

FIXED-POINT SETS OF AUTOHOMEOMORPHISMS OF COMPACT F -SPACES

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ABSTRACT. We investigate fixed-point sets of autohomeomorphisms of compact F -spaces. If the space in question is finite dimensional (in the sense of covering dimension), then the fixed-point set is a P -set; on the other hand there is an infinite-dimensional compact F -space with an involution whose fixed-point set is not a P -set.

In addition we show that under CH a closed subset of ω^* is a P -set iff it is the fixed-point set of an autohomeomorphism.

INTRODUCTION

In this note we investigate the fixed-point sets of autohomeomorphisms of compact F -spaces. In Vermeer [6, 7] the second author studied fixed-point sets of continuous self-maps of extremally and basically disconnected spaces. It was proved that whenever X is a compact κ -basically disconnected space (i.e., the Stone space of a κ -complete Boolean algebra) and $\phi : X \rightarrow X$ is injective and continuous, the fixed-point set of ϕ is a P_κ -set of X . In particular for a basically disconnected (i.e., ω_1 -basically disconnected) space the fixed-point set of a self-embedding is always a P -set.

The methods used to obtain the above-mentioned result do not readily generalize to the natural extension of the class of basically disconnected spaces: the class of F -spaces. The point is that these methods relied heavily on the fact that a countable increasing union of clopen sets in a basically disconnected space has a clopen closure and this last property hardly ever holds nontrivially in general F -spaces.

Here we use results about fixed-point free extensions of fixed-point free maps to obtain the result that the fixed-point set of an autohomeomorphism of a finite-dimensional compact F -space is a P -set of that space. This seems to be new, even for the space ω^* .

If we assume the Continuum Hypothesis, then we can even show that a closed subset of ω^* is a P -set iff it is the fixed-point set of an autohomeomorphism (even an involution) of ω^* . This gives an external characterization of the P -sets in ω^* and is a partial answer to Problem 218 of Hart and van Mill [4]. We finish the paper with an example of an infinite-dimensional compact F -space and an involution on it whose fixed-point set is not a P -set.

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1. PRELIMINARIES

By convention all spaces under consideration are completely regular. We call—as usual—a space X an F -space if every cozero set in it is C^* -embedded, i.e., if M is a cozero set of X and $f : M \rightarrow \mathbb{R}$ is a bounded continuous function, then f can be extended to a bounded continuous function from X to \mathbb{R} . For compact spaces this takes the following convenient form: A compact space X is an F -space iff for every F_σ -subset F of X the equality $\text{cl } F = \beta F$ holds. A rich supply of compact F -spaces can be gotten from the well-known fact that $\beta X \setminus X$ is an F -space whenever X is σ -compact and locally compact.

We also need the characterization of ω^* given by Parovičenko in [5]. This characterization is valid under the assumption of the Continuum Hypothesis (CH).

Theorem 1.1 (CH). *A compact space X is homeomorphic to ω^* if and only if it is a compact, zero-dimensional F -space of weight \mathfrak{c} without isolated points in which nonempty G_δ -sets have nonempty interiors.*

This theorem is particularly useful when one works with P -sets in ω^* ; we recall that a subset of a space is a P -set if every G_δ -set containing it is a neighbourhood of it or, equivalently, a set A is a P -set if for every F_σ -set F disjoint from it one has $A \cap \text{cl } F = \emptyset$.

For example, in the proof of Lemma 1.3 below we use the fact that $\omega^* \setminus \text{Int } A$ is homeomorphic to ω^* whenever A is a P -set of ω^* . A second application occurs in the proof of Theorem 2.2.

From van Douwen and van Mill [2] we quote the following theorem, the homeomorphism extension theorem for nowhere dense P -sets.

Theorem 1.2 (CH). *Let A and B be nowhere dense P -sets of ω^* and $h : A \rightarrow B$ a homeomorphism. Then there is an autohomeomorphism \tilde{h} of ω^* that extends h .*

We shall need the following mild extension of this theorem.

Lemma 1.3 (CH). *Let A and B be proper P -subsets of ω^* , and let $h : A \rightarrow B$ be a homeomorphism that maps the interior of A onto the interior of B . Then there is an autohomeomorphism \tilde{h} of ω^* that extends h .*

Proof. Consider $\omega^* \setminus \text{Int } A$ and $\omega^* \setminus \text{Int } B$. As noted above both spaces are homeomorphic to ω^* because A and B are P -sets.

The homeomorphism extension theorem for nowhere dense P -sets now gives us an extension $h' : \omega^* \setminus \text{Int } A \rightarrow \omega^* \setminus \text{Int } B$ of the restriction $h \upharpoonright \text{Fr } A$. To finish we let $\tilde{h} = h \cup h'$. \square

The final result that we need is from van Douwen [1]. We use the term ‘finite-dimensional’ in the sense of the covering dimension dim .

