

# The Minority Game: an introductory guide

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## Abstract

The Minority Game is a simple model for the collective behavior of agents in an idealized situation where they have to compete through adaptation for a finite resource. This review summarizes the statistical mechanics community efforts to clear up and understand the behavior of this model. Our emphasis is on trying to derive the underlying effective equations which govern the dynamics of the original Minority Game, and on making an interpretation of the results from the point of view of the statistical mechanics of disordered systems.

## 1 Introduction

There are  $10^{11}$  stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them astronomical numbers. Now we should call them economical numbers.

*Richard Feynman*

During the last years, the statistical mechanics community has turned its attention to problems in social, biological and economic sciences. This is due to the fact that in those sciences, recent research has focused on the emergence of aggregates like economic institutions, migrations, patterns of behavior, etc., as a result of the outcome of dynamical interaction of many individual agent decisions [2, 42, 62, 55]. The differences of this approach with traditional studies are twofold: firstly, the fact that collective recognizable patterns are due to *the interaction of many individuals*. On the other hand, in the traditional neoclassical picture, agents are assumed to be hyperrational and have infinite information about other agents' intentions. Under these circumstances, individuals jump directly into the steady state. However individuals are far from being hyperrational and their behavior often changes over time. Thus, new models consider explicitly *the dynamical approach towards the steady state* through evolution, adaptation and/or learning of individuals. The dynamical nature of the problem poses new questions like whether individuals are able to reach a steady state and under which circumstances this steady state is stable.

The problem is thus very appealing to statistical mechanics researchers, since it is the study of many interacting degrees of freedom for which powerful tools and intuitions had been developed. However, the dynamics of the system might not be relaxing and the typical energy landscape in which steady states are identified with minima of a Lyapunov function so that the dynamical process executes a march which terminates at the bottom of one of the valleys could not be applicable. The situation is reminiscent of other areas at the borderline of statistical mechanics like Neural Networks [1] where sometimes is not possible to build up a Lyapunov function which the dynamics tend to minimize. In fact, as we will see below there are strong analogies of the model considered here with that area of research.

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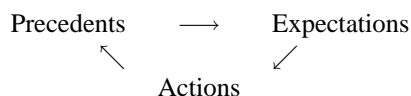
The model I consider here is called the Minority Game (MG) [12] which is the mathematical formulation of “El Farol Bar” problem considered by Brian Arthur [3]. The idea behind this problem is the study of how many individuals may reach a collective solution to a problem under adaptation of each one’s expectations about the future. As the models mentioned before, the MG is a dynamical system of many interacting degrees of freedom. However, the MG includes two new features which make it different: the first one is the minority rule, which makes a complete steady state in the community impossible. Thus, dust is never settled since individuals keep changing and adapting in quest of a non-existing equilibrium. The second ingredient is that the collectivity of individuals is heterogeneous, since individuals have different ways to tackle available information about the game and convert it into expectations about future. Effectively, the minority rule and heterogeneity translate into mean-field interaction, frustration and quenched disorder in the model, ideas which are somehow familiar to disordered systems in condensed matter [57, 61]. Actually, the MG is related to those systems and some of the techniques of disordered systems can be applied to understand its behavior.

At this point, I hope the reader is convinced that the MG is an interesting problem from the statistical mechanics point of view: is a complex dynamical disordered system which can be understood with techniques from statistical physics. In fact, most of the research about the MG problem has been done inside the physics community. However, the El Farol bar problem originates in the economic literature, although it represents a fierce assault on the conventions of standard economics. In this sense either the El Farol bar or the MG are interesting from the economic and game theory point of view as well. My intention here is to present the statistical mechanics approach to this problem and to explain how different concepts of economists and game theory translate into physics terminology. This biased review is the author’s effort to explain to physicists what is known about the MG. I make no attempt to be encyclopedic or chronologically historical.

## 2 The El Farol Bar problem

EL FAROL, 808 Canyon Rd., Santa Fe: El Farol restaurant is where locals like to hang out. The roomy, somewhat ramshackle bar area has an old Spanish-western atmosphere, but you can order some fine Spanish brandies and sherries. On most nights you can listen to live flamenco, country, folk, or blues. Dancers pack the floor on weekend nights in summer.  
*Fodor’s miniguide*

The El Farol Bar problem was posed as an example of inductive reasoning in scenarios of bounded rationality [3]. The type of rationality which is usually assumed in economics –perfect, logical, deductive rationality – is extremely useful in generating solutions to theoretical problems [25], but it fails to account for situations in which our rationality is bounded (because agents can not cope with the complexity of the situation) or when ignorance about other agents ability and willingness to apply perfect rationality lead to subjective beliefs about the situation. Even in those situations, agents are not completely irrational: they adjust their behavior based on the what they think other agents are going to do, and these expectations are generated endogenously by information about what other agents have done in the past. On the basis of these expectations, the agent takes an action, which in turn becomes a precedent that influences the behavior of future agents. This creates a feedback loop of the following type:



Inductive reasoning assumes that by feeding back the information about the game outcome, agents could eventually reach perfect knowledge about the game and arrive to an steady state. On the contrary, deductive reasoning (which is commonly applied in economics, for example), assume that the precedents

contain full information about the game and then there is not any dynamical approach to the steady state, which is attained in a single step:

Precedents (full information)  $\longrightarrow$  Actions (steady state)

The El Farol bar problem is posed in the following way:  $N$  people decide independently each week whether to go to a bar that offers entertainment on a certain night. Space is limited, and the evening is enjoyable if things are not too crowded –specifically, if fewer than  $aN$  with  $a < 1$  of the possible  $N$  are present. In other case, they stay at home. Thus agents have two actions: *go* if they expect the attendance to be less than  $aN$  people or *stay at home* if they expect it will be overcrowded. There is no collusion or prior communication among the agents; the only information available is the numbers who came in past weeks. Note that there is no deductively rational solution to this problem, since given only the numbers attending in the recent past, a large number of expectational models might be reasonable. Thus, not knowing which model other agents might choose, a reference agent cannot choose his in a well-defined way. On the other hand, expectations are forced to differ: if all believe most will go, nobody will go, invalidating that belief. In order to advance the attendance next week each agent is given a fixed number of *predictors* which map the past weeks’ attendance figures into next week. The idea behind those predictors is that when agents face complex scenarios like this they tend to look for patterns that occurred in the past and predict next move from that experience. Finally, each agent monitors his predictors by keeping an internal score of them which is updated each week by giving points or not to all of them whether they predicted the attendance or not. At each week, each agent uses his predictor with bigger score. Computer simulations of this model [3] shows that the attendance fluctuates around  $aN$  in a  $(aN, (1 - a)N)$  structure of people attending/not attending. Thus, predictors self-organize so that this structure emerges in the complex dynamics of the system.

Despite the fact that the El Farol bar problem deals apparently with a non-market context it can be considered as a kind of very simple “proto-market” or “market toy model” [45, 48, 63]: at each time step agents can buy (go to the bar) or sell an asset. After each time step, the price of the asset is determined by a simple supply-demand rule: if there are more buyers than sellers, the price is high, and if there are more sellers than buyers, the price is low. If the price is high, sellers do well, while if the price is low, buyers win the round. Thus, the minority group always win, irrespectively of whether they were buyers or sellers. However, some of the behavioral assumptions on which the El Farol bar problem (and the MG model) is based may be questionable when applied to financial markets [16]. Moreover, it still lacks some of the most basic features in a real market, e.g. the price which is determined by the aggregate supply-demand rule is not considered or the fact that agents do have different payoffs for their actions is not taken into account [63]. The relation to the markets is then on the conceptual level, as markets are interpreted as an adaptative competitive system in which minority membership plays an important role. Many other complex systems are driven by this minority rule, like vehicular traffic on roads, in which each agent would prefer to be on an uncongested road [50], packet traffic in networks, in which each packet will travel faster through lesser used routers [36] and ecologies of animals looking for food, in which individuals do best when they can find areas with few competitors [10, 23].

From the statistical mechanics point of view, we can say that in the El Farol bar problem, a system with  $N$  degrees of freedom has finally come to a stationary state in which the attendance fluctuates around  $aN$ . Some of the properties of the El Farol bar problem are very appealing to statistical mechanics, like:

- Many ( $N \gg 1$ ) interacting agents.
- Interaction is through the aggregate bar attendance, i.e. of the mean field type.
- The system of  $N$  agents is frustrated, since there is not a unique winning strategy in the problem.

- Quenched disorder, since agents use different predictors to generate expectations about the future of the game.

However, other features of the model are not usual in statistical mechanics, like the non-Markovian time dynamics induced by the fact that the predictors map past attendance into next week prediction which is released back to the agents. Moreover, the dynamics based on inductive reasoning may not be relaxing and, in principle, the steady state which the game attains could be out of equilibrium: there is no reason *a priori* to expect detailed balance or ergodicity in the system dynamics. At this stage it is not clear which of the main ingredients outlined before is relevant for the observed behavior. Thus, it is necessary first to give a precise mathematical definition of the model and try to clear up what are the ingredients truly responsible for the model main features.

### 3 The Minority Game

Truth always rests with the minority, and the minority is always stronger than the majority, because the minority is generally formed by those who really have an opinion, while the strength of a majority is illusory, formed by the gangs who have no opinion—and who, therefore, in the next instant (when it is evident that the minority is the stronger) assume its opinion . . . while Truth again reverts to a new minority.  
*Soren Kierkegaard*

In order to rationalize the discussion, Challet and Zhang [12] gave a precise mathematical definition for the El Farol bar problem which they called the Minority Game (MG). In the MG the predictors are replaced by well defined strategies from a given state space and the available information used by the agents is accurately defined.

