Global optimization of arborescent multilevel inventory systems

Roberto González [1], Edmundo Rofman, Claudia Sagastizábal [2].

ABSTRACT

We consider the numerical resolution of hierarchical inventory problems under global optimization. First we describe the model as well as the dynamical stochastic system and the impulse controls involved. Next we characterize the optimal cost function and we formulate the Hamilton-Jacobi-Bellman equations. We present a numerical scheme and a fast algorithm of resolution, with a result on the speed of convergence. Finally, we apply the discretization method to some examples where we show the usefulness of the proposed numerical method as well as the advantages of operating under global optimization.

Key words. Global optimization, inventory problems, discrete Hamilton-Jacobi-Bellman equations, quasi-Variational inequalities, subsolutions and supersolutions.

[1] Beppo Levi Institute, Pellegrini 250, Rosario, Argentina.
[2] INRIA, Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France.

1. Introduction. We consider in this paper inventory problems originated when a set of N installations, or nodes, organized according to a given hierarchy, share the distribution of a product. The hierarchy relates manufacturers, wholesalers and small traders. For the whole system to operate optimally, a compromise must be found between stocking and ordering costs. As for the exterior installations, they must also face –and satisfy– an external stochastic demand.

Numerical resolution of such hierarchical inventory optimization problems may be focused under two main approaches:

- Global or centralized optimization
- Decentralization techniques

depending whether the interest is centered around obtaining optimal controls and costs or around providing practical decision rules. When performing global optimization for inventory systems, the impulse controls —given by the purchasing decisions— are set by a *central* manager, exterior to the system. It decides on the ordering installations as well as the time and the amount of the order to be placed. As for decentralized methods, each node makes decisions independently. Generally such a behaviour gives rise to *suboptimal* policies, since fixed costs are not shared by many installations. Sometimes, however, under rather strong assumptions, a decentralized procedure may produce global optimal policies. This technique has been proposed in [4] and [5] for a serial stochastic inventory system. Unfortunately, such a decomposition is only possible when there are no fixed costs involved, a situation which appears rarely in practice.

A broad variety of heuristical rules for decision can be found in [13], [10] and [6], they are decentralized techniques of simple computation but are *not optimal*. A vast review for serial and arborescent systems can be found in [4], [19] and [15], [18] and [3] respectively.

Concerning decentralized techniques, it could be argued in their favor that solving N unidimensional problems is much simpler than solving the –probably huge– N-dimensional problem produced when applying global controls. While the often impractical character of global approaches is largely true for classical numerical techniques (see [8] and [12]), this is no longer valid for a wide class of problems when applying our approach. We outline a fast algorithm here (for more details, see [9]). Indeed, our fast algorithm turns out to be very suitable for treating large scale problems and obtaining global *optimal* policies without excessive computer requirements. — most of the numerical results we present here have been computed on a PC. What is especially important then is the new possibility of extending *global* optimization to more complex systems, using when necessary high performance computers.

We study here single-product-systems, where the N-nodes net has an arborescent structure, with different *levels* or sets of nodes having the same status, but not necessarily the same predecessor.

