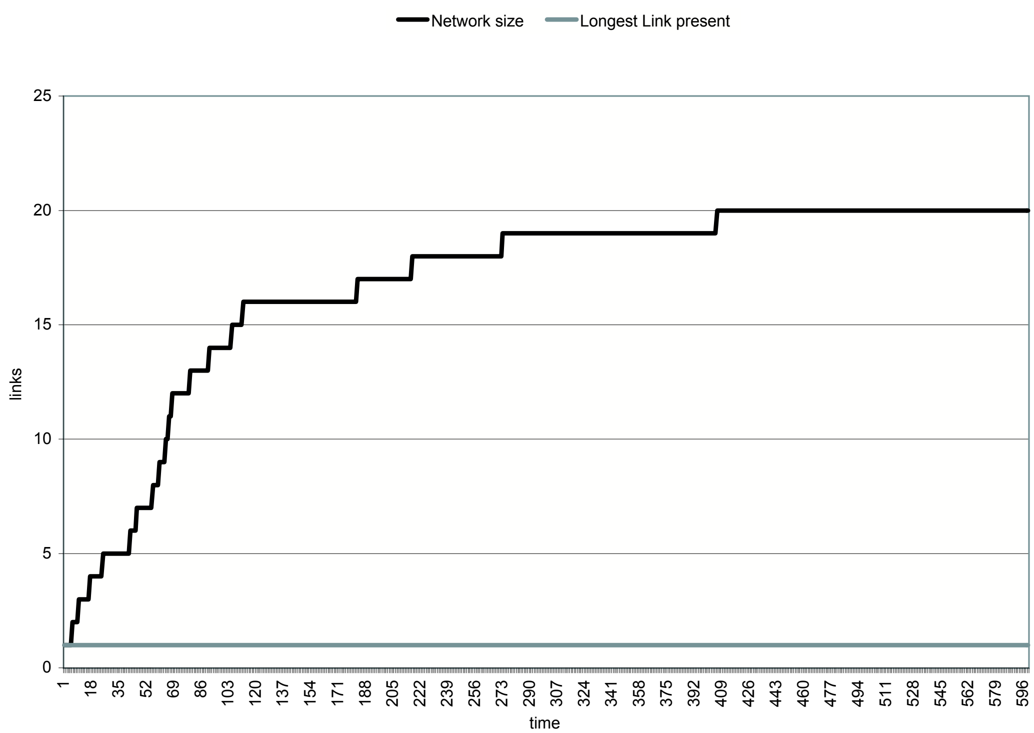

Figure 4-3: Network and longest link size; Linear model

Figure 4-4: Total net benefit and total social capital, 2000 periods; Linear model