In their definition of the MG model, the game consists of  $N$  (odd) agents playing as follows (see figure 1): at each time step of the game, each of the  $N$  agents takes an action  $a_i(t)$  with  $i = 1, \dots, N$ : he decides either to go to the bar ( $a_i(t) = 1$ ) or to stay at home ( $a_i(t) = -1$ ). The agents who take the minority action win, whereas the majority loses. After a round, the total action is calculated

$$A(t) = \sum_{i=1}^N a_i(t). \quad (1)$$

The minority rule sets the comfort level at  $A(t) = 0$ , so that agent is given a payoff  $-a_i(t)g[A(t)]$  at each time step with  $g$  an odd function of  $A(t)$ . Challet and Zhang's initial choice was  $g(x) = \text{sign}(x)$ , but other analytical functions are more suitable for mathematical treatment like  $g(x) = x/N$ . Most of the MG properties are qualitatively independent of the precise analytical form of  $g(x)$  [44]. The information about the winning group is released to the agents which is given by  $W(t+1) = \text{sign}A(t)$ .

The way agents choose their action  $a_i(t)$  is by inductive reasoning: we assume that the players have limited analyzing power and they can only retain the last  $m$  winning groups. Moreover they base their decision  $a_i(t)$  on these last  $m$  bits only. To this end they have a set of  $s \geq 2$  strategies. A strategy is just a mapping from the sequence of the last  $m$  winning groups to the action of the agent, i.e. a table which tells the agent what to do as a function of the input signal (the last  $m$  winning groups). An instance of one of those strategies is given in figure 1.

Since there are  $2^m$  possible inputs for each strategy, the total number of possible strategies for a given  $m$  is  $2^{2^m}$ . At the beginning of the game each agent is given a set of  $s$  strategies randomly drawn from the total  $2^{2^m}$  possible strategies. This assignment is different for each agent and thus, agents may or may not share the same set of strategies. Note that each strategy is determined by a  $2^m$  dimensional vector  $\vec{r}_i^\alpha$  whose components are the output of strategy  $\alpha$  of agent  $i$ . For example, in figure 1 the strategy is  $\vec{r}_i^\alpha = (-1, -1, +1, -1, -1, +1, -1, +1)$ . If the last winning groups were  $-1, +1, +1$ , then the prediction for this strategy is given by the fourth component of  $\vec{r}^\alpha$ . Thus, at each time step,

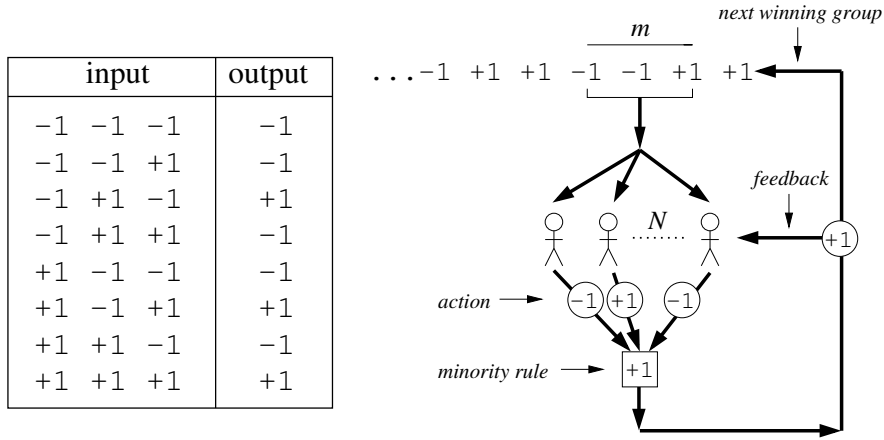


Figure 1: *Left*: An example of an strategy for  $m = 3$ . *Right*: Cartoon of the MG model for a given time step: in this case the strategy maps the last three winning groups ( $m = 3$ ) into the agent decision. Solid thick lines mimic how the information flows in the system: the  $N$  agents take the last  $m$  numbers ( $-1, -1, +1$  in this case) from the sequence of winning groups and perform an action accordingly. The  $N$  actions are transformed into the next winning group ( $+1$  in this case) through the minority rule. This information is shared with the agents for their own feedback [Eq. (2)] and becomes the next number in the sequence of winning groups.

the prediction of a strategy is given by its  $\mu(t) \in \{1, \dots, 2^m\}$  component, where  $\mu(t)$  is a number whose binary representation correspond to the last winning groups<sup>1</sup>. If we denote  $\vec{I}(t)$  the vector whose components are zero excepting the  $\mu(t)$  component which is one, then the prediction of strategy  $\vec{r}_i^\alpha$  is given by  $\vec{r}_i^\alpha \cdot \vec{I}(t)$ . For example, if the last three winning groups were  $-1, +1, +1$  then  $\mu(t) = 4$  and  $\vec{I}(t) = (0, 0, 0, 1, 0, 0, 0)$ . Thus, strategies are all possible  $2^m$ -dimensional vectors with  $\pm 1$  components.

Adaptation comes in the way agents choose at each time step one of their  $s$  strategies: they take the strategy within their own set of strategies whose performance over time to predict the next winning group is biggest. In order to do that each agent  $i$  assigns virtual points  $p_i^\alpha(t)$  to his strategy  $\alpha$  after each time step  $t$  when they predicted correctly the winning group:

$$p_i^\alpha(t+1) = p_i^\alpha(t) - \vec{r}_i^\alpha \cdot \vec{I}(t) g[A(t)] \quad (2)$$

where  $\alpha = 1, \dots, s$  and  $i = 1, \dots, N$ . However these points are only virtual points as they record only agents' strategies performance and serve only to rank strategies within each agent set. After time  $t$  agent  $i$  takes the first strategy in his personal ranking which tells him what to do in the future. If we denote agent  $i$  best strategy at time  $t$  in his ranking as  $\beta_i(t) \in \{1, \dots, s\}$ ,<sup>2</sup> then his action at time  $t$  is given by:

$$a_i(t) = \vec{r}_i^{\beta_i(t)} \cdot \vec{I}(t). \quad (3)$$

The fact that this personal ranking can change over time makes the agents adaptative: the ranking of each agent's strategies can change over time and then  $\beta_i(t)$  could be different at different times.

<sup>1</sup>In the binary representation of  $\mu(t)$  we make the correspondence  $-1 \leftrightarrow 0$  and  $+1 \leftrightarrow 1$ . Thus, if the last winning groups were  $-1, +1, +1$  the binary representation of  $\mu(t)$  is 011 and  $\mu(t) = 4$ .

<sup>2</sup>When two strategies have the highest number of points, the best strategy is chosen by coin tossing.

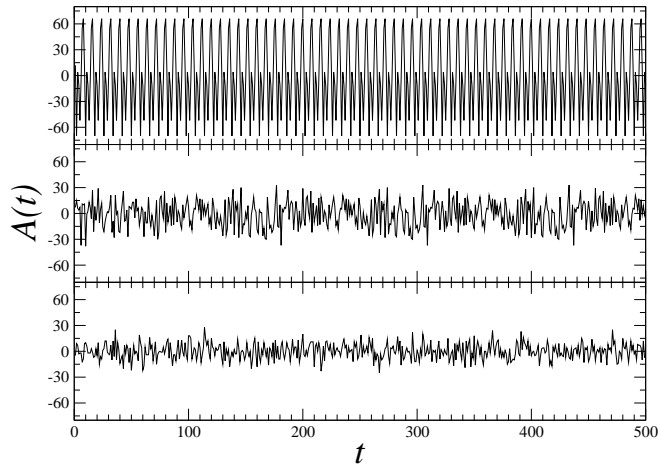


Figure 2: Time evolution of the attendance for the original MG with  $g(x) = x$  and  $N = 301$  and  $s = 2$ . Panels correspond to  $m = 2, 7$  and  $m = 15$  from top to bottom. Periodic patterns can be observed for  $m = 2$  and  $m = 7$ .

Finally, the information  $\vec{I}(t)$  is updated by adding the last winning group: the only nonzero component of  $\vec{I}(t+1)$  is the  $\mu(t+1)$  one which is given by [14]

$$\mu(t+1) = [2\mu(t) + (W(t+1) - 1)/2] \bmod P, \quad (4)$$

where  $W(t+1) = \text{sign}A(t)$  is the winning group.

Equations (2) and (3) together with the implicit ranking of each agent's strategies define the process of adaptation. Note that the minority rule is encoded in the function  $g(x)$  of the attendance and appears in the way public information  $\vec{I}(t)$  is built and also when virtual points are given to the strategies. Finally, interaction between agents in eq. (2) is through the attendance  $A(t)$  which is of the mean-field type. The heterogeneity among agents shows up at the set of  $\vec{r}_i^\alpha$  which could be different for different agents.

An interesting point is to observe that the variables which define the state of the game at each time step are  $p_i^\alpha(t)$  together with the public information  $\vec{I}(t)$ . But, on the other hand, eq. (4) and (2) introduce a non trivial feedback in the dynamics since the state of the system at time  $t$  depends on the last  $m$  winning groups. This is one of the characteristic features of the MG, since the dynamics given by eqs. (2)-(4) is non-local in time.