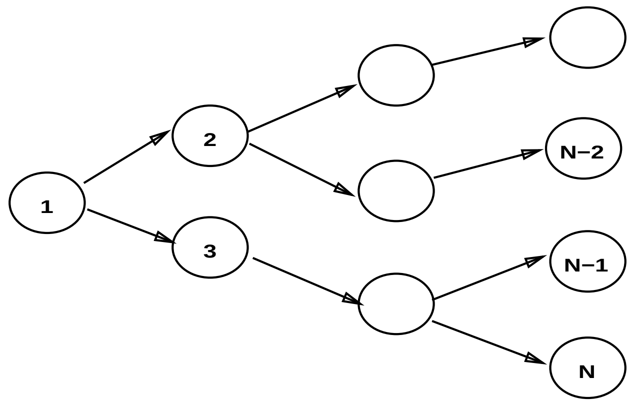

FIG. 1. *Arborescent chain*

The external stochastic demand may enter the system at any level of decision. Each node i places orders to its supplier $I(i)$, for $i = 2, \dots, N$, and the exterior supplies the highest installation 1. In the figure, N is the supplier of nodes $N-1$ and $N-2$: $N = I(N-1) = I(N-2)$. Node $N-2$ sells product to m, which must satisfy an external demand as well as those orders coming from 1 and 2. The purchasing decisions are made at any time and they modify instantaneously the state of the system $x = (x_1, x_2, \dots, x_N)$, where x_i represents the amount of stock at installation i. We are interested in the global control of stationary multi-level systems with state-space constraints. This means that we deal with *finite maximum stocking capacities*, that is: the trajectories of the controlled process must stay within a given subset of $\mathbb{R}^N$.

Our paper is organized as follows: we start with a general description of the problem, where we establish the main features of the model, we also describe the dynamical system and the controls involved. Next we study some properties of the optimal cost function and we formulate the associated Hamilton-Jacobi-Bellman system. We consider its solution via a constructive method which recursively generates a sequence of stopping-time problems for solving an equivalent fixed-point problem. We present a numerical scheme, we describe a fast algorithm of resolution and a result on the speed of convergence. Finally, we apply the discretization method to some examples where we show the usefulness of the proposed numerical method as well as the advantages of operating under global optimization.

2. Dynamic of the system.

2.1. General Description. We state first some notation and assumptions. Nodes receiving an external demand are gathered in the set $\mathcal{J} := \{j_1, j_2, \ldots\}$. Unsatisfied demands can be backlogged up to the maximum amount $|\underline{x}_j|$, for every $j \in \mathcal{J}$. At each installation i we denote the initial stock by x_i^0, by $\underline{x}_i$ its minimum capacity (negative stocks meaning accumulation of unsatisfied demands) and the maximum capacity by $\bar{x}_i$.

We denote stocking costs by $f : \mathbb{R}^N \to \mathbb{R}$; when $x \in \mathbb{R}^N$ has a negative coordinate, $f(\cdot)$ represents the backlogging cost. Purchasing, i.e. ordering, costs are denoted by $k : \mathbb{R}^N \to \mathbb{R}^+$; they represent the cost that installation N pays for ordering to the exterior as well as costs related to transfers between nodes. We suppose there exists a positive constant k_0

$$k(\nu) \geq k_0 > 0 \quad \text{for all } \nu\,, \tag{1}$$

this means that there exists a fixed ordering cost For every $j \in \mathcal{J}$, we also admit shortage costs, φ_j. They are originated when unsatisfied demands are so much accumulated that stock x_j falls beyond $\underline{x}_j$.

Let us now describe the dynamical system and its control. For every j, the demand has a Poisson distribution, with jump rate λ_j. The jump magnitude is a random variable, $\Delta\xi_j \in \mathbb{R}^+$, with conditional distribution given by the measure $m_j(\cdot)$. We suppose it is concentrated on a finite number of points: $\Delta\xi_j$ may just take a finite number of different values. Between two consecutive orders, each state x_j evolves as a one dimensional piecewise deterministic jump process (strictly speaking, it is a piecewise constant process; we refer to [1], [14], [16], [17] and references therein for a similar setting of the problem).

The demand arrives at times $\tau^\ell > 0$ for every unidimensional process ξ_j (hence $\ell = \ell(j)$). The stochastic process is then

$$\left[\begin{array}{lcl} x_j(0-) &=& x_j^0 \\ x_j(\tau_+^\ell) &=& \mathcal{P}_j\left(x_j(\tau_-^\ell) + \Delta\xi_j^\ell\right) \end{array}\right.$$

where x_j^0 is an arbitrary initial point, $\Delta\xi_j^\ell := \xi_j(\tau_+^\ell) - \xi_j(\tau_-^\ell)$ is the jump magnitude and

$$\mathcal{P}_j(\gamma_j) := \begin{cases} \gamma_j & \text{if } \gamma_j \in [\underline{x}_j, \bar{x}_j] \\ \underline{x}_j & \text{otherwise.} \end{cases}$$

Observe that $\mathcal{P}$ is the projection onto $[\underline{x}_j, \bar{x}_j]$ since the values over $\bar{x}_j$ are discarded by the admissible controls.

