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ON THE MAĆKOWIAK-TYMCHATYN THEOREM

K. P. HART AND B. J. VAN DER STEEG

Abstract. In this paper we give new proofs of the theorem of Maćkowiak and Tymchatyn that every metric continuum is a weakly-confluent image of some one-dimensional hereditarily indecomposable continuum of countable weight. The first is a model-theoretic argument; the second is a topological proof inspired by the first.

1. Introduction

In [5] Maćkowiak and Tymchatyn proved that every metric continuum is the continuous image of a one-dimensional hereditarily indecomposable continuum by a weakly confluent map. In [3] this result was extended to general continua, with two proofs, one topological and one model-theoretic. Both proofs made essential use of the metric result.

The original purpose of this paper was to (re)prove the metric case by model-theoretic means. After we found this proof we realized that it could be combined with any standard proof of the completeness theorem of first-order logic (see e.g., Hodges [4], 6.1]) to produce an inverse-limit proof of the general form of the Maćkowiak-Tymchatyn result. We present both proofs. The model-theoretic argument occupies sections 3 and 4, and the inverse-limit approach appears in section 5.

We want to take this opportunity to point out some connections with work of Bankston [1], who dualized the model-theoretic notions of existentially closed structures and existential maps to that of co-existentially closed compacta and co-existential maps. He proves that co-existential maps are weakly confluent, that co-existentially closed continua are one-dimensional and hereditarily indecomposable, and that every continuum is the continuous image of a co-existentially closed one. The map can in general not be chosen co-existential, because co-existential maps preserve indecomposability and do not raise dimension.

2. Preliminaries

2.1. **Maćkowiak-Tymchatyn theorem.** The theorem of Maćkowiak and Tymchatyn that we are dealing with in this paper states that every metric continuum is a weakly confluent image of a one-dimensional hereditarily indecomposable metric continuum.

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A continuum is *decomposable* if it can be written as a union of two proper subcontinua, it is called *indecomposable* if this is not the case. We call a continuum *hereditarily indecomposable* if every subcontinuum is indecomposable. This is equivalent to saying that whenever two subcontinua meet, one is contained in the other. As in [3] we can extend this notion for arbitrary compact Hausdorff spaces. So a compact Hausdorff space is hereditarily indecomposable if for every two subcontinua that meet, one is contained in the other. We call a continuous mapping between two continua *weakly confluent* if every subcontinuum in the range is the image of a subcontinuum in the domain.

Theorem 1 (Maćkowiak and Tymchatyn [5])**.** *Every metric continuum is a weakly confluent image of some one-dimensional hereditarily indecomposable continuum of the same weight.*

In [3] Hart, van Mill and Pol showed that the Maćkowiak and Tymchatyn result above implies the theorem for the non-metric case using model-theoretic means.

2.2. **Wallman space.** In the proof we will consider the lattice of closed sets of our metric continuum X and try to find, through model-theoretic means, another lattice in which we can embed our lattice of closed sets of X. This new lattice will be a model for some sentences which will make sure that its Wallman representation is a continuum with certain properties. So at the base of the proof is Wallman's generalization, to the class of distributive lattices, of Stone's representation theorem for Boolean algebras. Wallman's representation theorem is as follows.

Theorem 2 ([6])**.** *If L is a distributive lattice, then there is a compact T_1 space X with a base for its closed sets that is a homomorphic image of L. If L is also disjunctive then we can find a base for its closed sets that is an isomorphic image of L.*

We call the space X a Wallman space of L or a Wallman representation of L, notation: wL.

A lattice L is *disjunctive* if it models the sentence

$$\forall a\, b \exists x[(a \sqcap b \neq a) \to ((a \sqcap x = x) \wedge (b \sqcap x = \mathbf{0}))]. \tag{1}$$

Furthermore the space X in theorem 2 is Hausdorff if and only if the lattice L is a normal lattice. We call a lattice normal if it models the sentence

$$\forall a\, b \exists x\, y[(a \sqcap b = \mathbf{0}) \to ((a \sqcap x = \mathbf{0}) \wedge (b \sqcap y = \mathbf{0}) \wedge (x \sqcup y = \mathbf{1}))]. \tag{2}$$

Note that, if we start out with a compact Hausdorff space X and look at a base for its closed subsets which is closed under finite unions and intersections, i.e., a (normal, disjunctive and distributive) lattice, then the Wallman space of this lattice is just the space X.

Remark 1. *From now on we refer to a base for the closed subsets of some topological space which is closed under finite unions and intersections as a lattice base for the closed sets of the space X.*

The following theorem shows how to create an onto mapping from maps between lattices. In this theorem 2^X denotes the family of all closed subsets of the space X.

Theorem 3. [2] *Let X and Y be compact Hausdorff spaces and let $\mathcal{C}$ be a base for the closed subsets of Y that is closed under finite unions and intersections. Then Y is a continuous image of X if and only if there is a map $\phi : \mathcal{C} \to 2^X$ such that*

(1) $\phi(\emptyset) = \emptyset$, *and if* $F \neq \emptyset$ *then* $\phi(F) \neq \emptyset$
(2) *if* $F \cup G = Y$ *then* $\phi(F) \cup \phi(G) = X$
(3) *if* $F_1 \cap \cdots \cap F_n = \emptyset$ *then* $\phi(F_1) \cap \cdots \cap \phi(F_n) = \emptyset$.

So Y is certainly a continuous image of X if there is an embedding of some lattice base of the closed sets of Y into 2^X.

2.3. **Translation of properties.** Our model-theoretic proof of theorem 1 will be as follows. Given a metric continuum X, we will construct a lattice L such that some lattice base of X is embedded into L, the Wallman representation wL of L is a one-dimensional hereditarily indecomposable continuum and that for every subcontinuum in X there exists a subcontinuum of wL that is mapped onto it.

For this we need to translate things like being hereditarily indecomposable, being of dimension less than or equal to one and being connected in terms of closed sets only.

To translate hereditarily indecomposability we use the following characterization, due to Krasinkiewicz and Minc.

Theorem 4 (Krasinkiewicz and Minc)**.** *A compact Hausdorff space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed nonempty subsets.*

Which the authors translated in [3] into terms of closed sets only as follows.

Theorem 5. [3] *A compact Hausdorff space X is hereditarily indecomposable if and only if whenever four closed sets C, D, F and G in X are given such that $C \cap D = C \cap G = F \cap D = \emptyset$ one can write X as the union of three closed sets X_0, X_1 and X_2 such that $C \subset X_0$, $D \subset X_2$, $X_0 \cap X_1 \cap G = \emptyset$, $X_0 \cap X_2 = \emptyset$ and $X_1 \cap X_2 \cap F = \emptyset$.*

So a compact Hausdorff space is hereditary indecomposable if the lattice 2^X models the sentence

$$\begin{aligned}(3)\qquad &\forall a\,b\,c\,d\;\exists x\,y\,z[((a \sqcap b = \mathbf{0}) \wedge (a \sqcap c = \mathbf{0}) \wedge (b \sqcap d = \mathbf{0})) \to \\ &\to ((a \sqcap (y \sqcup z) = \mathbf{0}) \wedge (b \sqcap (x \sqcup y) = \mathbf{0}) \wedge (x \sqcap y = \mathbf{0}) \wedge \\ &\wedge (x \sqcap y \sqcap d = \mathbf{0}) \wedge (y \sqcap z \sqcap c = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))].\end{aligned}$$

A space X is of dimension less than or equal to one if the lattice 2^X models the sentence

$$\begin{aligned}(4)\qquad &\forall a\,b\,c\;\exists x\,y\,z[(a \sqcap b \sqcap c = \mathbf{0}) \to \\ &\to \quad ((a \sqcap x = a) \wedge (b \sqcap y = b) \wedge (c \sqcap z = c) \wedge \\ &\wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))].\end{aligned}$$

A space X is connected if the lattice 2^X models the sentence $\mathrm{conn}(\mathbf{1})$, where $\mathrm{conn}(a)$ is shorthand for the formula $\forall x\,y[((x \sqcap y = \mathbf{0}) \wedge (x \sqcup y = a)) \to (x = a) \vee (x = \mathbf{0}))]$.