## 4 Coordination due to adaptation

It is not worth an intelligent man's time to be in the majority. By definition, there are already enough people to do that.  
*Godfrey Harold Hardy*

Initial studies of the MG model relied in simulations [9, 12, 13, 45, 38]. Typical simulations of the MG in its original formulation  $g(x) = \text{sign}(x)$  are given in figure 2. As we mentioned in the introduction, the aggregate  $A(t)$  never settles down and it fluctuates around the comfort level,  $A(t) = 0$ , as observed by Arthur in his paper [3]. Thus we have  $\langle A(t) \rangle = 0$  where  $\langle \cdot \cdot \cdot \rangle$  is a time average for long times and  $\overline{\cdot \cdot \cdot}$  is an average over possible realizations of  $\vec{r}_i^\alpha$ . Despite its trivial mean value the possible

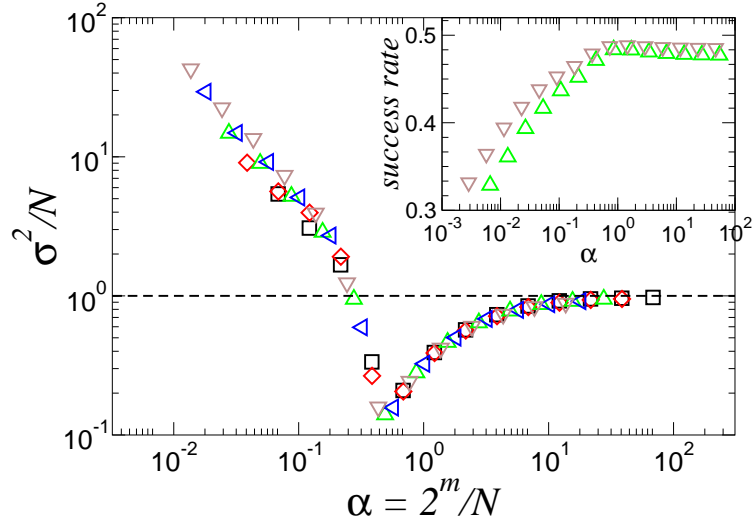


Figure 3: Volatility as a function of the control parameter  $\alpha = 2^m/N$  for  $s = 2$  and different number of agents  $N = 101, 201, 301, 501, 701$  ( $\square, \diamond, \triangle, \triangleleft, \triangleright$ , respectively). Inset: Agent's mean success rate as function of  $\alpha$ .

values of  $A(t)$  display a nontrivial shape and the fluctuations are important [9]. Moreover for small values of  $m$  the attendance display time periodic patterns which are lost for large values of  $m$ .

#### 4.1 Volatility

While the behavior of  $\overline{\langle A(t) \rangle}$  is somehow trivial, fluctuations of  $A(t)$  around its mean value given by the variance  $\sigma^2 = \overline{\langle [A(t) - \langle A(t) \rangle]^2 \rangle}$  have a more interesting behavior (see figure 3). First note that  $\sigma$  is related to the typical size of the losing group, so the smaller  $\sigma$ , the more winners are in the game. The variance  $\sigma^2$  is usually known as the volatility or (the inverse of) global efficiency. The behavior of  $\sigma^2$  as a function of the parameters of the model  $m$ ,  $s$  and  $N$  shows a quite remarkable behavior [45, 53]:

- It was found by extensive simulations that  $\sigma^2/N$  is only a function of  $\alpha = 2^m/N$  for each value of  $s$  (see figure 3). This finding not only identifies the control parameter in this model,  $\alpha$ , but also paves the way for the application of tools of statistical mechanics in the thermodynamic limit  $N \rightarrow \infty$ . Since qualitative results are independent of  $s \geq 2$  we take the simplest case  $s = 2$  for the rest of the chapter.
- For large values of  $\alpha$ ,  $\sigma^2/N$  approaches the value for the random choice game  $\sigma^2/N = 1$ , i.e., the game in which each agent randomly chooses  $a_i(t) = -1$  or  $a_i(t) = 1$  independently and with equal probability at each time step.
- At low values of  $\alpha$ , the average value of  $\sigma^2$  is very large, actually, it scales like  $\sigma^2/N \sim \alpha^{-1}$  which means that  $\sigma \sim N$  and thus the size of the losing group is much larger than  $N/2$ .
- At intermediate values of  $\alpha$ , the volatility  $\sigma$  is less than the random case, and it attains a minimum value at  $\alpha \simeq 1/2$ . In this region, the size of the losing group is close to  $N/2$  (which is the minimum possible size for the losing group).

The fact that  $\sigma$  gets below the random case for a given interval of values of  $\alpha$  suggests the possibility that agents coordinate in order to reach a state in which less resources are globally wasted [53]. In the market formalism, this means that agents can exploit information available and predict future market movements, so that global efficiency  $\sigma^2$  is minimized. In this sense *adaptation* is a good mechanism for the agents to achieve a better solution to the MG problem for  $\alpha = \alpha_c \simeq 1/2$ . The solution comes through coordination among agents, which is contrary to their selfish intrinsic nature. Global coordination as the better solution to a problem of competition for resources is common in different games and situations [25].

Note however that in the MG coordination is not complete and the best possible solution is not achieved: that in which agents alternate in groups of  $(N - 1)/2$  and  $(N + 1)/2$  which yields  $\sigma^2/N = 1/N$ . In this situation the mean success rate, i.e. the frequency an agent is successful in joining the minority is  $1/2$ . As shown in figure 3 the success rate in the MG is close to  $1/2$  around the minimum of  $\sigma^2/N$ , while it is substantially smaller in the region where the volatility is high.

## 4.2 Information

The fact that agents seem to exploit the available information led some authors to study the information contained in the time series of  $A(t)$ . Specifically, it was found that  $W(t + 1) = \text{sign}A(t)$  is independent of sequence of the last  $m$  attendances in the high volatility region, while there is a strong dependence for  $\alpha > \alpha_c$ . From the market point of view, the region where  $\alpha < \alpha_c$  is called efficient (although is socially inefficient), since the history of minority groups contains no predictive information for agents with memory  $m$ . On the contrary, in the region where  $\alpha > \alpha_c$  there is significant information available to the strategies of the agents playing the game with memory  $m$  and the market is not efficient in this sense, since there are arbitrage possibilities<sup>3</sup>. But in either case  $A(t)$  is not a random sequence (see figure 2) [9, 45]. To quantify this behavior, it was proposed in [45] to measure the conditional probability of  $W(t + 1)$  knowing  $\mu(t)$  through the mutual entropic information of  $W(t)$  and  $W(t + 1)$ . Another more direct measure of this quantity was suggested in [14] as

$$H = \frac{1}{2^m} \sum_{\nu=1}^{2^m} \langle W(t + 1) | \mu(t) = \nu \rangle^2 \quad (5)$$

where the average of  $W(t + 1)$  in (5) is conditioned to the requirement that the last  $m$  winning groups are given by  $\mu(t)$ . If there is no significant dependence between  $W(t + 1)$  and  $\mu(t)$  then as we have already seen  $\langle W(t + 1) \rangle = \langle \text{sign}A(t) \rangle = 0$  and  $H = 0$ . Loosely speaking  $H$  measures the information in the time series of  $A(t)$  available to the agents. In the market context,  $H \neq 0$  indicates the presence of information or arbitrages in the signal  $A(t)$ . As seen in figure 4,  $H = 0$  for  $\alpha < \alpha_c \simeq 0.3$  while  $H \neq 0$  for  $\alpha > \alpha_c$ . When  $\alpha \rightarrow \infty$  we recover  $H = 0$  since in that case the system of  $N$  agents behave almost randomly and we expect no correlations in the sequence of  $W(t)$ .

Despite the fact that  $H$  measures the information content in the sequence of  $A(t)$  it does it only through the correlation of the next winning group  $W(t + 1)$  with the last  $m$  winning groups. Moreover,  $H$  also measures the asymmetry of the *response* of the agents to the information available to them: when  $H \neq 0$  the system of  $N$  agents reacts differently to the available information, while when  $H = 0$  all given information is treated equally by the agents. Thus,  $H$  has a twofold role in the game as a measure of the information available in the sequence of  $A(t)$  and the response of the system to a given piece of information  $\mu(t)$ .

<sup>3</sup>The existence of arbitrage opportunities in a market implies that there exist price discrepancies of the same stock which can be exploited by agents to make profit by selling and buying them simultaneously. The market is efficient whenever these arbitrage opportunities are exploited instantaneously and therefore eliminated from the market.



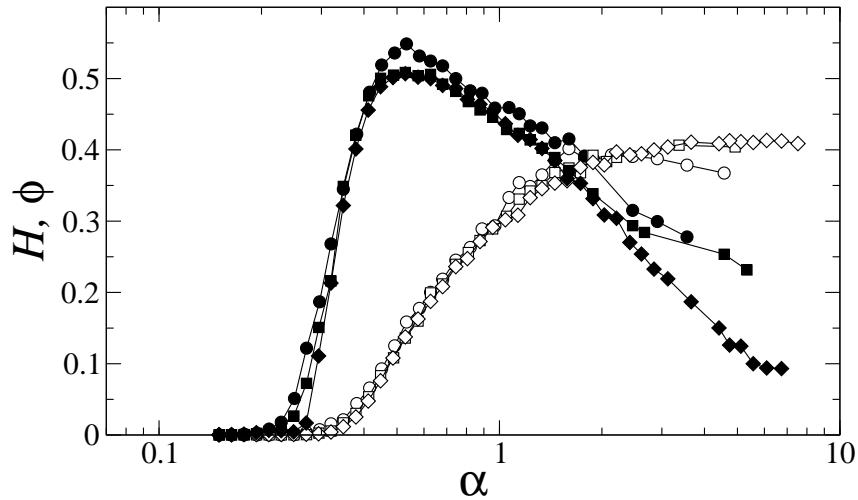


Figure 4: Information  $H$  (open symbols) and fraction of frozen agents  $\phi$  (full symbols) as a function of the control parameter  $\alpha = 2^m/N$  for  $s = 2$  and  $m = 5, 6, 7$  (circles, squares and diamonds respectively).