Controls will be purchasing orders of impulsive type set at times θ^ℓ and of amounts $\nu(\cdot)$. There is no delay for delivering. The set of admissible controls $\mathcal{A}$ will be the set of policies adapted to the capacity constraints of the demanding nodes as well as to the available stocks of their suppliers.

Accordingly, the corresponding trajectory $x(t)$ is a piecewise constant function of t, with jumps at times τ^k and θ^ℓ:

$$x_j(t) = x_j + \sum_{k=1}^{\infty} \Delta x_j^k \chi_{[0,t)}(\tau^k) + \sum_{\ell=1}^{\infty} \nu_j(\theta^\ell)\chi_{[0,t)}(\theta^\ell),$$

where χ is the characteristic function of a set, $\nu(t)$ an admissible control and x is any starting point. We have set $\Delta x_j^k := x_j(\tau_j^k+) - x_j(\tau_j^{k-1}-)$. For better legibility, in the sequel we extend $\mathcal{J}$ to $\{1, 2, \ldots, N\}$, defining by zero the demand parameters corresponding to "interior" nodes.

As for the expected cost associated to every decision, for an infinite horizon, it must include the (actualized) stocking cost as well as the ordering and shortage costs. This total average cost is expressed in the following formula:

$$J(x,\nu(\cdot)) = E\left[\int_0^\infty e^{-\alpha s} f(x(s))ds + \sum_{\substack{j=1,N \\ \ell=1,\infty}} e^{-\alpha\theta^\ell} k(\nu(\theta^\ell)) + \sum_{\substack{j=1,N \\ \ell=1,\infty}} e^{-\alpha\tau^\ell} \Phi_j(x_j(\tau_-^\ell) + \Delta\xi_j^\ell))\right] \tag{2}$$

where $\Phi_j(\gamma_j) := \begin{cases} \varphi_j(\underline{x}_j - \gamma_j) & \text{if } \gamma_j < \underline{x}_j \\ 0 & \text{otherwise} \end{cases}$ and α is the discount factor.

This expected cost is decomposed in three parts: an integral cost corresponding to the free evolution of the system, and two impulsive costs related respectively to the purchasing and to the stock rupture phenomena.

We want the system to operate optimally, the optimal cost is therefore given by

$$V(x) := \inf\{J(x,\nu(\cdot)) \; : \; \nu(\cdot) \in \mathcal{A}\}. \tag{3}$$

2.2. Characterization of the optimal cost. The process we have defined is a strong Markov process hence a version of Ito's formula holds (see [7]). Namely, for any $\psi \in C^1(\Omega)$,

$$E[\psi(x(T))] = \psi(x^0) + E\left[\int_0^T \sum_{k=1}^{N} \lambda_j \int_0^\infty [\psi(x(t) + \Delta\xi_j) - \psi(x(t))] m_j(d\Delta\xi_j)dt\right],$$

where $\Omega := \prod_{i=1}^{N} [\underline{x}_i, \bar{x}_i]$.

In order to find an analityc characterization of the optimal cost, we apply formally the Dynamic Programming Principle. We consider the cost to be paid in the interval $[0, T]$ for any small T: two exclusive alternatives are possible, whether the system evolves freely or a control is placed. Accordingly, we obtain

$$V(x) = \min \begin{cases} E\left[\int_0^T e^{-\alpha t} f(x(t))dt + \sum_{\substack{j=1,N \\ \ell=1,\infty}} e^{-\alpha\tau_-^\ell}\Phi_j(x_j(\tau_-^\ell) + \Delta\xi_j^\ell)\chi_{(0,T)}(\tau_-^\ell)e^{-\alpha T}V(x(T))\right] \\ \min_\nu [k(\nu) + V(x+\nu)] \end{cases}$$

(4)

We refer to [2] for more details.