Remark 2. *For the next two sections, section 3 and section 4 we fix some metric continuum X and we will show there exists a hereditarily indecomposable one-dimensional continuum Y of weight $\mathrm{w}(X)$ such that X is weakly confluent image of Y.*

3. A continuous image of an hereditarily indecomposable one-dimensional continuum of the same weight

Using theorem 2 and 3 of the previous section we see that to get a hereditarily indecomposable one-dimensional continuum of weight $w(X)$ that maps onto X we must find a countable distributive, disjunctive normal lattice L such that it is a model of the sentences 3, 4 and conn($\mathbf{1}$), and furthermore some lattice base for the closed sets of X is embedded into this lattice L.

Fix a lattice base $\mathcal{B}$ for the closed sets of X.

For some countable set of constants K we will construct a set of sentences Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$. We will make sure that Σ is a consistent set of sentences such that, if we have a model $\mathfrak{A} = (A, \mathcal{I})$ for Σ then

$$L(\mathfrak{A}) = \mathcal{I} \restriction K$$

is the universe of some lattice model in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$ which is normal, distributive and disjunctive and models the sentences 3, 4 and conn($\mathbf{1}$). To make sure that $\mathcal{B}$ is embedded into $L(\mathfrak{A})$ we simply add the diagram of the lattice $\mathcal{B}$ to the set Σ and make sure that there are constants in K representing the elements of $\mathcal{B}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ are given by there interpretations under $\mathcal{I}$ in the model $\mathfrak{A}$.

Let K be the following countable set of constants

$$K = \bigcup_{-1 \le n < \omega} K_n = \bigcup_{-1 \le n < \omega} \{k_{n,m} : m < \omega\}. \tag{5}$$

We will define the sentences of Σ in an ω-recursion. So Σ will be the set $\bigcup_{n<\omega} \Sigma_n$.

For definiteness we define $K_{-1} = \mathcal{B}$ and $\Sigma_0 = \triangle_{\mathcal{B}}$, the diagram of $\mathcal{B}$.

3.1. Construction of Σ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$.

Suppose we already defined the sentences up to Σ_{5n}.

(1) Σ_{5n+1} will be a set of sentences that will make sure that the supremum and infimum of any pair of constants in $\bigcup_{m \le 5n} K_m$ are defined, using the new constants from K_{5n+1}.

We will also make sure that the set Σ_{5n+1} makes sure that distributivity holds for any triple of elements from $\bigcup_{m \le 5n} K_m$.

And we will make sure that the family of sets of sentences $\{\Sigma_{5n+1} : n < \omega\}$ will prevent the existence of a counterexample for conn($\mathbf{1}$) in $\bigcup_{m \le 5n} K_m$.

(2) Σ_{5n+2} will be a set of sentences that will make sure that for every $a, b \in \bigcup_{m \le 5n} K_m$ there exists some $c \in \bigcup_{m \le n} K_{5m+2}$ such that the formula that is sentence 1 without quantifiers will hold for these a, b and c.

(3) Σ_{5n+3} will be a set of sentences that will make sure that for every $a, b \in \bigcup_{m \le 5n} K_m$ there exist $c, d \in \bigcup_{m \le n} K_{5m+3}$ such that the formula that is sentence 2 without quantifiers will hold for these a, b, c and d.

(4) Σ_{5n+4} will be a set of sentences that will make sure that the according to the elements of $K_{5n+4} \cup \bigcup_{m \le 5n} K_m$ the dimension of the Wallman space of $L(\mathfrak{A})$ for any model $\mathfrak{A}$ of Σ will be less than or equal to one.

(5) $\Sigma_{5(n+1)}$ will be a set of sentences that will make sure that for any $a, b, c \in \bigcup_{m < 5n} K_m$ there exist $x, y, z \in K_{5(n+1)}$ such that the formula, which is the sentence 3 without quantifiers, holds for this a, b, c and x, y, z.

We now show how to define the sets of sentences of $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \bigcup_{m<5n+4} K_m$ as described in 1 - 5.

We have a natural order $\lhd$ on the set $K = \bigcup_m K_m$ defined by

$$k_{n,m} \lhd k_{r,t} \leftrightarrow [(n < r) \vee ((n = r) \wedge (m < t))].$$

Let $\{p_l\}_{l<\omega}$ be an enumeration of

$$\{p \in [\bigcup_{m \leq 5n} K_m]^2 : p \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}.$$

$$\begin{aligned}
\Sigma^0_{5n+1} &= \{\bigsqcap p_l = k_{5n+1,2l} : l < \omega\} \\
\Sigma^1_{5n+1} &= \{\bigsqcup p_l = k_{5n+1,2l+1} : l < \omega\} \\
\Sigma^2_{5n+1} &= \{a \sqcup a = a, a \sqcap a = a : a \in \bigcup_{m \leq 5n} K_m\} \\
\Sigma^3_{5n+1} &= \{a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c, a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c : a, b, c \in \bigcup_{m \leq 5n} K_m\} \\
\Sigma^4_{5n+1} &= \{a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c) : a, b, c \in \bigcup_{m \leq 5n} K_m\} \\
\Sigma^5_{5n+1} &= \{a \sqcup (a \sqcap b) = a, a \sqcap (a \sqcup b) = a : a, b \in \bigcup_{m \leq 5n} K_m\} \\
\Sigma^6_{5n+1} &= \{[((a \sqcup b = \mathbf{1}) \wedge (a \sqcap b = \mathbf{0})) \rightarrow ((a = \mathbf{0}) \vee (a = \mathbf{1}))] : a, b \in \bigcup_{m \leq 5n} K_m\}
\end{aligned}$$

Define Σ_{5n+1} by

$$\Sigma_{5n+1} := \bigcup_{i \leq 6} \Sigma^i_{5n+1}.$$

This set of sentences will make sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be a distributive lattice and also a model of the sentence $\mathrm{conn}(\mathbf{1})$.

$$\begin{aligned}
\Sigma_{5n+2} = \ &\{[(\max_{\lhd} p_l \sqcap \min_{\lhd} p_l = \mathbf{0}) \rightarrow ((\max_{\lhd} p_l \sqcap k_{5n+2,2l} = \mathbf{0}) \wedge \\
&\wedge (\min_{\lhd} p_l \sqcap k_{5n+2,2l+1} = \mathbf{0}) \wedge (k_{5n+2,2l} \sqcup k_{5n+2,2l+1} = \mathbf{1}))] : l < \omega\}
\end{aligned}$$

This set of sentences will make sure that any (lattice) model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ will be normal.

The following set of sentences makes sure that any model of Σ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$ which is also a lattice is a disjunctive lattice.

$$\begin{aligned}
\Sigma^0_{5n+3} = \{[(\max_{\lhd} p_l \sqcap \min_{\lhd} p_l \neq \max_{\lhd} p_l) &\rightarrow ((k_{5n+3,2l} \sqcap \max_{\lhd} p_l = k_{5n+3,2l}) \wedge \\
&\wedge (k_{5n+3,2l} \sqcap \min_{\lhd} p_l = \mathbf{0}))] : l < \omega\} \\
\Sigma^1_{5n+3} = \{[(\min_{\lhd} p_l \sqcap \max_{\lhd} p_l \neq \min_{\lhd} p_l) &\rightarrow ((k_{5n+3,2l+1} \sqcap \min_{\lhd} p_l = k_{5n+3,2l+1}) \wedge \\
&\wedge (k_{5n+3,2l+1} \sqcap \max_{\lhd} p_l = \mathbf{0}))] : l < \omega\}
\end{aligned}$$

And define Σ_{5n+3} by

$$\Sigma_{5n+3} = \Sigma^0_{5n+3} \cup \Sigma^1_{5n+3}.$$

Let ζ denote the following formula in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$

$$\begin{aligned}\zeta(a,b,c;x,y,z) = [(a \sqcap b \sqcap c = \mathbf{0}) \quad \rightarrow \quad & ((a \sqcap x = a) \wedge (b \sqcap y = b) \wedge (c \sqcap z = c) \wedge \\ & \wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1}))]\end{aligned}$$

Let $\{q_l\}_{l<\omega}$ be an enumeration of the set

$$\{q \in [\bigcup_{m \leq 5n} K_m]^3 : q \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}$$

For every $l < \omega$ write $q_l = \{q_l(0), q_l(1), q_l(2)\}$.