### 4.3 Phase transition

The fact that  $H = 0$  for  $\alpha < \alpha_c$  and  $H \neq 0$  for  $\alpha > \alpha_c \simeq 0.3$  suggests the possibility that there is a phase transition at  $\alpha = \alpha_c$  which separates those two efficient and inefficient phases [14]. Thus  $H$  is the order parameter of the system which measures the symmetry of  $W(t+1)$  in the system, a symmetry which is broken at  $\alpha = \alpha_c$ . Since the phase transition occurs at  $\alpha = \alpha_c$ , it means that for a fixed value of  $m$ , the agents who enter the market exploit the  $A(t)$  predictability and hence reduce it. From the point of view of the community,  $\alpha_c$  separates two regions in which adaptation is successful in achieving a good global solution to the MG problem. The following table summarizes the different names for the phases given in different contexts:

Context	$\alpha < \alpha_c$	$\alpha > \alpha_c$
Volatility/ Global Waste	Inefficient (Worse than random)	Efficient (Better than random)
Information/ Arbitrage in $A(t)$	Efficient (no information, $H = 0$ )	Inefficient (arbitrage, $H \neq 0$ ).

Note that the phase transition is both explained in terms of the information which is available to the agents and the minimization of the total waste, while only  $H$  is an order parameter. Other quantities have been identified as order parameters to explain the phase transition found numerically at  $\alpha = \alpha_c$ . For example in [14] it was considered the fraction of agents  $\phi$  who always use the same strategy, i.e. those agents who have  $\beta_i(t) = \beta_i$ , independently of time. As we see in figure 4  $\phi$  is zero in the  $\alpha < \alpha_c$  phase which means that all the agents change their strategy at least once. This is also the case for large values of  $\alpha$ , since agents behave more or less randomly. However close to the transition point the fraction  $\phi$  is large and most of the agents stick to one of their strategies forever. In terms of  $\phi$  the phase transition can be understood as an “unbinding” transition, since for  $\alpha < \alpha_c$  strategy points  $p_i^\alpha(t)$  remain very close to

each other and then agents can switch easily from one strategy to the other. However, for  $\alpha > \alpha_c$  one of each agent's strategies wins in the long run and its virtual score  $p_i^\alpha(t)$  is much larger than the others.

Finally, while there is no significant finite size corrections to the order parameter  $H$ , the fraction of frozen agents  $\phi$  seems to drop discontinuously to zero at  $\alpha_c$  when the number of agents is changed. Despite both order parameters indicate that the system undergoes a phase transition at  $\alpha = \alpha_c$  it is not yet clear which is the nature of the phase transition (first, second order?) and whether it is an equilibrium or non-equilibrium phase transition.

## 5 Spanning the strategy space

Nobody goes there anymore. It's too crowded.  
Yogi Berra 1925-, American Baseball Player

The universality of the behavior found in the last section for different number of agents, strategies or memory, suggests that it should depend on a generic feature of the game. Due to the minority rule, agents tend to differentiate: if they use the same strategy then all of them will lose. Thus, somehow the minority rule forces the agents to choose that strategy among theirs that makes them singular in the community. This translates into a repulsion of agents chosen strategy inside the strategy space. But the set of available strategies only contains  $2^{2^m}$  strategies. Thus one should expect that agents are very different between them if the number of strategies is bigger than  $N$ , while they should behave more like a crowd if their strategy heterogeneity is small. In other words we should observe a change in the system behavior when  $N \sim 2^{2^m}$ . However as we saw in the last section this change of behavior is observed at  $N \sim 2^m$ , i.e.  $\alpha = \mathcal{O}(1)$ .

As note by various authors, while  $2^{2^m}$  is the number of possible strategies,  $2^m$  is the dimension of the space in which strategies live and thus there are only  $2^m$  completely different strategies [13, 32, 38]<sup>4</sup>. If agents tend to differentiate they can only do it by choosing those completely different strategies and this is only possible if  $N < 2^m$ . For  $N > 2^m$  groups of agents with the same set of similar strategies emerge. Those agents end up using their same best strategy and become a *crowd*, since they react to the available information in the same way. The different behavior of  $\sigma^2/N$  can be understood in terms of this crowds [32, 38]:

- When  $N < 2^m$ , agents carry a considerable fraction of all possible strategies. This means that for a large fraction of agents their best strategy  $\beta_i(t)$  is the same at time  $t$ . Then the size of this crowd is of the order of  $\mathcal{O}(N)$ , which means that the variance is of order  $\sigma^2/N \sim N$ .
- On the other hand, when  $N \ll 2^m$  the crowds will be low populated since the  $\beta_i(t)$  are almost different for any agent due to the high heterogeneity, which makes agents act independently. Thus,  $\sigma^2/N \sim 1$ .
- Finally, for moderate values of  $\alpha$  crowds form which share  $\beta_i(t)$  and with size of order of  $N$ . But at the same time there will be an *anti-crowd* which is a group of people which have the same best strategy but which is completely different to the one the crowd is using. Note that only at moderate values of  $\alpha$  this is possible, since there is a nonzero probability that two groups of agents cannot share any of their strategies. In this case crowds and their anticrowds do completely the opposite: considering the extreme case in which the size of the crowd is  $N/2$  we have that  $\sigma^2/N \simeq 0$ .

More quantitative approximations to the size of the crowds and anticrowds are possible which reproduce the observed volatility behavior [38]. Another way to characterize the formation of crowds and/or

<sup>4</sup>Two strategies are said to be completely different if none of their components match. A quantitative measure of this is the Hamming distance, see Eq. (6).

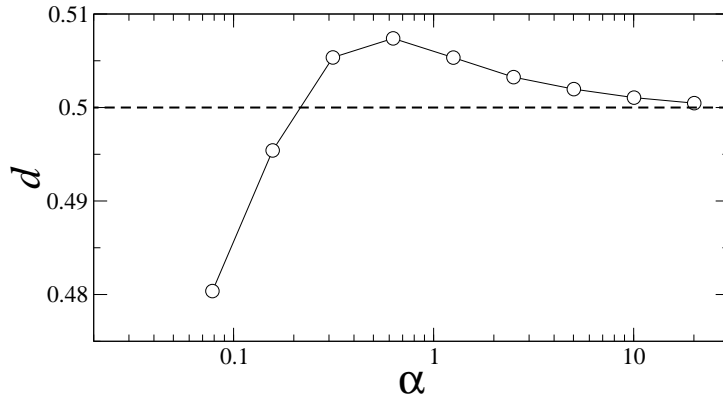


Figure 5: Distance of agents' best strategies as a function of  $\alpha$ . Parameters used are  $N = 51$ ,  $s = 2$

anticrowds is to measure the distance between the best strategies of all the agents [13]

$$d = \frac{1}{2^m(N-1)^2} \sum_{i \neq j} \langle \|\vec{r}_i^{\beta_i(t)} - \vec{r}_j^{\beta_j(t)}\|_1 \rangle \quad (6)$$

where  $\|\cdot\|_1$  is the Hamming distance (1-norm) in the  $2^m$ -dimensional space of the strategies. As seen in figure 5, agents tend to differentiate between them and thus maximizing the distances between their best strategies. However, this is not possible for small values of  $\alpha$ , since then agents are forced by their limited memory to share their strategies. The situation is better for large  $\alpha$ , since agents' set of strategies do not overlap and then the distance between them is random  $d \simeq 1/2$  [37]. However, for moderate values of  $\alpha$  differentiation makes the system of agents to organize in groups with completely different best strategies. In the case the crowds and anticrowds are of size  $(N-1)/2$  and  $(N+1)/2$  respectively, and knowing that the distance between two completely different strategies is  $2^m$  we get  $d = N/[2(N-1)]$  which is bigger than the random case.

In summary, the observed behavior in the MG is the result of the minority rule in the strategy space: the minority rule imposes a repulsion of agents' chosen strategies at each time. However, the volume of the strategy space is fixed by  $m$  and thus repulsion is only effective when the dimension  $2^m$  is much bigger than the number of strategies present in the game. Note that at intermediate values of  $\alpha$ , repulsion of best strategies leads to coordination, since the optimal way to achieve differentiation is by joining an existing crowd or anticrowd.