So, similarly to what has been done in [16], (4) leads to

$$V(x) = \min \begin{cases} \dfrac{f(x)}{\alpha} + \sum_{j=1}^N \dfrac{\lambda_j}{\alpha} \int [V(\mathcal{P}_j(x_j + \Delta\xi_j)) - V(x) + \Phi_j(x_j + \Delta\xi_j)]m_j(d\Delta\xi_j) \\ \\ \min_\nu [k(\nu) + V(x+\nu)]\,. \end{cases}$$

Now, since $\int V(x)m_j(d\Delta\xi_j) = V(x)$ for all j, setting $\Lambda := \sum \lambda_j$, and shortening $q_j := \Delta\xi_j$ we obtain

$$(5) \quad V(x) = \min \begin{cases} \dfrac{f(x)}{\alpha+\Lambda} + \sum_{j=1}^N \dfrac{\lambda_j}{\alpha+\Lambda} \int [V(\mathcal{P}_j(x_j + q_j)) + \Phi_j(x_j + q_j)]m_j(dq_j) \\ \\ \min_{\nu\in A_x} [k(\nu) + V(x+\nu)]. \end{cases}$$

3. Equivalent formulations for the optimal problem.

3.1. Associated quasi-variational inequalities. We study here the system of inequalities associated to (5). Namely, we consider the following quasi-variational system:

$$(QVI) \qquad w(x) \le \min \begin{cases} \dfrac{f(x)}{\alpha+\Lambda} + \sum_{j=1}^N \dfrac{\lambda_j}{\alpha+\Lambda} \int [w(\mathcal{P}_j(x_j + q_j)) + \Phi_j(x_j + q_j)]m_j(dq_j) \\ \min_{\nu\in A_x} [k(\nu) + w(x+\nu)] \end{cases}$$

We will prove that solving (5) is equivalent to finding a particular solution of (QVI) (i.e. the maximum subsolution, see definition below). Next, we transformate the (QVI) into an equivalent fixed-point problem.

We introduce now some notation. We shorten the functional espace of continuous functions by $\mathcal{V}$. We define the linear operator $I : \mathcal{V} \to \mathcal{V}$ by

$$(I(u))(x) := \frac{1}{\alpha+\Lambda} f(x) + \sum_{j=1}^{N} \frac{\lambda_j}{\alpha+\Lambda} \int [u(\mathcal{P}_j(x_j+\Delta\xi_j)) + \Phi_j(x_j+\Delta\xi_j)] m_j(d\Delta\xi_j)\,.$$

For every ψ in $\mathcal{V}$, we define $\sigma : \mathcal{V} \to \mathcal{V}$ as the maximum subsolution of the stopping-time problem:

$$u \le W_\psi(u) := \min\{\psi, I(u)\}\,. \tag{6}$$

We call a *subsolution* any $u \in \mathcal{V}$ such that $u \le W_\psi(u)$ (respectively, u is a *supersolution* when $u \ge W_\psi(u)$).

LEMMA 3.1. *W_ψ defined in* (6) *is monotone and contractive, with contraction factor $\eta := \frac{\Lambda}{\alpha+\Lambda} < 1$.*

Proof. Let us first prove that $W_\psi(\cdot)$ is contractive:

$$\forall u, v \in \mathcal{V} \quad \|W_\psi(u) - W_\psi(v)\| \le \eta \|u - v\|\,.$$

Given a point $x \in \Omega$, we proceed by inspection of the different possibilities for $W_\psi(x)$.

When $(W_\psi(v))(x) = \psi(\mathrm{x})$, we have

$$(W_\psi(u) - W_\psi(v))(x) \le \psi(x) - \psi(x) = 0\,. \tag{7}$$

In the other case, when $(W_\psi(v))(x) = (I(v))(x)$,

$$\begin{aligned}(W_\psi(u) - W_\psi(v))(x) &\le (I(u) - I(v))(x) \\ &\le \textstyle\sum_{j=1}^{N} \frac{\lambda_j}{\alpha+\Lambda} \int [u(\mathcal{P}_j(x_j+\Delta\xi_j) - v(\mathcal{P}_j(x_j+\Delta\xi_j)] m_j(d\Delta\xi_j)\end{aligned} \tag{8}$$

therefore $W_\psi(u) - W_\psi(v) \le \frac{\Lambda}{\alpha+\Lambda}\|u - v\|$.