Now define Σ_{5n+4} by

$$\Sigma_{5n+4} = \{\zeta(q_l(0), q_l(1), q_l(2); k_{5n+4,3l}, k_{5n+4,3l+1}, k_{5n+4,3l+2}) : l < \omega\}.$$

This will make sure that the Wallman space of any lattice model of Σ will be at most one-dimensional.

For making sure that the Wallman space of any model of Σ will be hereditarily indecomposable we introduce the following formulas in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\}$:

$$\begin{aligned}\phi(a,b,c,d) &= [(a \sqcap b = \mathbf{0}) \wedge (a \sqcap d = \mathbf{0}) \wedge (b \sqcap c = \mathbf{0})] \\ \psi(a,b,c,d;x,y,z) &= [(x \sqcup y \sqcup z = \mathbf{1}) \wedge (x \sqcap z = \mathbf{0}) \wedge \\ & \quad \wedge (a \sqcap (y \sqcup z) = \mathbf{0}) \wedge (b \sqcap (x \sqcup y) = \mathbf{0}) \wedge \\ & \quad \wedge (x \sqcap y \sqcap d = \mathbf{0}) \wedge (y \sqcap z \sqcap c = \mathbf{0})] \\ \theta(a,b,c,d;x,y,z) &= \phi(a,b,c,d) \rightarrow \psi(a,b,c,d;x,y,z) \qquad (6)\end{aligned}$$

Let $\{r_l\}_{l<\omega}$ be an enumeration of the set

$$\{r \in {}^4[\bigcup_{m \leq 5n} K_m] : \operatorname{ran}(r) \setminus \bigcup_{m \leq 5(n-1)} K_m \neq \emptyset\}.$$

Let $\Sigma_{5(n+1)}$ be the set of sentences defined by:

$$\Sigma_{5(n+1)} = \{\theta(r_l(0), r_l(1), r_l(2), r_l(3); k_{5(n+1),3l}, k_{5(n+1),3l+1}, k_{5(n+1),3l+2}) : l < \omega\}$$

Here the formula θ is as in equation 6.

3.2. **Consistency of Σ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$.** In this section we show that Σ is a consistent set of sentences.

We will find for $\Sigma' \in [\Sigma]^{<\omega}$ a metric space $X(\Sigma')$ and an interpretation function $\mathcal{I} : K \rightarrow 2^{X(\Sigma')}$ such that $(X(\Sigma'), \mathcal{I}) \models \Sigma' \cup \triangle_{\mathcal{B}}$. The interpretations of $\sqcap$, $\sqcup$, $\mathbf{0}$ and $\mathbf{1}$ will always be $\cap$, $\cup$ (the normal set intersection and union), $\emptyset$ and $X(\Sigma')$ respectively.

For $\Sigma' = \emptyset$ we let $X(\emptyset) = X$ and we interpret every constant from K_{-1} as its corresponding base element in $\mathcal{B}$. Extend the interpretation function by assigning the empty set to all constants of $K \setminus K_{-1}$. It is obvious that $(2^{X(\emptyset)}, \mathcal{I}) \models \triangle_{\mathcal{B}}$.

Remark 3. *As the interpretation of $\sqcap$ and $\sqcup$ in the metric continuum $X(\Sigma')$ will always be the normal set intersection and set union, all the sentences in Σ^i_{5n+1} for some $n < \omega$ and $i \in \{3,4,5,6\}$ are true in the model $(2^{X(\Sigma')}, \mathcal{I})$. So we can ignore these sentences and for the remainder of this section concentrate on the remaining sentences of Σ.*

We can define a well order $\sqsubset$ on the set $\Sigma \setminus \{\Sigma^i_{5n+1} : n < \omega \text{ and } i \in \{3,4,5,6\}\}$ by stating that $\phi \sqsubset \psi$ if and only if there are $n < m < \omega$ such that $\phi \in \Sigma_n$ and $\psi \in \Sigma_m$ or there are $k < l < \omega$ and $n < \omega$ such that $\phi, \psi \in \Sigma_n$ and ϕ is a sentence that mentions p_k (q_k or r_k respectively) and ψ is a sentence that mentions p_l (q_l or r_l respectively).

Suppose Σ' is a finite subset of Σ such that for all of its proper subsets Σ'' there exists a metric continuum $X(\Sigma'')$ and an interpretation function $\mathcal{I} : K \to 2^{X(\Sigma'')}$ such that $(X(\Sigma''), \mathcal{I}) \models \Sigma'' \cup \triangle_{\mathcal{B}}$.

Let θ be the $\sqsubset$-maximal sentence in $\Sigma' \setminus \{\Sigma^i_{5n+1} : n < \omega \text{ and } i \in \{3,4,5,6\}\}$. We will show that there exists a metric space $X(\Sigma')$ and an interpretation function $\mathcal{I} : K \to 2^{X(\Sigma')}$ such that $(X(\Sigma'), \mathcal{I}) \models \Sigma' \cup \triangle_{\mathcal{B}}$.

Let $\Sigma'' = \Sigma' \setminus \{\theta\}$.

3.2.1. $\theta \in \bigcup_{m<\omega}\{\Sigma_{5n+1} \cup \Sigma_{5m+2} \cup \Sigma_{5m+3}\}$. We can simply let $X(\Sigma') = X(\Sigma'')$ and either (re)interpret the new constant as the intersection or union of two closed sets in $X(\Sigma'')$ if θ is in some Σ_{5n+1} or, if θ is an element of some Σ_{5m+2} or Σ_{5m+3}, using the fact that the space $X(\Sigma'')$ is normal find (re)interpretations for the newly added constants, in an obvious way.

3.2.2. $\theta \in \{\Sigma_{5m+4} : m < \omega\}$. Suppose the preamble of θ is true in the model $(2^{X(\Sigma'')}, \mathcal{I})$, where θ is the following sentence

$$\begin{aligned}\theta \quad = \quad & [(a \sqcap b \sqcap c = \mathbf{0}) \to (a \sqcap x = a) \wedge (b \sqcap y = b) \wedge \\ & \wedge (c \sqcap z = c) \wedge (x \sqcap y \sqcap z = \mathbf{0}) \wedge (x \sqcup y \sqcup z = \mathbf{1})].\end{aligned}$$

If a has a zero interpretation then we can choose $x = \mathbf{0}$, $y = \mathbf{1}$ and $z = \mathbf{1}$, and this interpretation of x, y and z makes sure that θ holds in the model $(2^{X(\Sigma'')}, \mathcal{I})$. So we may assume that a, b and c have non zero interpretations.

As the space $X(\Sigma'')$ is metric, we can assume that we have a metric ρ on $X(\Sigma'')$. Moreover we can assume that ρ is bounded by 1.

Consider the following function f from $X(\Sigma'')$ to $\mathbb{R}^3$

$$f(x) = (\kappa_a(x), \kappa_b(x), \kappa_c(x)),$$

where $\kappa_a : X(\Sigma'') \to [0,1]$ is defined by

$$\kappa_a(x) = \frac{\rho(x,a)}{\rho(x,a) + \rho(x,b) + \rho(x,c)},$$

and κ_b and κ_c are like κ_a, but with a interchanged with b and c respectively. Then $f[X(\Sigma'')]$ is a subset of the triangle $T = \{(t_1,t_2,t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 1 \text{ and } t_1, t_2, t_3 \geq 0\}$.