## 6 Inventing the future

The best way to predict the future is to invent it.  
Alan Kay

One of the main characteristics of the MG model is that agents adapt and learn from their past history to achieve a better personal record. As we saw in the previous section, the memory of the agents is crucial, because it determines the strategy space dimension and also because the memory is associated with the ability of agents to spot patterns in the available information so they can coordinate better. Somehow, the sequence of the winning groups contains information about the strategies of all the agents and having more memory gives an agent the possibility to exploit this information. For example,

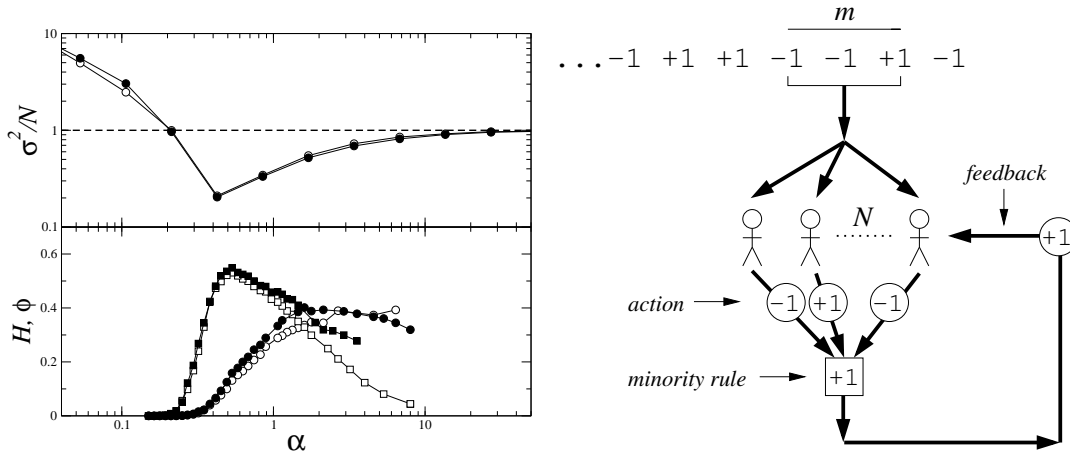


Figure 6: *Left*: Comparison of the model variance  $\sigma^2$ , information  $H$  (squares) and  $\phi$  (circles) with endogenous information (open symbols) and exogenous information (full symbols). *Right*: Cartoon of the MG model using exogenous information. As in figure 1 solid thick lines mimic the flow of information in the system. In this case the winning group obtained through the minority rule is only used by the agents to rank their strategies and may differ from the one in the exogenous sequence of winning groups (as in the figure).

for a fixed number of agents  $N$ , increasing their memory  $m$  could lead to better coordination and higher individual success. In the limit  $\alpha \gg 1$  however, more memory implies increasing complexity and random behavior.

Since information is fed back into the system, early studies of the MG concentrated in the possibility that the system of  $N$  agents could exploit this information to achieve better coordination. However, A. Cavagna in [5] showed by means of extensive simulations that if the information  $\mu(t)$  which is given to the agents at each time step is chosen randomly and independently of time from its possible values  $\mu(t) \in \{1, \dots, 2^m\}$  the behavior of the MG remains the same regarding time averaged extensive quantities like the volatility, the information  $H$  or  $\phi$  (see Fig. 6). The crucial point is that every agent reacts to the same piece of information whether this information is endogenous or exogenous, false or true. As a consequence, there is no room in this model for any kind of forecasting of the future based on the understanding of the past: coordination in the MG does not come through exploitation of available information in the sequence of the winning groups.

Leaving aside the fact that changing the real sequence of  $\mu(t)$  for random numbers gives the same quantitative behavior for different quantities, this finding has significant importance for the theoretical understanding of the MG model. Specifically, equation (4) drops from the definition of the model which is simply given by equations (2) and (3) together with the ranking of each agent's strategies at any time step. In this sense, this results are useful in the theoretical understanding of the MG model, since it shows that the complicated feedback of the endogenous information in the system is just an irrelevant complication and thus, can be dropped in the way to explain the rich structure of this adaptative model.

This fact does not mean that the memory parameter  $m$  is not relevant in the model: indeed, the model displays a phase transition at a given value of  $m$  for a fixed number of agents. However, from the results of A. Cavagna in [5] it is reasonable to conclude that the parameter  $m$  only relates to the dimension  $D = 2^m$  of the strategies space. Indeed, as we have just shown in section 5, the description of the phase transition observed in the MG can be just based on geometrical considerations about the

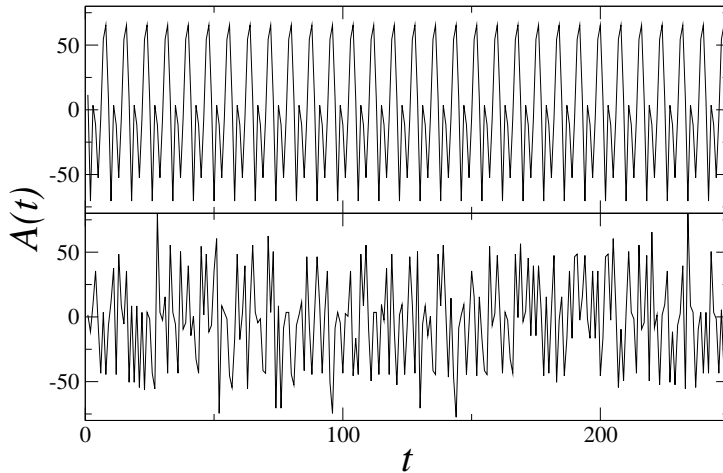


Figure 7: Comparison of the bar attendance in the symmetric phase with endogenous (upper panel) and exogenous information (lower panel). Parameters used are  $N = 301$ ,  $m = 2$ ,  $s = 2$ .

distribution of strategies in the space of strategies.

As we can see in figure 6 the order parameter  $H$  does not change substantially when exogenous information is considered instead of endogenous one. But, as we said in section 4,  $H$  is related to correlations in the sequence of winning groups. Since the sequence of the winning groups is exogenous and random we expect in  $H = 0$  for all values of  $\alpha$ . The fact that this is not true is due to the twofold role that  $H$  plays in the original MG:  $H$  is the response of the system to a given piece of information and does not depend on the origin of the information. But (see figure 6):

- When endogenous information is considered, the response of the system to the available information does show up in the sequence of winning groups.
- When exogenous information is considered, the response of the system is only considered to get the winning group and to keep record of the performance of the strategies and is not included in the sequence of the winning group.

In either case,  $H \neq 0$  means that for a given piece of information, there is a possibility for agents to predict the response of the system. In this sense, memory is not irrelevant in the MG (see the discussion about this in [5, 6, 54]).

Despite the fact that the phase transition and macroscopic properties of the MG are the same with exogenous information, some properties of the MG are lost. As we show in figure 2 the attendance and the size of the minority groups are correlated in time and display time periodic patterns for  $\alpha < \alpha_c$ . This is lost when the information agents react to is exogenous as seen in figure 7, although the typical size of the minority groups is the same and then  $\sigma^2/N$  is independent on the nature of the information used. Thus, any extension of the MG game which depends on the periodic nature of the bar attendance should give different results when exogenous information is considered [17, 43, 64].

The fact that the memory parameter  $m$  is only relevant to determine the dimension of the strategies space led to the generalization of equation (2) to consider general  $D$ -dimensional strategies and vectors  $\vec{I}(t)$  for any value of  $D$  and not only for those with  $D = 2^m$ , which is only an unnecessary complication [7].

## 7 Maximizing the audience

In section 4 we saw that agents adapt to achieve a personal better solution to the MG problem which in turn makes the total waste smaller than the random solution for  $\alpha > \alpha_c$ . Thus, although agents are selfish and tend to place themselves into the minority group irrespectively of other agent's action, their dynamics tend to maximize the global efficiency, that is, to maximize the bar attendance. The concept that the system is trying to minimize a given quantity is very appealing to statistical mechanics. If this quantity exists, the system can be studied by considering its minima and perturbations around them.

However, due to the minority rule, the MG never settles down: specifically, some of the agents keep on changing their strategies forever and nor  $\beta_i(t)$ , neither  $p_i^\alpha(t)$  come to a rest<sup>5</sup>. Thus, there can not be any quantity based on these degrees of freedom that the MG dynamics tend to minimize. However, since the actions of the agents  $a_i(t)$  depend on the points of the strategies and  $p_i^\alpha(t)$  depend on the past history of actions, one can wonder whether there is any time pattern in the long run that agents follow. Specifically, it might be that  $a_i(t)$  never come to a rest, but  $m_i(t) = \sum_{\tau=0}^t a_i(\tau)$  can converge when  $t \rightarrow \infty$  to a given quantity if agent  $i$  does follow any behavioral pattern. For example, agent  $i$  can be within the fraction  $\phi$  of people who always chooses one of his strategies and then  $m_i(t)$  is the average of the possible outcomes of that strategy for all possible realizations of the information. This is the key point in the first attempt at solution of the MG model found in [15, 47], which we outline in this section.