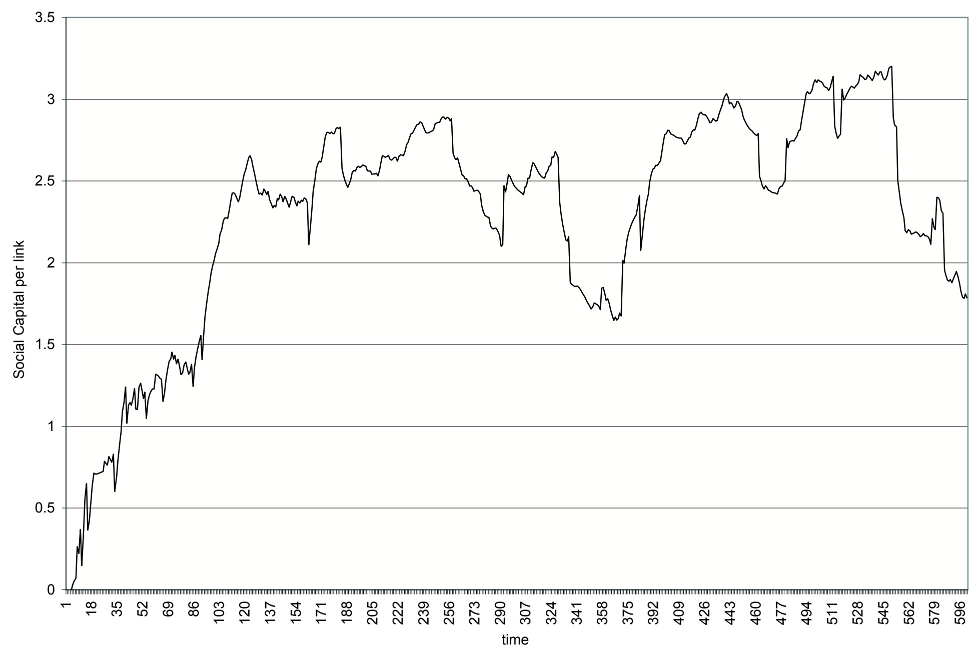

Figure 4-5: Social capital per link; Quadratic model

Quadratic Model

The second model will show how agents are able to build up enough social capital to make links farther away than their direct neighbors. However, this system this system is not nearly as well-behaved as the system described in the Linear Model. We let $\delta = 0.6, \gamma = 0.51, v = 1$, $\kappa_1 = 1, \kappa_2 = 0.4, \kappa_3 = 0.2$ and $h(d_{j,t}) = 0.45\,(d_{j,t} - 2)^2$. For all $i, j \in N$, $t \in \mathbb{N}_0$ and $v_{ij,t} = v$ Equation (4.1) now reduces to

$$b_{ij,t} = I_{ij,t} + 0.4S_{ij,t} + 0.2R_{j,t} - I_{ij,t} \cdot 0.45\,(d_{j,t} - 2)^2 - I_{ij,t}c_{ij} \tag{4.3}$$

We start the system with 10 players and an empty network and let it run for 10,000 periods. There is no stable state resulting. Hence the system remain in permanent volatility. Figure 4-7 best illustrates the interplay between the longest link present and the network size. In the area around period 169 and in the low 400's we see an interval of time where the chain network persists. Directly proceeding from those chain periods is a network that is composed of the chain and an additional link with the length of two units. We can see this because as the size

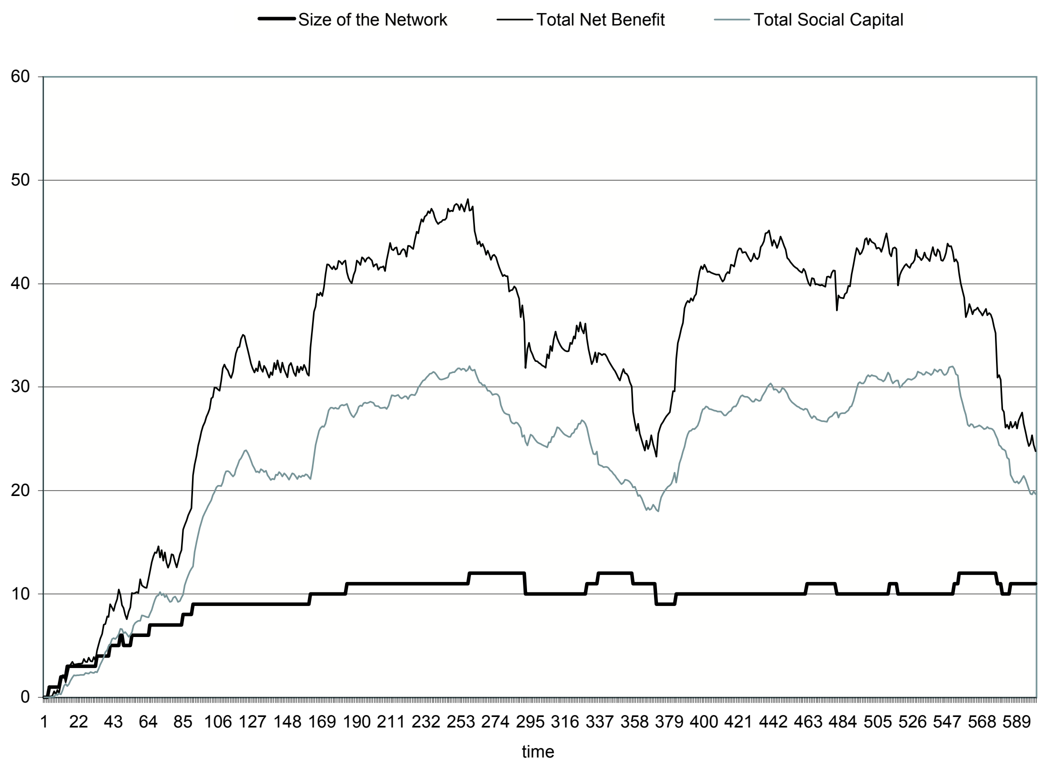

Figure 4-6: Comparison of size of network, total social capital and total net benefits; Quadratic model.