The space $X(\Sigma'')$ is embedded in the space $X(\Sigma'') \times T$ by the graph of f (in other words the embedding is defined by $x \mapsto (x, f(x))$). Let us denote this embedding by g.

Consider the space $\partial T \times [0,1]$, where $\partial T = T \setminus \mathrm{int}(T)$ in $\mathbb{R}^3$. Let h be the map from $\partial T \times [0,1]$ onto T defined by

$$h((x,t)) = x(1-t) + t(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

The map h restricted to $\partial T \times [0,1)$ is a homeomorphism between $\partial T \times [0,1)$ and $T \setminus \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.

We define $X(\Sigma')$ as the space

$$X(\Sigma') = (\mathrm{id} \times h)^{-1}[g[X(\Sigma'')]].$$

Let us (re)interpret the constants k in K in the following way:

$$\mathcal{I}(k) := \mathcal{I}(k) \times (\partial T \times [0,1]) \cap X(\Sigma') \ (= (\mathrm{id} \times h)^{-1}[g[\mathcal{I}(k)]]).$$

Remark 4. *For future reference we note that, as the inverse images of points $(x, (t_1, t_2, t_3))$ under the map $\mathrm{id} \times h$ are points for $(x, (t_1, t_2, t_3))$ in $X(\Sigma'') \times T$ for which $(t_1, t_2, t_3) \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and equal to $\{x\} \times \partial T \times \{1\}$ for those $(x, (t_1, t_2, t_3))$ in $X(\Sigma'') \times T$ for which $(t_1, t_2, t_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we have that the map $\mathrm{id} \times h : X(\Sigma'') \times (\partial T \times [0,1]) \to X(\Sigma'') \times T$ is monotone. Furthermore it is also closed.*

We did nothing to disturb the truth or falsity of the sentences Σ'' in the model $(2^{X(\Sigma'')}, \mathcal{I})$ as $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$ and $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$ for any function f and any sets A and B.

So we have that $(2^{X(\Sigma')}, \mathcal{I})$ is a model for Σ''.

Let A be the line segments between $(0,1,0)$ and $(0,0,1)$, B the line segment between $(1,0,0)$ and $(0,0,1)$ and C the line segment between $(1,0,0)$ and $(0,1,0)$. Now we (re)interpret x, y and z as follows

$$\begin{aligned}
\mathcal{I}(x) &:= X(\Sigma'') \times (A \times [0,1]) \cap X(\Sigma') \\
\mathcal{I}(y) &:= X(\Sigma'') \times (B \times [0,1]) \cap X(\Sigma') \\
\mathcal{I}(z) &:= X(\Sigma'') \times (C \times [0,1]) \cap X(\Sigma')
\end{aligned}$$

As is easily seen, this interpretation of the constants x, y and z makes the sentence θ a true sentence in the model $(2^{X(\Sigma')}, \mathcal{I})$. So $(2^{X(\Sigma')}, \mathcal{I}) \models \Sigma'$.

3.2.3. $\theta \in \{\Sigma_{5(m+1)} : m < \omega\}$. Suppose the preamble of θ is true in the model $(2^{X(\Sigma'')}, \mathcal{I})$, where θ is the sentence

$$\theta = \phi(a, b, c, d) \to \psi(a, b, c, d; x, y, z),$$

as in equation 6.

If the interpretation of a is zero we can simply take $x = y = \mathbf{0}$ and $z = \mathbf{1}$ to make $(2^{X(\Sigma'')}, \mathcal{I})$ a model of θ. So again we may assume that the interpretations of a, b, c and d are nonzero.

To show that Σ' is a consistent set of sentences we are going to use an idea from [3].

With the aid of Urysohn's lemma we can find a continuous function $f : X(\Sigma'') \to [0,1]$ such that $f(\mathcal{I}(a)) \subset \{0\}$, $f(\mathcal{I}(b)) \subset \{1\}$, $f(\mathcal{I}(c)) \subset [0, \frac{1}{2}]$ and $f(\mathcal{I}(d)) \subset [\frac{1}{2}, 1]$.

Let P denote the (closed and connected) subset of $[0,1] \times [0,1]$ given by

$$P = \{\frac{1}{4}\} \times [0, \frac{2}{3}] \cup [\frac{1}{4}, \frac{1}{2}] \times \{\frac{2}{3}\} \cup \{\frac{1}{2}\} \times [\frac{1}{3}, \frac{2}{3}] \cup [\frac{1}{2}, \frac{3}{4}] \times \{\frac{1}{3}\} \cup \{\frac{3}{4}\} \times [\frac{1}{3}, 1].$$

Let $X^+ \subset [0,1] \times X(\Sigma'')$ denote the pre-image of the set P under the function $\mathrm{id} \times f$:

$$X^+ = \{(t, x) \in [0,1] \times X(\Sigma'') : (t, f(x)) \in P\}.$$

As P is closed and $\mathrm{id} \times f$ is continuous we have that X^+ is a compact metric space. Define the (continuous) map $\pi : X^+ \to X(\Sigma'')$ by $\pi((t, x)) = x$ for every $(t, x) \in X^+$.

Lemma 1. *There exists a unique component C of X^+ such that $\pi[C] = X(\Sigma'')$.*

Proof. Suppose we have closed sets F and G such that $X^+ = F + G$. Define subsets A_i, B_i of X, where $i \in \{0, 1, 2\}$, by

$$\begin{aligned} A_0 &= \{x \in X(\Sigma'') : (\frac{1}{4}, x) \in F\},\ B_0 = \{x \in X(\Sigma'') : (\frac{1}{4}, x) \in G\} \\ A_1 &= \{x \in X(\Sigma'') : (\frac{1}{2}, x) \in F\},\ B_1 = \{x \in X(\Sigma'') : (\frac{1}{2}, x) \in G\} \\ A_2 &= \{x \in X(\Sigma'') : (\frac{3}{4}, x) \in F\},\ B_2 = \{x \in X(\Sigma'') : (\frac{3}{4}, x) \in G\} \end{aligned}$$

It is clear that $A_i \cap B_i = \emptyset$ for every $i \in \{0, 1, 2\}$.

Claim 1. *The following holds*

(1) *For every $x \in (A_0 \cap B_1) \cup (B_0 \cap A_1)$ we have $f(x) < \frac{2}{3}$.*
(2) *For every $x \in (A_1 \cap B_2) \cup (B_1 \cap A_2)$ we have $f(x) > \frac{1}{3}$.*

Proof. As the proofs of the statements are very similar we will only prove the first statement.

If $x \in A_0 \cap B_1$ or $x \in B_0 \cap A_1$ then $f(x) \leq \frac{2}{3}$. As $f(x) = \frac{2}{3}$ is impossible, we are done. □

Let us define the following closed sets A^* and B^* of $X(\Sigma'')$ by

$$\begin{aligned} A^* = \bigcup\ &\{f^{-1}[0, \frac{1}{3}] \cap A_0, f^{-1}[\frac{2}{3}, 1] \cap A_2, A_0 \cap A_1 \cap A_2, \\ &A_0 \cap B_1 \cap B_2, B_0 \cap B_1 \cap A_2\} \\ B^* = \bigcup\ &\{f^{-1}[0, \frac{1}{3}] \cap B_0, f^{-1}[\frac{2}{3}, 1] \cap B_2, B_0 \cap B_1 \cap B_2, \\ &B_0 \cap A_1 \cap A_2, A_0 \cap A_1 \cap B_2\} \end{aligned}$$

The sets A^* and B^* are disjoint closed subsets of $X(\Sigma'')$ and their union is the whole of $X(\Sigma'')$. As $X(\Sigma'')$ is connected one of these sets must be empty. So without loss of generality we can assume that $B^* = \emptyset$.