First of all, let us rewrite the equations of the MG at this stage. Since the observed behavior of the MG is independent of  $s \geq 2$  we chose the minimal case  $s = 2$ . Then introducing  $\sigma_i(t) = \text{sign}[\delta p_i(t)]$ , where  $\delta p_i(t) = p_i^1(t) - p_i^2(t)$  is the difference of points between agent  $i$  two strategies, equation (2) reads

$$\delta p_i(t+1) = \delta p_i(t) - \vec{\xi}_i \cdot \vec{I}(t) g[A(t)] \quad (7)$$

and

$$A(t) = \sum_{j=1}^N a_j(t) = \sum_{j=1}^N [\vec{\omega}_j + \sigma_j(t) \vec{\xi}_j] \cdot \vec{I}(t) \quad (8)$$

where  $\vec{\omega}_i = (\vec{r}_i^1 + \vec{r}_i^2)/2$  and  $\vec{\xi}_i = (\vec{r}_i^1 - \vec{r}_i^2)/2$ . With this notation,  $\beta_i(t) = (\sigma_i(t) + 3)/2$ . Another unnecessary complication is the binary payoff function  $g(x) = \text{sign}(x)$  for the strategies. Thus, we focus on the linear case  $g(x) = x/D$  which allows for a simple treatment. In this case we have

$$\delta p_i(t+1) = \delta p_i(t) - \frac{1}{D} \vec{\xi}_i \cdot \vec{I}(t) \left\{ \sum_{j=1}^N [\vec{\omega}_j + \sigma_j(t) \vec{\xi}_j] \cdot \vec{I}(t) \right\} \quad (9)$$

and one is allowed to neglect the fluctuations (law of large numbers)

In this form, the authors in [15, 47] argued that if the information is just a random number then in the long run each of the possible values of  $\vec{I}(t)$  is visited with equal probability. Thus, in the limit  $N \rightarrow \infty$  and  $D \rightarrow \infty$  and making a temporal coarse-graining over time steps  $\tau = t/D$  that contain infinite number of updatings like (7), the right hand side of (7) can be replace by its mean value over time [49]:

$$\delta p_i(t+\tau) = \delta p_i(t) - \vec{\xi}_i \cdot \left\{ \sum_{j=1}^N [\vec{\omega}_j + m_j(t) \vec{\xi}_j] \right\}, \quad (10)$$

where  $m_i(t) = \sum_{t'=t-\tau}^t \sigma_i(t')$  and we have neglected fluctuations in favor of mean values according to the law of large numbers. Finally, making the approximation that the points at time  $t$  are related to

<sup>5</sup>This can be observed in the order parameter  $\phi$ , which is never equal to one.

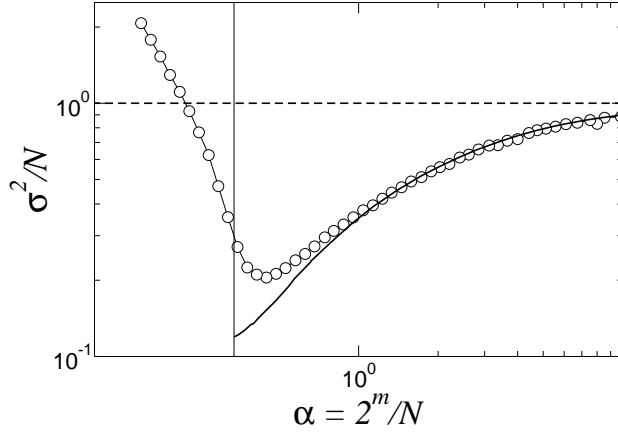


Figure 8: Comparison of the results for the volatility in the original MG (circles) with the solution obtained for equation (13) through the evaluation of the minima of  $\mathcal{H}$

$m_i(t)$  through the soft condition  $m_i(t) = \tanh[\Gamma \delta p_i(t)]$  (where  $\Gamma$  is a constant) we obtain the following dynamical equation in the continuum approximation  $\tau \rightarrow 0$

$$\frac{dm_i}{d\tau} = -2\Gamma(1 - m_i^2) \left[ \sum_{j=1}^N \vec{\omega}_j \cdot \xi_i + \sum_j \vec{\xi}_i \cdot \vec{\xi}_j m_j \right]. \quad (11)$$

This can be easily written as a gradient descent dynamics  $dm_i/d\tau = -\Gamma(1 - m_i^2)(\partial\mathcal{H}/\partial m_i)$  where

$$\mathcal{H}\{\{m_i\}\} = \frac{1}{2} \left[ \sum_{i=1}^N (\vec{\omega}_i + \vec{\xi}_i m_i) \right]^2. \quad (12)$$

Note that  $\mathcal{H}$  is a positive function of  $m_i$  and then  $\mathcal{H}$  is a Lyapunov function of the dynamics given by (11). Making the same type of temporal coarse-graining we get

$$\sigma^2 = \overline{\mathcal{H}} + \sum_{i=1}^N \overline{\xi_i^2} (1 - m_i^2). \quad (13)$$

This equations implies that the dynamics of the agents tends to minimize  $\mathcal{H}$  instead of  $\sigma^2$  and thus the properties of the MG are described by the ground states of  $\mathcal{H}$ . It is easy to see that  $\overline{\mathcal{H}}$  is related to the order parameter  $H$  at least when  $H$  is very small, i.e. close to the phase transition. Then, adaptive agents tend to minimize the arbitrage/information rather than their collective losses  $\sigma^2$ .

Taking into account the random character of  $\vec{\omega}_i$  and  $\vec{\xi}_i$ , the Lyapunov function  $\mathcal{H}$  is reminiscent of of the Sherrington-Kirkpatrick model [56] augmented by a random field. Ground state properties of  $\overline{\mathcal{H}}$  can be analyzed using standard tools from statistical mechanics of disordered systems. Averages over different realizations of the disorder  $\vec{\xi}_i$  and  $\vec{\omega}_i$  are obtained through the replica trick [15, 47]. Despite the approximations made to get to equations (11) and (12), the calculations showed that there is a phase transition at  $\alpha_c \simeq 0.33740$  and gave accurate predictions for the values of  $\sigma^2$  for  $\alpha > \alpha_c$  (see figure 8): in fact  $\overline{\mathcal{H}} = 0$  for  $\alpha \leq \alpha_c$  while  $\overline{\mathcal{H}} \neq 0$  when  $\alpha \geq \alpha_c$ . For  $\alpha \leq \alpha_c$  however, the solution given by exploring the ground state of  $\mathcal{H}$  has some mathematical problems and fails to reproduce the observed behavior.

The fact that the game for  $\alpha > \alpha_c$  is described by the minimization of  $\mathcal{H}$  shed some light also about the real nature of the MG: first of all, the dynamics tend to minimize a Lyapunov function which depends on the history of agents decisions instead of agents instantaneous decisions. And second, the function which agents tend to minimize has some disorder which is encoded in the “quenched” patterns  $\vec{\xi}_i$  and  $\vec{\omega}_i$ . This last feature of the MG resembles the dynamics of attractor neural networks (ANN) [1] where  $D$  patterns are stored through the Hebbian learning rule and the recurrent network retrieve the information stored in the neurons. The control parameter  $\alpha = D/N$  measures then the ratio between the number of stored patterns  $D$  and the number of neurons  $N$ . In general, information retrieval in ANN is possible only under some conditions about the dynamics and  $\alpha$ . This comparison is very appealing, since in principle agents in the MG would like to retrieve time patterns or other agents strategies to achieve a better personal record. Thus, is MG just an ANN of neurons trying to retrieve other agents’ strategies? We will complete this analogy in the following sections.

## 8 No news, good news

The irrelevance of the information contained in the sequence of the winning groups found in section 6 poses a new question: how can it be that changing the model from being a closed system with endogenous information to an open one with exogenous information does not change its aggregate behavior? As we saw, the reason for that is that the aggregate behavior of the MG is encoded in the response of the system to the input information. Averaging this response over all possible values of the information will give us the mean response of the system and then we can update all strategies’ points concurrently. For example, we found in section 7 that coarse-graining the dynamics of the MG over many time steps leads to a coupled set of equations for  $p_i^\alpha(t)$  in which the information  $\vec{I}(t)$  is averaged out. The information is then simply a go-between quantity that connects agents strategies dynamics and in principle it is disposable.

In order to get the averaged dynamics of the strategies for all possible values of  $\alpha$  and not just for  $\alpha > \alpha_c$  as in the previous section, several techniques were used [28, 29, 34, 49, 58]. Here we outline one of them, which is by means of the Kramers-Moyal expansion [27]. This is a general technique to obtain diffusion approximations of stochastic processes. Specifically, equations (9) define a stochastic process than can be approximated by a drift together with a diffusion in the space of strategies’ points. In this case, due to the fact that the random process  $\vec{I}(t)$  appears twice in equation (9), only the drift term remains non-zero in the Kramers-Moyal approximation which yields

$$\delta p_i(t+1) = \delta p_i(t) - \vec{\xi}_j \cdot \left\{ \sum_{j=1}^N \vec{\omega}_j + \sigma_j(t) \vec{\xi}_j \right\}. \quad (14)$$

Comparing equation (14) with (9) we observe that information has been averaged out in favor of an effective coupling between agents’ strategies, as we expected. Thus, rather than allowing the strategy payoff valuations to be changed at each round, only the accumulated effect on a large number of market decisions is used to change an agent’s strategy payoff valuations. We regard equation (14) as the equivalent of what in the neural network literature would be called the *batch* version of the conventional *on-line* MG [34].