Theorem 1.4. *Let X be a finite-dimensional paracompact space and $f : X \rightarrow X$ a closed self-map for which there is a natural number k such that $|f^{-1}(x)| \leq k$ for all $x \in X$. Then f has a fixed point if and only if βf has a fixed point.*

2. FINITE-DIMENSIONAL SPACES

We get our first result by a judicious application of van Douwen’s theorem.

Theorem 2.1. *Let X be a compact finite-dimensional F -space and $\phi : X \rightarrow X$ a continuous and injective map. The fixed-point set F of ϕ is a P -set of X .*

Proof. Let K be an F_σ -subset of X that is disjoint from F . We must show that $\text{cl } K$ is disjoint from F . To this end we take the set $L = \bigcup_{k \in \mathbb{Z}} \phi^k[K]$. Observe that L is also an F_σ -set that is disjoint from F ; that L is an F_σ -set is clear. To see that L contains no fixed points of ϕ combine the facts that K contains none and that ϕ is injective. It is also clear that $\phi[L] \subseteq L$. Finally we observe that $\phi \upharpoonright L$ is closed: use the fact that $\phi^{-1}[L] = L$.

Now, because X is an F -space, we have $\text{cl } L = \beta L$. Then van Douwen's theorem implies that $\text{cl } L$ contains no fixed points of ϕ either. It follows that $\text{cl } L \cap F = \emptyset$, so certainly $\text{cl } K \cap F = \emptyset$. □

For the space ω^* we can reverse the implication, provided we assume CH.

Theorem 2.2 (CH). *A closed subset A of ω^* is a P -set iff it is the fixed-point set of some autohomeomorphism of ω^* .*

Proof. Let A be a P -set of ω^* . We shall find an autohomeomorphism ϕ of ω^* of which A is the fixed-point set; indeed, ϕ will be an involution, i.e., ϕ^2 is the identity.

Consider $\omega^* \times 2$ and identify, for every $x \in A$, the points $\langle x, 0 \rangle$ and $\langle x, 1 \rangle$ (we glue the two copies of ω^* together along the copies of A). Because A is a P -set, the resulting quotient space Q is homeomorphic to ω^* : it satisfies the conditions from Parovičenko's theorem.

Define an autohomeomorphism ψ of Q by sending $\langle x, i \rangle$ to $\langle x, 1 - i \rangle$ for every x . Clearly ψ^2 is the identity and the copy A_Q of A in Q is the fixed-point set of ψ .

It remains to turn ψ into an autohomeomorphism of ω^* whose fixed-point set is A itself.

The identity $\text{Id} : A_Q \rightarrow A$ is a homeomorphism that maps the interior of A_Q onto the interior of A and so by Lemma 1.3 it may be extended to a homeomorphism $h : Q \rightarrow \omega^*$.

In the end we take $\phi = h \circ \psi \circ h^{-1}$ of course. □

3. INFINITE-DIMENSIONAL SPACES

In this section we give an example of compact infinite-dimensional F -space X and an autohomeomorphism ϕ of X whose fixed-point set is not a P -set. Again ϕ can be taken to be an involution.

Our starting point is the following example, considered by van Douwen in [1]. Let $\mathbb{S} = \bigoplus_n S^n$, where S^n is the standard n -sphere. Next let $\phi : \mathbb{S} \rightarrow \mathbb{S}$ be the sum of the antipodal mappings. Now ϕ has no fixed points, yet $\beta\phi$ does have fixed points; this can be seen as follows: if $\beta\phi$ would have no fixed points, then there would be a finite closed cover $\{F_1, \dots, F_n\}$ of $\beta\mathbb{S}$ such that $\beta\phi[F_i] \cap F_i = \emptyset$ for all i . However, the Lusternik-Schnirelman-Borsuk Theorem (Dugundji and Granas [3, Theorem 4.4]) implies that $\phi[F_i] \cap F_i \cap S^n \neq \emptyset$ for some i .

To begin we take for every n the closed n -ball B^n . Remove the origin and call the result X_n . The antipodal map e_n on X_n has no fixed points and, as $\dim X_n = n$, neither do βe_n and $e_n^* = \beta e_n \upharpoonright X_n^*$ (apply Theorem 1.4). Also note that X_n^* is an F -space.

Write $X = \bigoplus_n \beta X_n$ and $e = \bigoplus_n \beta e_n$. The map e has no fixed points but βe has many of them: for any sequence $\langle S_n \rangle_n$ of spheres centered at the origins of the B^n we get fixed points of βe in the closure of $\bigoplus_n S_n$.

Now take any neighbourhood of $(\bigoplus_n X_n^*)^*$ in βX ; it contains a tail of a sequence of spheres as in the preceding paragraph and hence a fixed point of βe . But then $(\bigoplus_n X_n^*)^*$ contains fixed points of βe as well.

Our example is the closure of $\bigoplus_n X_n^*$ in βX , and the map ϕ is the restriction of βe . It is clearly an F -space, and the (nonempty) fixed-point set of ϕ is contained in the nowhere dense G_δ -set $(\bigoplus_n X_n^*)^*$.

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