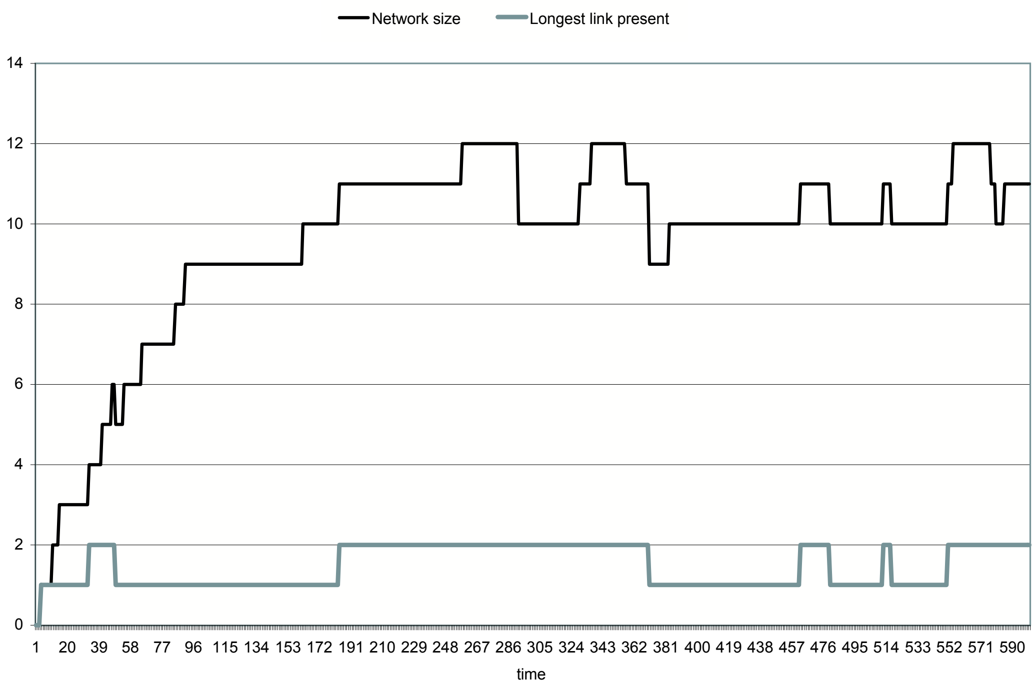

Figure 4-7: Network and longest link size, Quadratic model

of the network jumps to 11, the longest link jumps to 2. As with the linear model, players are not able to afford a long link without some historical benefit. In this case, the social capital of each of the players was accumulated which in turn enhanced their reputations which finally allowed them to afford a more expensive, distant link.

The size of the network formed is depicted by the bold straight lines of Figure 4-6. The size of the graph mostly increases from 1 player to 12 and then it continues to vary in size from 9 to 12 players. The correlation between the size of the network with the total social capital and total net benefit is indicated by Figure 4-6. As the network builds in size sometimes to 10 players, other times to 11 players, it is able to accumulate more capital. When one more link is formed, it has the effect of costing other connected members of the network tp lose social capital due to increased congestion. Every time this simulation reached a network size of 12, total social capital and total net benefit began to tumble.

Figure 4-5 differes from Figure 4-1 in that it is much more volatile. Social capital per link changes quite dramatically over time. The major downward dips in social capital follow an