Equations (14) are the simplest version of the dynamics to reproduce the observed behavior in the MG for all values of  $\alpha$ , as shown in figure 9. Since fluctuations due to the information are averaged out, there is a small quantitative difference in the aggregates (like  $\sigma^2$ ), but its main features are qualitatively



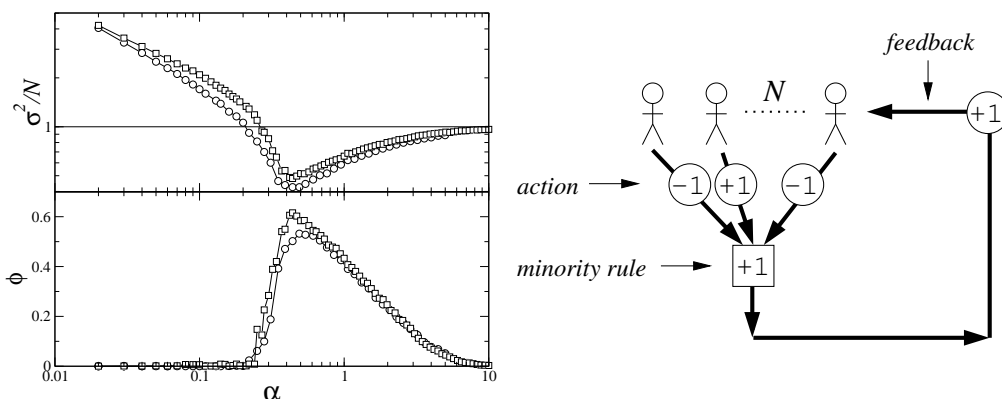


Figure 9: *Left*: Results of the MG using equations (14) (squares), compared with simulations of the original MG (circles). Parameters used are  $N = 100$  and  $s = 2$ . *Right*: Cartoon of the MG model after averaging out the information: feedback performed by the agents to keep track of their strategies performance provides an effective interaction in the system. As in Fig. 1, solid thick lines mimic the flow of information in the system.

conserved. Within this approximation, the volatility is given by

$$\sigma^2 = \bar{\Omega} + 2 \sum_{i=1}^N \overline{h_i \langle \sigma_i(t) \rangle} - \sum_{i,j=1}^N \overline{J_{ij} \langle \sigma_i(t) \sigma_j(t) \rangle} \quad (15)$$

where  $\Omega = \sum_{j=1}^N \vec{\omega}_i \cdot \vec{\omega}_j / D$ ,  $h_i = \sum_{j=1}^N \vec{\omega}_j \cdot \vec{\xi}_i / D$  and

$$J_{ij} = -\frac{1}{D} \vec{\xi}_i \cdot \vec{\xi}_j. \quad (16)$$

Although there are some quantitative differences between the original MG and the effective dynamics in the strategy space given by equations (9), the approximations given by (14) reconciliates the model with its endogenous nature. On the other hand, note that equations (14) look like equations (10) with the accumulated agent temporal pattern  $m_i(t)$  instead of the instantaneous  $\sigma_i(t)$ . In fact, equations (14) can be written in the continuum limit like

$$\frac{d(\delta p_i)}{dt} = -\frac{\partial \mathcal{H}[\{\sigma_i\}]}{\partial \sigma_i} \quad (17)$$

where  $\mathcal{H}[\{\sigma_i\}]$  is given by equation (12) with  $\sigma_i(t)$  instead of  $m_i(t)$ . This time,  $\mathcal{H}$  with  $\sigma$ 's instead of  $m$ 's is not a Lyapunov function of (14) in general, since the gradient is with respect to  $\sigma$ , not  $p$ . But, as we saw in section 7,  $\mathcal{H}$  is a Lyapunov function for  $\alpha > \alpha_c$  for  $m_i = \langle \sigma_i(t) \rangle$  [28, 58].

Since equations (14) are the simplest dynamics of the MG, we are in the position to single out the main features responsible for the behavior of the MG. To this end, let us rewrite equations (14) the following way:

$$\sigma_i(t+1) = \text{sign} \left[ \frac{\delta p_i(0)}{t} - \Omega_i + \sum_{j=1}^N J_{ij} x_j(t) \right] \quad (18)$$

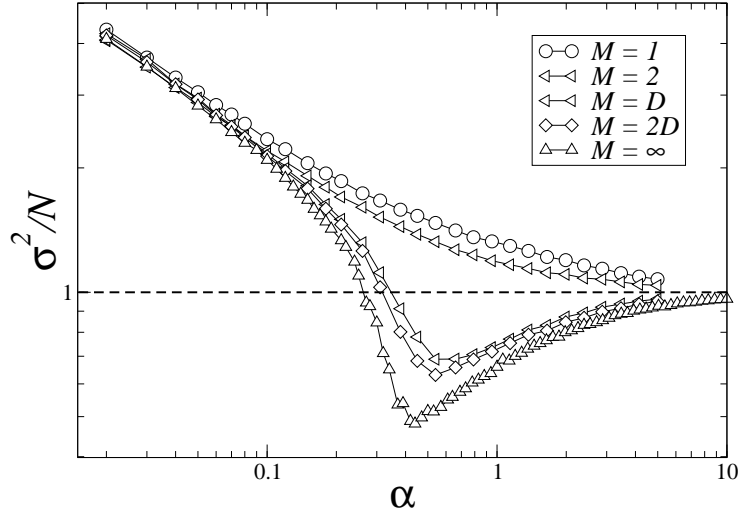


Figure 10: Volatility obtained through simulations of equations (19) for different values of  $M$  with  $N = 100$ .

where  $x_i(t) = \frac{1}{t} \sum_{\tau=0}^t \sigma(\tau)$  is the time average of agent  $i$  actions over time. If we assume that  $\delta p_i(0) = 0$ , equation (18) looks like an attractor neural network [1] (specifically a Hopfield model), where the patterns  $\vec{\xi}$  have been learned through the *anti-Hebbian* rule (16) instead of the usual Hebbian rule  $J_{ij} = \vec{\xi}_i \cdot \vec{\xi}_j$ , and each neuron is subject to external biases  $\Omega_i$ . In ANNs anti-Hebbian rule is studied in models of paramagnetic *unlearning* [35, 51], the process by which spurious states from the space of possible configurations of  $\sigma_i(t)$  are removed since they do not correspond to any stored pattern<sup>6</sup>; the opposite sign in the anti-Hebbian rule hinders rather than assists information retrieval in ANN. In the MG, the origin of the anti-Hebbian rule is different: it comes through the minority rule, which forces the behavior of agents in the long run to be different to any other agents behavior.

More generally we can look at systems like

$$\sigma_i(t+1) = \text{sign} \left[ -\Omega_i + \sum_{j=1}^N J_{ij} x_j(t) \right] \quad (19)$$

where  $x_i(t) = \frac{1}{M} \sum_{\tau=0}^M \sigma_i(t-\tau)$  is the time average of agent  $i$  behavior over the last  $M$  time steps. This kind of time delayed ANN were considered as possible candidates of models for storing and generating time sequences of patterns [41, 46]. In the MG, the time delayed dynamics is not set up to retrieve or create any particular time pattern but rather comes from the inductive reasoning process, which keeps record of the performance of the strategies along the game.

Thus, equations (19) define a new type of ANN whose main ingredients are:

- $D$  heterogeneous patterns are stored through the anti-Hebbian rule (16) due to the minority rule,
- each neuron reacts to an external field given by  $\Omega_i$ , and
- dynamics is time delayed over the last  $M$  steps because of the adaptation process,

<sup>6</sup>Some biologists have suggested that this unlearning procedure correspond to REM sleep, which is widely observed among mammals [35].

which are necessary to reproduce the observed behavior of the MG. Finally one may ask whether it is relevant to keep the whole historic performance of each strategy, i.e.,  $M \rightarrow \infty$  or just the last  $M < \infty$  time steps<sup>7</sup>. In figure 10 we see that if only the last  $M$  strategy performances are kept with  $M$  small, then the system never attains better result than the random case. Actually the cases  $M = 1, 2$  reproduces quite accurately the high volatility region  $\alpha \ll \alpha_c$  which means that in this phase, agents react only to the last winning group. For  $\alpha \geq \alpha_c$  keeping the record of strategies over large time periods  $M \gg D$  is the only way for the agents to achieve a better-than-random solution to the problem.

Finally, it is appealing to rewrite equations (18) like a learning process [1]. After all, inductive reasoning in the MG is the way agents learn how to act in the game. If the time dependence in (19) is considered as a learning process or training of a set of  $N$  interacting perceptrons, then the learning process is given by

$$\sigma_i(t+1) = \text{sign}[-\Omega_i + \sum \tilde{J}_{ij}(t)\sigma_j] \quad (20)$$

$$\tilde{J}_{ij}(t) = \frac{t}{t+1}\tilde{J}_{ij}\sigma_j(t+1)\sigma_j(t) + \frac{1}{t+1}J_{ij} \quad (21)$$

This type of interacting  $N$  perceptrons (one for each agent) have been considered recently in the context of the MG [40], although with a different learning rule for the perceptrons and with homogeneous agents (i.e.  $\vec{\xi}_i = \vec{\xi}_j, \vec{\omega}_i = \vec{\omega}_j$ ). However, as we can see in (21), the MG has its unique learning process which comes through adaptation of agents along iteration of the game.

In summary, after clearing up the MG model we ended up with a set of heterogenous agents (or perceptrons) which are learning through the adaptation process (21) the minority task given by (20). It would be interesting to extend the training of  $N$  perceptrons given by (21) and to consider the general problem of learning in a set of interacting  $N$  perceptrons under the basic MG features: heterogeneity, minority rule and time delayed interactions.