We see now that $\pi[F] = X(\Sigma'')$ and that $\pi[G] \subset f^{-1}[\frac{1}{3}, \frac{2}{3}]$. It follows that whenever C is a clopen subset of X^+ then either $\pi[C] = X(\Sigma'')$ and $\pi[X^+ \setminus C] \subset f^{-1}[\frac{1}{3}, \frac{2}{3}]$ or it is the other way around. This shows that $\mathcal{F} = \{C : C \text{ is clopen and } \pi[C] = X(\Sigma'')\}$ is an ultrafilter in the family of clopen subsets of X^+; its intersection $\bigcap \mathcal{F}$ is the unique component of X^+ that is mapped onto $X(\Sigma'')$. We let C be this component. This ends the proof of lemma 1. □

Let $X(\Sigma')$ be the unique component C of X^+ that is mapped onto $X(\Sigma'')$ by the map π with the subspace topology.

In $2^{X(\Sigma')}$, the constants x, y and z that will make the sentence θ true will have the following (re)interpretations:

$$\begin{aligned} \mathcal{I}(x) &= \{(t, x) \in X(\Sigma') : t \in [0, \frac{3}{8}]\}, \\ \mathcal{I}(y) &= \{(t, x) \in X(\Sigma') : t \in [\frac{3}{8}, \frac{5}{8}]\} \text{ and} \\ \mathcal{I}(z) &= \{(t, x) \in X(\Sigma') : t \in [\frac{5}{8}, 1]\}. \end{aligned}$$

The (re)interpretation of the constants in K will be as follows.

$$\mathcal{I}(k) := [0, 1] \times \mathcal{I}(k) \cap X(\Sigma') \ (= \pi^{-1}[\mathcal{I}(k)] \cap X(\Sigma')).$$

As π maps C onto $X(\Sigma'')$ we have that $(2^{X(\Sigma')}, \mathcal{I})$ is a model of Σ', as the truth or falsity of sentences in Σ'' are not affected by the new interpretation of the constants.

4. The Maćkowiak-Tymchatyn theorem

Apart from the weakly confluent property of the continuous map we have proven the Maćkowiak-Tymchatyn theorem, theorem 1. To make sure that the continuous map following from the previous section is weakly confluent, we must consider all the subcontinua of the space X.

We let $\hat{K}$ be the following set

$$\hat{K} = \bigcup_{-2 \le n < \omega} \hat{K}_n = \bigcup_{-2 \le n < \omega} \{k_{n,\alpha} : \alpha < |2^X|\}.$$

We will construct a set

$$\hat{\Sigma} = \bigcup_{-1 \le n < \omega} \hat{\Sigma}_n$$

of sentences in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ similar as in the previous section such that given any model $\mathfrak{A} = (A, \mathcal{I})$ of $\hat{\Sigma}$, the set $L(\mathfrak{A}) = \mathcal{I} \restriction \hat{K}$ will be the universe of some normal distributive and disjunctive lattice such that

1. $L(\mathfrak{A})$ is a model of the sentences 3, 4 and $\mathrm{conn}(\mathbf{1})$,
2. the lattice 2^X is embedded into $L(\mathfrak{A})$ so there exists a continuous map f from $wL(\mathfrak{A})$ onto X,
3. for every subcontinuum of X there exists a subcontinuum of $wL(\mathfrak{A})$ that is mapped onto it by f.

4.1. A weakly confluent map.

We let $\hat{K}_{-1} = \{k_{-1,\alpha} < |2^X|\}$ correspond to the set $2^X = \{x_\alpha : \alpha < |2^X|\}$ in such a way that the set $\mathcal{C}(X)$ of all the subcontinua of X corresponds to the set $\{x_\alpha : \alpha < \beta\}$ for some ordinal number $\beta < |2^X|$. Let the set of sentences $\hat{\Sigma}_0$ in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}_{-1}$ correspond to $\triangle_{2^X}$, the diagram of the lattice 2^X.

We want to define a set of sentences Σ_{-1} in $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K_{-2} \cup K_{-1}$ that will make sure that if $\mathfrak{A}$ is a model of $\hat{\Sigma}$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ then we have for every subcontinuum in X a subcontinuum of $wL(\mathfrak{A})$ that will be mapped onto it by the continuous onto map we get by the fact that 2^X is embedded in the lattice $L(\mathfrak{A})$.

$$\begin{aligned}
\hat{\Sigma}^0_{-1} &= \{\mathrm{conn}(k_{-2,\alpha}) \wedge (k_{-2,\alpha} \sqcap k_{-1,\alpha} = k_{-2,\alpha})) : \alpha < \beta\} \\
\hat{\Sigma}^1_{-1} &= \{(\mathrm{conn}(k_{-2,\alpha}) \wedge (k_{-2,\alpha} \sqcap k_{-1,\gamma} = k_{-2,\alpha})) \rightarrow \\
&\quad \rightarrow (k_{-1,\alpha} \sqcap k_{-1,\gamma} = k_{-1,\alpha}) : \alpha < \beta,\ \gamma < |2^X|\} \\
\hat{\Sigma}^2_{-1} &= \{k_{-2,\gamma} = \mathbf{0} : \beta \le \gamma < |2^X|\}.
\end{aligned}$$

And define the set of sentences Σ_{-1} as

$$\hat{\Sigma}_{-1} = \hat{\Sigma}^0_{-1} \cup \hat{\Sigma}^1_{-1} \cup \hat{\Sigma}^2_{-1}. \tag{7}$$

Suppose $\mathfrak{A}$ is a model of $\hat{\Sigma}$. The set $\hat{\Sigma}^0_{-1}$ will make sure that for every subcontinuum C of X there is some subcontinuum C' of $wL(\mathfrak{A})$ that is mapped into C by the continuous onto map f we get from theorem 3 and the fact that 2^X is embedded into $wL(\mathfrak{A})$. The set $\hat{\Sigma}^1_{-1}$ will then make sure that C' is in fact mapped onto C by the map f.

Let us further construct the sets $\hat{\Sigma_n}$ for $0 < n < \omega$ in the same manner as we have constructed the set Σ_n in the previous section. So that if we have a model $\mathfrak{A}$ of $\hat{\Sigma}$, the lattice $L(\mathfrak{A})$ will be a normal distributive and disjunctive lattice that models the sentences 3, 4 and conn($\mathbf{1}$).

To prove the consistency of $\hat{\Sigma}$ it is enough to prove the following lemma.

Lemma 2. *For every finite $\Sigma' \in [\hat{\Sigma}]^{<\omega}$ there is a metric continuum $X(\Sigma')$, and an interpretation function $\mathcal{I} : \hat{K} \to 2^{X(\Sigma')}$ such that $(2^{X(\Sigma')}, \mathcal{I})$ is a model for Σ'.*

Proof. Suppose we have a metric continuum $X(\Sigma'')$ for every subset Σ'' of a given $\Sigma' \in [\Sigma]^{<\omega}$ such that there is an interpretation function $\mathcal{I}_{\Sigma''} : \hat{K} \to 2^{X(\Sigma'')}$ such that $(2^{X(\Sigma'')}, \mathcal{I}_{\Sigma''}) \models \Sigma''$. We want to show that there exists a metric continuum $X(\Sigma')$ and an interpretation function $\mathcal{I}_{\Sigma'} : \hat{K} \to 2^{X(\Sigma')}$ such that $(2^{X(\Sigma')}, \mathcal{I}_{\Sigma'}) \models \Sigma'$.

Let θ be an $\sqsubset$-maximal sentence in Σ' that is of interest (see remark 3). If θ is an element of $\hat{\Sigma}_0$, $\hat{\Sigma}_{5n+1}$, $\hat{\Sigma}_{5n+2}$ or $\hat{\Sigma}_{5n+3}$ then we can choose $X(\Sigma') = X(\Sigma'')$ and redefine the interpretation function $\mathcal{I}$ in a natural way to obtain the wanted result. So let us suppose that θ is an element of $\hat{\Sigma}_{-1}$, $\hat{\Sigma}_{5n+4}$ or $\hat{\Sigma}_{5(n+1)}$ for some $n < \omega$.