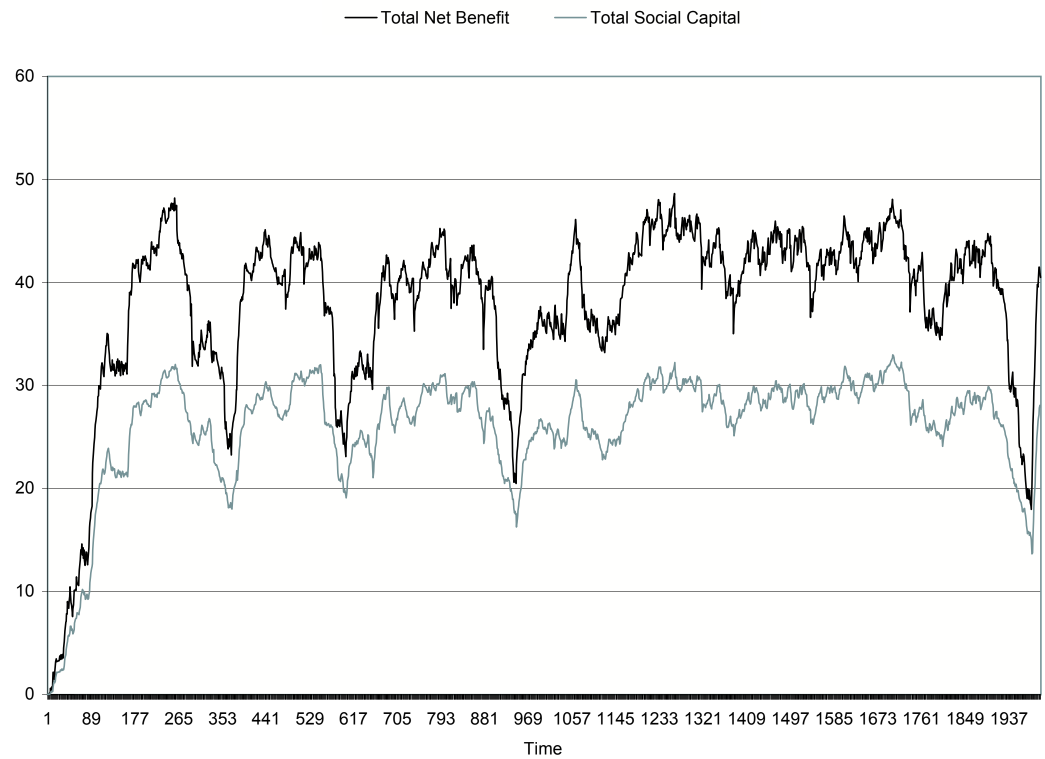

Figure 4-8: Total net benefit and total social capital, 2000 periods; Quadratic model

interval of relatively large networks. Figure 4-8 illustrates the turbulent level of social capital for several hundred periods after the 9-12 link network has formed. The remaining 8000 periods (not pictured) look very much like a continuation of Figure 4-8.

4.3 Discussion

Both models of social capital accumulation show that social capital can be built without creating the incentive for all players to be connected. The primary difference between the two simulation models presented is how the congestion aspect was handled. In the first model it was a simple, linear function of the degree of connectedness of a player. This had a minor role in limiting the amount of social capital accumulated. In this model the coefficients on $S_{ij,t}$ and $R_{j,t}$ played a central role in keeping the state of net benefits nearly steady. However, the quadratic model used the congestion function $0.45\,(d_{j,t}-2)^2$ as guard against the proliferation of many long links. This congestion function favors players participating in a chain because they bear no congestion costs. If it is our objective to observe a system that builds social capital, maintains more distant links, and does not create the complete network then we will have to use either a quadratic congestion function or cost function.

One extension to the deterministic model is the addition of imperfect agents. Players would be imperfect in the realization of $v_{ij,t}$. The population of players could be distributed into two groups, players that pay $v_{ij,t}$ with probability $1-\varepsilon_1$ and players that pay $v_{ij,t}$ with probability $1-\varepsilon_2$, where $\varepsilon_1 < \varepsilon_2$. The net benefit for player i for a relationship with player j in the deterministic system represented by Equation (4.1) would become

$$b_{ij,t} = \kappa_1 I_{ij,t} E\left(v_{ij,t}\right) + \kappa_2 S_{ij,t} + \kappa_3 R_{j,t} - I_{ij,t} h\left(d_{j,t}\right) - I_{ij,t} c_{ij}, \tag{4.4}$$

for the system with imperfect agents. $E\left(v_{ij,t}\right) = (1-\varepsilon_1)v_{ij,t}$ if player j was of type one and $E\left(v_{ij,t}\right) = (1-\varepsilon_1)v_{ij,t}$ if player j was of type two. The players could make choices on the reliability of links as well as the duration. We would expect to see social capital used to maintain relationships with inconsistent agents. The primary goal of adding this complication would be to test the hypothesis that players with lower reliability maintain less social capital than other players and thereby are more likely to lose a link when an ε_2 event occurs than for an ε_1 player.

4.4 Conclusion

We presented two simulation models of social capital formation. The first model was primarily a model of social capital accumulation and the second showed an additional benefit of accumulated capital - the ability to link to more distant players.

One final observation on social capital that should be emphasized is that social capital is indeed a form of capital. As a form of capital it can create some inertia in an economic or social system. By the same token that accumulated social capital can help form costly links, it can also become an incentive for players to stay in costly relationships. When social capital is taken into consideration, links that seem irrational when examined in one period, may be perfectly rational when the history of the link is taken into consideration. Bourdieu [11] opens his often cited article with the following sentences that we would like to use to close this discussion:

> The social world is accumulated history, and if it is not to be reduced to a discontinuous series of instantaneous mechanical equilibria between agents who are treated as interchangeable particles, one must reintroduce into it the notion of capital and with it, accumulation and all its effects...It is what makes the games of society – not least, the economic game – something other than simple games of chance offering at every moment the possibility of a miracle...Capital, which, in its objectified forms, takes time to accumulate and which, as a potential capacity to produce profits and to reproduce itself in identical or expanded form, contains a tendency to persist in its being, is a force inscribed in the objectivity of things so that everything is not equally possible or impossible.

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