A similar inequality for $W_\psi(v) - W_\psi(u)$ can be shown, mutatis mutandis.

Let $u, v \in \mathcal{V}$ such that $u \le v$. We consider as before the two possible values for $(W_\psi(v))(x)$. From (7) and (8) we get $(W_\psi(u))(x) - (W_\psi(v))(x) \le 0 \quad \text{for all}\, x \in \Omega$ in both circumstances; we conclude $W_\psi(\cdot)$ is increasing. □

THEOREM 3.2. *Let $\sigma(\psi) \in \mathcal{V}$ be the unique solution of the fixed-point problem $u = W_\psi(u)$. Then*

(i) $\sigma(\psi)$ solves (6)

(ii) $\sigma(\psi)$ is the maximum subsolution and the minimum supersolution of (6)

(iii) $\sigma(\cdot)$ is increasing.

Proof. (i) is straightforward. Let us prove (ii). Because of Lemma 3.1, we have $W_\psi(\cdot)$ has a unique fixed-point $\sigma(\psi)$. This fixed-point can be computed recursively:

$$\sigma(\psi) = \lim_{m\to\infty} [W_\psi(u)]^m \,, \tag{9}$$

for u any initial point in $\mathcal{V}$.

We already know $\sigma(\psi)$ is a subsolution. Let $u \in \mathcal{V}$ be another subsolution: $u \leq W_\psi(u)$. Then the monotonicity of $W_\psi(\cdot)$ implies $W_\psi(u) \leq [W_\psi(u)]^2$; recursively, $u \leq [W_\psi(u)]^m \leq [W_\psi(u)]^{m+1}$ for any m. Passing to the limit, (9) implies that $\sigma(\psi)$ is the maximum subsolution. A symmetric argument can be used for proving $\sigma(\psi)$ is also the minimum supersolution.

We consider now (iii). Observe that $W_\psi(\cdot)$ is increasing in ψ: for $\psi_1 \leq \psi_2 \in \mathcal{V}$, we have $W_{\psi_1}(u) = \min\{\psi_1, I(u)\} \leq \min\{\psi_2, I(u)\} = W_{\psi_2}(u)$. Passing to the limit again, we obtain the desired inequality:

$$\sigma(\psi_1) = \lim_{m\to\infty} [W_{\psi_1}(u)]^m \leq \lim_{m\to\infty} [W_{\psi_2}(u)]^m = \sigma(\psi_2)\,. \qquad \square$$

3.2. The fixed-point problem. We consider now $\mathcal{M} : \mathcal{V} \to \mathcal{V}$ defined by

$$\forall x \in \Omega \quad (\mathcal{M}u)(x) := \min_{\nu\in\mathcal{A}}\{k(\nu) + u(x+\nu)\}\,,$$

and we analize its composition with σ, namely the operator $M := \sigma \circ \mathcal{M}$. We will show that M has a unique fixed-point which is the maximum subsolution of (QVI). Finally we present an algorithm for finding the solution and we establish its rate of convergence; similar to the bound obtained by Hanouzet-Joly in [11]. We always suppose $k(\cdot)$ has a positive inferior bound (recall (1)) and that f, φ_j for $j \in \mathcal{J}$, are Lipschitz continuous.

We start with some properties of M which will be useful in the sequel.