If θ is an element of $\hat{\Sigma}_{-1}$. Then, as θ is the $\sqsubset$-maximal sentence in Σ' of interest, no $\phi \in \Sigma'$ is an element of $\hat{\Sigma}_{5n+4}$ or $\hat{\Sigma}_{5(n+1)}$ for any $n < \omega$. As 2^X is a normal, distributive and disjunctive lattice and as X is a continuum, we have that the lattice 2^X with an obvious interpretation function $\mathcal{I}$ is even a model for $\triangle_{2^X} \cup \hat{\Sigma}_{-1} \cup \Sigma'$.

Suppose now that θ is an element of $\hat{\Sigma}_{5n+4}$ for some $n < \omega$. If we look at the construction in subsection 3.2.2 we know that the function $(id \times h)$ is a closed monotone map from $X(\Sigma'')$ onto $X(\Sigma')$. So inverse images of connected sets are connected and all the sentences of $\hat{\Sigma}_{-1}$ in Σ'' that were true (false) in the model $(2^{X(\Sigma'')}, \mathcal{I}_{\Sigma''})$, stay true (resp. false) in the model $(2^{X(\Sigma')}, \mathcal{I})$ as we get from subsection 3.2.2.

Finally, suppose that θ is an element of some $\hat{\Sigma}_{5(n+1)}$. Lets take a look at the construction of $X(\Sigma')$ in subsection 3.2.3. Let π be the map of $X(\Sigma')$ onto $X(\Sigma'')$ as given in subsection 3.2.3. Consider the following lemma.

Claim 2. *For every connected subset A of $X(\Sigma'')$ there exists a connected set $C(A) \subset C = X(\Sigma')$ such that $\pi[C(A)] = A$.*

Proof. Suppose we have $A \subset X(\Sigma'')$ connected. If we look at the image of A under the function f there are a number of possibilities:

(1) $f[A] \subset [0, \frac{2}{3}]$ and $f[A] \cap [0, \frac{1}{3}) \neq \emptyset$ ($f[A] \subset [\frac{1}{3}, 1]$ and $f[A] \cap (\frac{2}{3}, 1] \neq \emptyset$),
(2) $f[A] \subset [\frac{1}{3}, \frac{2}{3}]$,
(3) $f[A] \setminus [0, \frac{2}{3}] \neq \emptyset \neq f[A] \setminus [\frac{1}{3}, 1]$

In case 1 we have $\{\frac{1}{4}\} \times A$ ($\{\frac{3}{4}\} \times A$) is a connected subset of X^+ which must intersect the component C, as every other component is mapped onto some subset of $X(\Sigma'')$, which is mapped into $[\frac{1}{3}, \frac{2}{3}]$ by the function f.

In case 2 we have that the component C must intersect at least one of the connected subsets $\{\frac{1}{4}\} \times A$, $\{\frac{1}{2}\} \times A$ or $\{\frac{3}{4}\} \times A$, as C is mapped onto $X(\Sigma'')$, $X(\Sigma'')$ is connected and f is continuous.

In case 3 we can, as above, assuming $A^+ (= \pi^{-1}[A]) = F + G$, construct closed and disjoint subsets A^* and B^* of A which cover it. Again the image under π is either all of A or a proper subset of A. The (unique) piece that maps onto the whole of A must intersect the set C, and so is contained in it.

This ends the proof of the claim. □

We have that $(2^{X(\Sigma')}, \mathcal{I}_{\Sigma'}) \models \Sigma' \setminus \hat{\Sigma}_{-1}$, we now define a new interpretation function $\mathcal{I}$ on $\hat{K}$ to $2^{X(\Sigma')}$ such that $(2^{X(\Sigma')}, \mathcal{I})$ will be a model for Σ'. Note that the set of constants that are mentioned in the set Σ' is a finite subset of $\hat{K}$, and let $\hat{K}(\Sigma')$ denote this finite subset. We will define the interpretation under $\mathcal{I}$ of the constants in $\hat{K}(\Sigma')$ 'from the bottom up'.

By claim 2 for every $k_{-2,\alpha} \in \hat{K}(\Sigma')$ such that $C = \mathcal{I}_{\Sigma''}(k_{-2,\alpha})$ is a connected subset of $X(\Sigma'')$ we can find a connected subset C' of $X(\Sigma')$ that maps onto C by the map π. Let the $\mathcal{I}$ interpretation of the constant $k_{-2,\alpha}$ be this connected set C' in $X(\Sigma')$.

For all those k in $\hat{K}(\Sigma') \cap \hat{K}_{-2}$ that have no connected interpretation in $X(\Sigma')$ and for all the constants k in $\hat{K}(\Sigma') \cap \hat{K}_{-1}$ the interpretation under $\mathcal{I}$ will be the same as the interpretation under $\mathcal{I}_{\Sigma'}$. So for those $k \in \hat{K}(\Sigma')$ we have

$$\mathcal{I}(k_{-1,\alpha}) = \mathcal{I}_{\Sigma'}(k_{-1,\alpha}) = \pi^{-1}[\mathcal{I}_{\Sigma''}(k_{-1,\alpha})] \cap X(\Sigma').$$

The interpretations of the rest of the constants in $\hat{K}(\Sigma')$ will follow from the interpretations of the constants we have just defined, because the interpretation of every constant depends on just a finite set of other constants and we just have to make sure that we define their interpretation in the right order.

As $\mathcal{I}(k) \subset \mathcal{I}_{\Sigma'}(k)$ for all $k \in \hat{K}(\Sigma')$ and for all $k \in \hat{K}(\Sigma') \cap \hat{K}_{-2}$ such that $\mathcal{I}_{\Sigma'}(k)$ is connected $\mathcal{I}(k)$ is also connected we have that all the sentences of $\hat{\Sigma}_{-1}$ true (false) in $(2^{X(\Sigma'')}, \mathcal{I}_{\Sigma''})$ are true (false) in $(2^{X(\Sigma'')}, \mathcal{I})$. The true or falseness of the other sentences in Σ' have not been affected by the new interpretation function $\mathcal{I}$, and we have completed the proof. □

4.2. **The Maćkowiak-Tymchatyn theorem.** As we have seen in the previous section the set of sentences $\hat{\Sigma}$ in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup \hat{K}$ is consistent. Let $\mathfrak{A}$ be a model for $\hat{\Sigma}$. This model gives us a normal distributive and disjunctive lattice $L(\mathfrak{A})$ which models the sentences 3, 4 and conn($\mathbf{1}$). There also exists an embedding of 2^X into this lattice $L(\mathfrak{A})$ (remember that we showed that, with K_{-2} an enumeration of the lattice 2^X the set Σ is a consistent set of sentences in the language $\{\sqcap, \sqcup, \mathbf{0}, \mathbf{1}\} \cup K$). All this implies that the Wallman space $wL(\mathfrak{A})$, is a one-dimensional hereditarily indecomposable continuum which admits a weakly confluent surjection onto the metric continuum X.

Now we only have to make sure that there exists such a space that is of countable weight to complete the proof of the Maćkowiak Tymchatyn theorem.

Theorem 6. [3] *Let $f : Y \to X$ be a continuous surjection between compact Hausdorff spaces. Then f can be factored as $h \circ g$, where $Y \xrightarrow{g} Z \xrightarrow{h} X$ and Z has the same weight as X and shares many properties with Y (for instance, if Y is one-dimensional so is X or if Y is hereditarily indecomposable, so is X).*

Proof. Let $\mathcal{B}$ a minimal sized lattice-base for the closed sets of X, and identify it with its copy $\{f^{-1}[B] : B \in \mathcal{B}\}$ in 2^Y. By the Löwenheim-Skolem theorem there is an elementary sublattice of 2^Y, of the same cardinality as $\mathcal{B}$ such that $\mathcal{B} \subset D \prec 2^Y$. The space wD is as required. □

Applying this theorem to the space $wL(\mathfrak{A})$ and the weakly confluent map $f : wL(\mathfrak{A}) \to X$ we get a one-dimensional hereditarily indecomposable continuum wD which admits a weakly confluent map onto the space X and moreover the weight of the space wD equals the weight of the space X. This is exactly what we were looking for.