## 9 Dynamics matter

The success of the solution obtained using the replica trick [15, 47] and outlined in section 7, established the concept that inductive reasoning of  $N$  interacting agents in the MG is described by the minimization of a global function, the response  $\mathcal{H}$ . Thus, the MG seems to be similar to an equilibrium problem of disordered systems. But a simple numerical experiment shows that this is not the case [28]: Suppose that agents are confident about one of their strategies and they start by giving it a period of grace, i.e. an initial number of points different from zero. This means that  $p_i^\alpha(0) \neq 0$  for some  $\alpha$  in equation (2). Simulations in figure 11 show that this tiny modification of the MG model has a profound impact in the results for  $\alpha \leq \alpha_c$ . Specifically, the high volatility region is lost and a region in which volatility is very small emerges. Moreover, the steady state volatility for  $\alpha \leq \alpha_c$  depends on the initial condition  $p_i^\alpha(0)$  [22, 28, 34]<sup>8</sup>.

The fact that the steady state reached by agents through adaptation depends on the initial condition for  $\alpha \leq \alpha_c$  means that this region is non-ergodic and that the observed change of behavior at  $\alpha = \alpha_c$  is due to a non-equilibrium phase transition instead of an equilibrium transition. A precise mathematical description of this was reached through the generating functional methods of De Dominicis [24] by A. C. C. Coolen and coworkers [20, 21, 22, 34]. This technique allows to carry out the average over

<sup>7</sup>A similar model (named time horizon MG), was considered in [33] in which, in the original MG model,  $p_i^\alpha(t)$  are just the points of strategy  $\alpha$  of agent  $i$  for the last  $M$  time steps.

<sup>8</sup>Due to the dependence on the initial condition for  $\alpha < \alpha_c$ , experiments in which the number of agents  $N$  or the memory  $D$  are changed quasi-statically lead to hysteresis behavior in the MG [58], again demonstrating the non-equilibrium behavior in the region  $\alpha < \alpha_c$

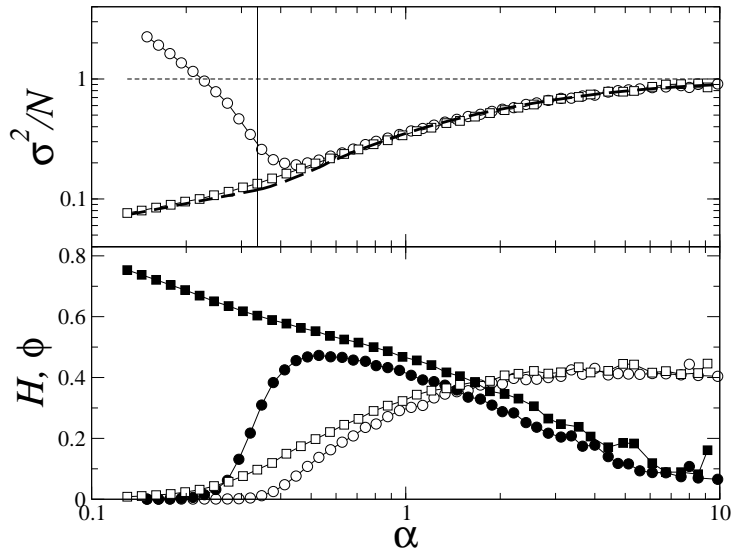


Figure 11: Simulations of the original MG model (with endogenous information) for different initial conditions and  $m = 7$ ,  $s = 2$  and  $p_i^1(0) = 0$  (circles) and  $p_i^1(0) = 100$  (squares). *Upper panel*: dashed lines correspond to the stationary solution for  $\alpha > \alpha_c$  and  $\sigma^2/N \sim \alpha^{1/2}$  for  $\alpha \leq \alpha_c$ . *Lower panel*: full symbols correspond to  $H$  while open symbols refer to  $\phi$ .

the quenched disorder in the system into the dynamical equations directly, instead of performing it in the equilibrium partition function. The outcome of this averaging is a set of closed equations for the correlation  $C(t, t')$  and response  $R(t, t')$  functions of the variables involved in the dynamical equations [ $p_i^\alpha(t)$  and  $\sigma_i(t)$ ] at different times. This system of dynamical equations for  $C(t, t')$  and  $R(t, t')$  is difficult to solve in general, but some analytical progress can be made if one looks at two particular situations:

- For instance, one can investigate asymptotic stationary solutions, i.e. when  $C(t, t) = C(t - t')$  and  $R(t, t') = R(t - t')$ . and assume that the system has not anomalous response, i.e. that fluctuations around the stationary state are damped in finite time and the susceptibility is finite. In this case, the results obtained in section 7 through the evaluation of the minima of  $\mathcal{H}$  are recovered. Moreover, this stationary solution predicts self-consistently when it breaks down: the susceptibility of the solution diverges at  $\alpha = \alpha_c$  and the assumption of stationary in the solution is only valid for  $\alpha > \alpha_c$ .
- For  $\alpha \leq \alpha_c$  one can investigate the first time steps of the correlation and response function dynamics. It is found that the volatility depends strongly on the initial conditions: for  $s = 2$  and depending on  $\delta p_i(0)$  a transition from high volatility  $\sigma^2/N \sim \alpha^{-1}$  to low volatility  $\sigma^2/N \sim \alpha^{1/2}$  is found (see Fig. 11).

These results point out that the phase transition found in the MG is a non-equilibrium transition and that only for  $\alpha > \alpha_c$  the adaptation dynamics is relaxing and ergodic: for any initial condition a final unique steady state is reached which is described by the minima of  $\mathcal{H}$ . For  $\alpha \leq \alpha_c$  the system is non-ergodic, since the system asymptotic behavior depends strongly on the initial conditions. After all, the MG is posed without requiring detailed balance or any kind of Fluctuation-Dissipation theorem about

its dynamics, so there is in principle no reason to expect that the MG should have the usual relaxing and ergodic dynamics of other models in statistical mechanics. Finally, it is interesting to note [22] that ergodicity in the MG is not broken at the macroscopic level as in replica symmetry breaking solutions in spin-glass; it is done at the microscopic level since the dynamical rules do not fulfill any type of detailed balance.

The generating functional analysis proved to be the right tool to study the MG since its focus is on the dynamics of disordered systems, specially on spin glasses. Since the existence of a partition function is not needed, complicated cases like steady states with non-equilibrium fluctuation-dissipation relationships or violations of detailed balance can be tackled within this formalism. Indeed, the MG model features do not root in physical considerations so there is no reason *a priori* to expect detailed balance or fluctuation-dissipation relationships.

## 10 Outlook

The land of statistical physics is broad, with many dales, hills, valleys and peaks to explore that are of relevance to the real world and to our ways of thinking about it.  
*Michael E. Fisher in Rev. Mod. Phys.* **70**, 653 (1998).

Through this journey we have learned that adaptation or inductive reasoning as that of the MG model leads to a system of many interacting and heterogeneous degrees of freedom whose dynamics is not local in time. The system may or may not be ergodic and, generally speaking, equilibrium statistical mechanics should be abandoned. However there are some similarities with statistical mechanics of disordered systems and some of the tools and concepts of disordered system do apply to the MG. The MG is just a model of adaptation of  $N$  agents which can be generalized to other types of learning as we saw in section 8. For those types of dynamical problems, the suitable tool from statistical physics is the generating functional approach we mentioned in section 9 which gives not only information about the properties of the system when a steady state is reached, but also about the dynamical nature of the phase transition. In fact, the generating function approach is also useful in different modifications of the MG [20, 21, 22, 26].

Adaptation turned out to be a good way to solve the MG problem for  $\alpha > \alpha_c$ . But it might not be the best way to do it in the region  $\alpha \leq \alpha_c$ . As a matter of fact, simple modifications of the original MG model like taking a non-zero initial condition or what is called the Thermal Minority Game [7, 8, 18], in which agent's decisions are taken with some degree of stochasticity (instead of using their best one always), lead to better solutions ( $\sigma^2/N \ll 1$ ) of the MG problem than the original model.

However, part of the success of the application of analytical techniques in the MG is due to its fixed rules. For example, in real life we do not expect to come across a game in which players have the same space of strategies, or in which the initial condition for all of them is that in which none of their strategies is preferred. Relaxing some of these rules in the MG does not modify its behavior; unfortunately, others do. During the last years we have witnessed a burst of spin-off models whose core is given by the MG [11]. Of particular importance are those in which refinements of the model were towards the description of real markets [19, 30, 31, 48] or real problems in ecology [10]. The validity of the MG as a behavioral model of agents trying to spot patterns in aggregate outcomes has been even tested through experiments with people [4, 52] showing that indeed people tend to coordinate despite its selfish nature in the game.

On the other hand, there has been some extensions of the MG which has been used in the context of time series analysis and prediction by doing *reverse engineering*: since the MG is, as we saw in section 8, a precise learning rule, why not training a set of  $N$  agents with a real life time series? In that case, agents strategies are not fixed at the beginning of the game but rather, they are chosen to reproduced the given time series. This has been applied to financial times series in order to forecast future price movements [39].

The virtue of the MG relies on the incorporation of the minority rule and agents' heterogeneity in the model as well as an agent-based approach to the solution of a given problem. In this regard, during the last years there has been a lot of attention in the solution of general problems using multi-agent models [59, 60]. In fact coupled layers of perceptrons are just a simple example of this approach [1]. The general principles of those multi-agent systems is that agents individually have incomplete information or capabilities for solving the problem but can reach a solution by interacting among them. The success of understanding of the MG using tools from statistical mechanics encourages to tackle more general heterogeneous multi-agent systems problems within the theory of disordered systems in condensed matter.

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