THEOREM 3.3. *With the notation above, the following properties hold:*

(i) $M : \mathcal{V} \to \mathcal{V}$ *is increasing.*

(ii) M *is concave :* $\forall u, v \in \mathcal{V}$ *and for any* $\theta \in [0,1]$,

$$\theta M(u) + (1-\theta)M(v) \leq M(\theta u + (1-\theta)v)$$

(iii) For every $u \in \mathcal{V}$, *take*

$$K(u) > \max\left\{\|u\|, \frac{1}{\alpha+\Lambda}(\|f\| + \sum_j \lambda_j\|\Phi_j\|)\right\} \geq 0 \tag{10}$$

and

$$\delta(u) := \min\left\{1, \frac{k_0}{2K}, \frac{1}{2} - \frac{\|f\| + \sum_j \lambda_j\|\Phi_j\|}{2\alpha K}\right\} \in]0,1]\,. \tag{11}$$

Then there exist $\underline{u}$,$\bar{u}$ in $\mathcal{V}$ such that $\underline{u} \le u \le \bar{u}$ and

$$\underline{u} + \delta(\bar{u} - \underline{u}) \le M(\underline{u}) \tag{12}$$

$$\forall v \le \bar{u} \quad M(v) \le \bar{u}\,, \tag{13}$$

throughout the inequalities are considered in a pointwise sense.

Proof. (i): Let $v_1, v_2 \in \mathcal{V}$ such that $v_1 \le v_2$; by definition, $\mathcal{M}(v_1) \le \mathcal{M}(v_2)$. From Theorem 3.2($iii$) $\sigma(\cdot)$ is increasing, hence so is M. (ii): We prove first that $\mathcal{M}(\cdot)$ is concave. For this, take $u, v \in \mathcal{V}$ and $\theta \in [0,1]$. We have

$$\begin{aligned} \mathcal{M}\left(\theta u + (1-\theta)v\right)(x) &= \min_{\nu}\{k(\nu) + \left(\theta u + (1-\theta)v\right)(x+\nu)\} \\ &= k(\hat{\nu}) + \left(\theta u + (1-\theta)v\right)(x+\hat{\nu}) \\ &= \theta\left[k(\hat{\nu}) + u(x+\hat{\nu})\right] + (1-\theta)\left[k(\hat{\nu}) + v(x+\hat{\nu})\right] \\ &\ge \theta\left(\mathcal{M}(u)\right)(x) + (1-\theta)\left(\mathcal{M}(v)\right)(x)\,, \end{aligned}$$

where $\hat{v}$ denotes the control realizing the minimum in (9). We analyze now the different values M can take.

When $M(u) \le \mathcal{M}(u)$ and $M(v) \le \mathcal{M}(v)$, the convex sum of $M(u)$ and $M(v)$ gives

$$\theta M(u) + (1-\theta)M(v) \le \theta\mathcal{M}(u) + (1-\theta)\mathcal{M}(v) \le \mathcal{M}(\theta u + (1-\theta)v)\,. \tag{14}$$

Whereas for the case $M(u) \le I(M(u))$ and $M(v) \le I(M(v))$, again from their convex sum we get

$$\theta M(u) + (1-\theta)M(v) \le \theta I(M(u)) + (1-\theta)I(M(v))\,. \tag{15}$$

Combine now (14) and (15) with the linearity of $I(\cdot)$ to conclude that $\theta M(u) + (1-\theta)M(v)$ is a subsolution of (6). Now, set $\psi := \mathcal{M}\left(\theta u + (1-\theta)v\right)$ and apply Theorem 3.2(ii):

$$\theta M(u) + (1-\theta)M(v) \le \sigma(\psi) = \sigma\left(\mathcal{M}(\theta u + (1-\theta)v)\right) = M(\theta u + (1-\theta)v)\,.$$