5. A topological proof of the Maćkowiak-Tymchatyn theorem

5.1. **the Maćkowiak-Tymchatyn theorem.** After the above proof was found we realized that it could be transformed into a purely topological proof, which we shall now describe.

Let X be a metric continuum. We are going to define a inverse sequence of metric continua with onto bonding maps $\{\langle X_n, f_n\rangle : n < \omega\}$,

$$X = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} X_n \xleftarrow{f_{n+1}} \cdots ,$$

in such a way that the inverse limit space X_ω

$$X_\omega = \varprojlim\{\langle X_n, f_n\rangle : n < \omega\}$$

is a hereditarily indecomposable one-dimensional continuum of weight $w(X)$ such that $\pi_0 : X_\omega \to X$ is a weakly confluent and onto. Here, for every $n < \omega$ the continuous function π_n is defined by $\pi_n = \mathrm{proj}_n \restriction X_\omega : X_\omega \to X_n$, where $\mathrm{proj}_n : \Pi_{n<\omega} X_n \to X_n$ is the projection.

Let us furthermore define maps $f^n_m : X_n \to X_m$ for $m < n$ as

$$f^n_m = \begin{cases} f_{m+1} \circ f_{m+2} \circ \cdots \circ f_n, & \text{if } m+1 < n \\ f_{m+1} \text{ if } m+1 = n. \end{cases}$$

The following lemma is well known.

Lemma 3. *The family of all sets of the form $\pi_n^{-1}(F)$, where F is an closed subset of the space X_n and n runs over a subset N cofinal in ω, is a base for the closed sets of the limit of the inverse sequence $\{\langle X_n, f_n\rangle : n < \omega\}$. Moreover, if for every $n < \omega$ a base $\mathcal{B}_n$ for the closed sets of space X_n is fixed, then the subfamily of those $\pi_n^{-1}(F)$ for which $F \in \mathcal{B}_n$, also is a base for the closed sets of X_ω.*

To make sure that the space X_ω is one-dimensional, it is sufficient to show that $\{\pi_k^{-1}(F) : F \in \mathcal{B}_k \text{ and } k < \omega\}$ is a model of sentence 4.

Let $s : \omega \to \omega \times \omega$ be an onto map in such a way that for every $n, m < \omega$ we have $s^{-1}(\langle n, m\rangle) \geq \max\{n, m\}$. For instance, we take an onto map $g : \omega \to \omega \times \omega \times \omega$ and given $g(n) = \langle p, q, r\rangle$ we define $s(n)$ by

$$s(n) = \begin{cases} \langle p, q\rangle \text{ if } n \geq \max\{p, q\}, \\ \langle 0, 0\rangle \text{ otherwise.} \end{cases}$$

Let $X_0 = X$ and suppose we have defined the pairs $\langle X_m, f_m\rangle$ and the bases $\mathcal{B}_m$ for every $m < n$. And suppose that we have also defined an enumeration of all the triples of $\mathcal{B}_m$ that have empty intersection for every $m < n$. Let $\{G^m_k : k < \omega\}$ be an enumeration of the set $\{G \in [\mathcal{B}_m]^3 : \bigcap G = \emptyset\}$ for $m < n$, write $G^m_k = \{a^m_k, b^m_k, c^m_k\}$.

The way we now define the space X_n and the onto map $f_n : X_n \to X_{n-1}$ will be as follows.

Suppose $s(n) = \langle k, m\rangle$, we consider the closed sets $(f^{n-1}_k)^{-1}(a^k_m)$, $(f^{n-1}_k)^{-1}(b^k_m)$ and $(f^{n-1}_k)^{-1}(c^k_m)$ of X_{n-1}. They have empty intersection.

If there exist sets x, y and z in $2^{X_{n-1}}$ such that

$$(f_k^{n-1})^{-1}(a_m^k) \subset x, (f_k^{n-1})^{-1}(b_m^k) \subset y \text{ and } (f_k^{n-1})^{-1}(c_m^k) \subset z,$$
$$x \cap y \cap z = \emptyset \text{ and } x \cup y \cup z = X_{n-1},$$

then we let $X_n = X_{n-1}$, $f_n = \mathrm{id}_{X_n}$ and we choose a countable base $\mathcal{B}_n$ for the closed sets of X_n such that $\mathcal{B}_{n-1} \cup \{x, y, z\} \subset \mathcal{B}_n$.

If there do not exist such sets x, y and z in 2^{X_n} then we use the construction in subsection 3.2.2 to find a (metric) continuum X_n and a continuous onto map $f_n : X_n \to X_{n-1}$, such that in X_n there are closed sets x, y and z in X_n such that

$$(f_k^n)^{-1}(a_m^k) \subset x,\ (f_k^n)^{-1}(b_m^k) \subset y,\ (f_k^n)^{-1}(c_m^k) \subset z,$$
$$x \cap y \cap z = \emptyset \text{ and } x \cup y \cup z = X_n$$

Let $\mathcal{B}_n$ be some countable base for the closed sets of X_n such that $\{(f_n)^{-1}(F) : F \in \mathcal{B}_{n-1}\} \subset \mathcal{B}_n$ and $x, y, z \in \mathcal{B}_n$.

After we have chosen the base $\mathcal{B}_n$ we can choose some enumeration of all the triples of $\mathcal{B}_n$ that have empty intersection.

We do not get into trouble by considering base elements of some base for the closed sets of X_ω which have not yet been defined, because this will not happen by the way the function s is defined and the bases $\mathcal{B}_n$ are chosen.

The limit $X_\omega(s)$ of the inverse sequence $\{\langle X_n, f_n\rangle : n < \omega\}$ is a continuum, as all the spaces X_n are continua, moreover, as the base $\{\pi_n^{-1}(F_k^n) : k, n < \omega\}$ of the space $X_\omega(s)$ models the sentence 4 we have that $X_\omega(s)$ is one-dimensional. As all the spaces X_n are compact and all the bonding maps f_n are onto, we have that $\pi_0 : X_\omega(s) \to X$ is a continuous onto map.

In a similar way we can construct a function $t : \omega \to \omega \times \omega \times \omega$, an onto map in such a way that for all $k, l, m < \omega$ we have $t^{-1}(\langle k, l, m\rangle) \geq \max\{k, l, m\}$, and use it together with the construction in subsection 3.2.3 to define, given $X_0 = X$, X_n and f_n so that $X_\omega(t)$, the inverse limit of the sequence $\{\langle X_n, f_n\rangle : n < \omega\}$ is a hereditarily indecomposable continuum which admits a continuous onto map, π_0 onto the space X.

We can combine these two constructions by defining the function r by letting $r(2n)$ equal $s(n)$ and $r(2n+1)$ equal $t(n)$ for every $n < \omega$. Define $X_0 = X$ and use the construction in subsection 3.2.2 if n is even and the construction in subsection 3.2.3 if n is odd to construct X_n and f_n.

Let $X_\omega(r)$ be the inverse limit of the inverse sequence $\{\langle X_n, f_n\rangle\}$ we have constructed with the aid of the function r as described in the previous paragraph.

As $\mathcal{B} = \{\pi_n^{-1}(F_k^n) : n, k < \omega\}$ is a base for the closed sets of $X_\omega(r)$ we see that $w(X_\omega(r)) = w(X) = \aleph_0$. The space $X_\omega(r)$ is a one-dimensional hereditarily indecomposable continuum as, by construction $\mathcal{B}$ is a model of the sentences 4 and 3. So by the following claim we have proven the Maćkowiak-Tymchatyn theorem.

Claim 3. *The map π_0 is a weakly confluent map from $X_\omega(r)$ onto X.*

Proof. Suppose we have a subcontinuum C of the space X, we want to find a subcontinuum C' of $X_\omega(r)$ such that $\pi_0[C'] = C$. As, by construction, the f_{2n}'s are monotone closed maps from X_{2n} onto X_{2n-1} and the f_{2n+1}'s are weakly confluent, we can define an inverse sequence $\{\langle Y_n, g_n\rangle : n < \omega\}$ such that $Y_0 = C$ and Y_n is, for every n, some subcontinuum of $f_n^{-1}(Y_{n-1})$ that is mapped onto Y_{n-1} by the map f_n and the map g_n is the restriction of the map f_n to the subspace Y_n of X_n.

Let C' be the inverse limit of the inverse sequence $\{\langle Y_n, g_n\rangle : n < \omega\}$, so

$$C' = \varprojlim\{\langle Y_n, g_n\rangle : n < \omega\}.$$

We have that C' is a closed subspace of the space $X_\omega(r)$. Furthermore it is a continuum as it is an inverse limit of continua, so it is a subcontinuum of $X_\omega(r)$. As π_n maps C' onto the Y_n's we have proven the claim. □

5.2. **The extended Maćkowiak-Tymchatyn theorem.** Given a continuum X we will construct an inverse sequence $\{\langle X_\alpha, f_\alpha\rangle : \alpha < w(X)\}$ such that the inverse limit space Y is a hereditarily indecomposable continuum of weight $w(X)$ and $\dim(Y) = 1$ and there exists a weakly confluent map of the space Y onto X. This is a somewhat different proof than is given in the paper of Hart, Van Mill and Pol (see [3]).

$$X = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_\alpha} X_\alpha \xleftarrow{f_{\alpha+1}} \cdots \quad (\alpha < w(X)).$$

We are going to make sure that every X_α is a continuum of weight $w(X)$ and that there exist some base $\mathcal{B}_\alpha$ for the closed sets of X_α of cardinality $w(X)$ such that the base $\{\pi^{-1}(B) : B \in \mathcal{B}_\alpha, \alpha < w(X)\}$ for the closed sets of the space Y will show that Y is the desired continuum.

For $\beta < w(X)$ a limit ordinal we let X_β be the inverse limit of the sequence $\{\langle X_\gamma, f_\gamma\rangle : \gamma < \beta\}$, and we let $\mathcal{B}_\beta$ be the set $\{(\pi^\beta_\alpha)^{-1}(B) : B \in \mathcal{B}_\alpha,\ \alpha < \beta\}$. This is a base for the closed sets of X_β and $|\mathcal{B}_\beta| \leq w(X)$. Furthermore X_β is a continuum as it is an inverse limit of continua.

Suppose we have defined the continua X_β for $\beta \leq \alpha$ for some $\alpha < w(X)$, as well as the bases $\mathcal{B}_\beta$ for the closed sets of these spaces and for every $\beta \leq \alpha$ we also have defined an enumeration $\{G^\beta_\tau : \tau < \Gamma_\beta\}$ of the triples of elements of $\mathcal{B}_\beta \setminus \{\emptyset\}$ that have empty intersection, we write $G^\beta_\tau = \{a^\beta_\tau, b^\beta_\tau, c^\beta_\tau\}$. Here Γ_β is some ordinal number less than or equal to $w(X)$.

As in the previous section we can find a function $s : w(X) \to w(X) \times w(X)$ such that for every $\alpha, \beta < w(X)$ we have $s^{-1}(\langle\alpha, \beta\rangle) \geq \max\{\alpha, \beta\}$. To find $X_{\alpha+1}$ and f_α we do almost the same thing as we have done in the previous section. If $s(\alpha) = \langle\beta, \gamma\rangle$ we consider the closed sets $a = f^\alpha_\beta(a^\beta_\gamma)$, $b = f^\alpha_\beta(b^\beta_\gamma)$ and $c = f^\alpha_\beta(c^\beta_\gamma)$ of the space X_α.

If there exist x, y and z in 2^{X_α} such that $a \subset x$, $b \subset y$, $c \subset z$, $x \cap y \cap z = \emptyset$ and $x \cup y \cup z = X_\alpha$ then we let $X_{\alpha+1} = X_\alpha$ and $f_{\alpha+1} = \mathrm{id}_{X_{\alpha+1}}$.

If there are no such x,y and z in 2^{X_α} then we will do as in subsection 3.2.2, but as in that section we used a metric for X we have to slightly alter the proof there. As X_α is normal and $a \cap b \cap c = \emptyset$ we can find a continuous function $f_a : X_\alpha \to [0, 1]$ such that $f_a(a) \subset \{0\}$ and $f_a(b \cap c) \subset \{1\}$. Now, as $f_a^{-1}(\{0\}) \cap b \cap c = \emptyset$ we can find a continuous function $f_b : X_\alpha \to [0, 1]$ such that $f_b(b) \subset \{0\}$ and $f_b(f_a^{-1}(\{0\}) \cap c) \subset \{1\}$. Finally, since $f_a^{-1}(\{0\}) \cap f_b^{-1}(\{0\}) \cap c = \emptyset$ we can find a continuous function $f_c : X_\alpha \to [0, 1]$ such that $f_c(c) \subset \{0\}$ and $f_c(f_a^{-1}(\{0\}) \cap f_b^{-1}(\{0\})) \subset \{1\}$. Now define the function $f : X_\alpha \to \mathbb{R}^3$ by

$$f(x) = (\kappa_a(x), \kappa_b(x), \kappa_c(x)),$$

where $\kappa_a : X_\alpha \to [0, 1]$ is defined by

$$\kappa_a(x) = \frac{f_a(x)}{f_a(x) + f_b(x) + f_c(x)},$$

and κ_b and κ_c are likewise defined. The function f maps X_α into the triangle that is the convex hull of the points $\{(0,0,1),(0,1,0),(1,0,0)\}$ in $(R)^3$ just as in subsection 3.2.2 and from this point on we can follow the method in subsection 3.2.2 to find a continuum $X_{\alpha+1}$ and a continuous onto map $f_{\alpha+1}: X_{\alpha+1} \to X_\alpha$ such that there exist $x, y, z \in 2^{X_{\alpha+1}}$ such that $f^{-1}_{\alpha+1}(a) \subset x$, $f^{-1}_{\alpha+1}(b) \subset y$ and $f^{-1}_{\alpha+1}(c) \subset z$, $x \cap y \cap z = \emptyset$ and $x \cup y \cup z = X_{\alpha+1}$. Now let $\mathcal{B}_{\alpha+1}$ be a base for the closed sets of $X_{\alpha+1}$ such that $\{(f_{\alpha+1})^{-1}(B) : B \in \mathcal{B}_\alpha\} \cup \{x, y, z\} \subset \mathcal{B}_{\alpha+1}$ and $|\mathcal{B}_{\alpha+1}| \leq w(X)$. Enumerate the set of triples of $\mathcal{B}_{\alpha+1} \setminus \{\emptyset\}$ with empty intersection as $\{G^{\alpha+1}_\tau : \tau < \Gamma_{\alpha+1}\}$, where $\Gamma_{\alpha+1}$ is some ordinal number less than or equal to $w(X)$.

In a similar way we can find an (transfinite) inverse sequence such that the inverse limit is a hereditarily indecomposable continuum of the same weight as X and for which the map π_0 is a continuous onto map between the limit and the space X.

As in the previous section we can combine these two (we take care of the hereditary indecomposability at even ordinal stages and we take care that the dimension of the limit space will not exceed one at the odd ordinal stages), to find a transfinite inverse sequence such that the inverse limit is a one-dimensional hereditarily indecomposable continuum that admits a continuous map onto the space X. After some thought, as in the previous section we see that this continuous map is in fact a weakly confluent map.

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Faculty of Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands

E-mail address, K. P. Hart: `k.p.hart@its.tudelft.nl`

E-mail address, B.J. van der Steeg: `b.j.vandersteeg@its.tudelft.nl`

URL, K. P. Hart: `http://aw.twi.tudelft.nl/~hart`