(iii) For any $u \in \mathcal{V}$, it holds $-\|u\| \le u \le \|u\|$. Then with $K := K(u)$ defined by (10), take $\underline{u} \equiv -K$ and $\bar{u} \equiv K$. Clearly, $\underline{u}$ and $\bar{u} \in \mathcal{V}$ and

$$\underline{u} \le u \le \bar{u}\,.$$

For proving (12), observe first that $\delta := \delta(u)$ gives

$$\underline{u} + \delta(\bar{u} - \underline{u}) = (2\delta - 1)K \tag{16}$$

It also holds

$$\underline{u} + \delta(\bar{u} - \underline{u}) = -K + \delta 2K \leq -K + \frac{k_0}{2K} 2K = -K + k_0 \leq \underline{u}(x+\nu) + k(\nu)\,,$$

for all $x \in \Omega$ and for all $\nu \in \mathcal{A}$; in particular for $\hat{\nu}$ such that $k(\hat{\nu}) + \underline{u}(x+\hat{\nu}) = \mathcal{M}(\underline{u})(x)$, it follows $\underline{u} + \delta(\bar{u} - \underline{u}) \leq \mathcal{M}(\underline{u})$. We will show that $\underline{u} + \delta(\bar{u} - \underline{u})$ is a subsolution:

$$\underline{u} + \delta(\bar{u} - \underline{u}) \leq W_\psi(\underline{u} + \delta(\bar{u} - \underline{u}))\,,$$

with $\psi := \mathcal{M}(\underline{u})$. Since $\sigma(\psi) = M(\underline{u} + \delta(\bar{u} - \underline{u}))$ is the maximum subsolution of (6), we will get $\underline{u} + \delta(\bar{u} - \underline{u}) \leq M(\underline{u})$.

For this, recall that

$$(2\delta - 1)K < -\frac{1}{\alpha + \Lambda}\{\|f\| + \sum_j \|\Phi_j\|\}\,. \tag{17}$$

Let us consider $I(\cdot)$:

$$\begin{array}{rcl} I(\underline{u} + \delta(\bar{u} - \underline{u})) & = & \frac{1}{\alpha+\Lambda} f + \sum_j \frac{\lambda_j}{\alpha+\Lambda} \int \{\underline{u} + \delta(\bar{u} - \underline{u}) + \Phi_j(\mathcal{P}_j(x_j + \Delta\xi_j))\} m(d\Delta\xi_j) \\ & \geq & \frac{1}{\alpha+\Lambda} f + \frac{\Lambda}{\alpha+\Lambda}\{\underline{u} + \delta(\bar{u} - \underline{u})\} + \sum_j \frac{\lambda_j}{\alpha+\Lambda} \Phi_j(\mathcal{P}_j(x_j + \Delta\xi_j)) m(d\Delta\xi_j) \\ & \geq & -\frac{1}{\alpha+\Lambda}\|f\| + \eta\{\underline{u} + \delta(\bar{u} - \underline{u})\} - \sum_j \frac{\lambda_j}{\alpha+\Lambda}\|\Phi_j\| \\ & = & -\frac{1}{\alpha+\Lambda}\{\|f\| + \sum_j \lambda_j \|\Phi_j\|\} + \eta(2\delta - 1)K \\ & > & (1+\eta)(2\delta - 1)K \qquad \text{[from(17)]} \\ & = & (1+\eta)\{\underline{u} + \delta(\bar{u} - \underline{u})\} \qquad \text{[from(16)]}\,. \end{array}$$

Since $1 + \eta > 1$, we are done.

Let us prove (13). Let $v \in \mathcal{V}$ such that $v \leq \underline{u}$. We already know $M(v) \leq M(\bar{u})$, by the monotonicity of M. We only need to prove then that $M(\bar{u}) \leq \bar{u}$. We have, for $\psi := \mathcal{M}(\bar{u})$ that, for any initial point $u \in \mathcal{V}$,

$$M(\bar{u}) = \lim_{m \to \infty} W_\psi^m(u)\,,$$

in particular, for $u := \bar{u}$. By definition of $\bar{u}$, i.e. of K, $W_\psi(\bar{u})$ cannot be equal to $\mathcal{M}(\bar{u})(= K_0 + \bar{u})$ but to $I(\bar{u})$ which in turn is bounded by $\bar{u}$. We get then $M(\bar{u}) = \lim_{m\to\infty} W_\psi^m(\bar{u}) \leq \bar{u}$ and this ends the proof. □

3.2.1. Convergence rates. Let us consider a standard algorithm for solving fixed-point problems, which we call *Algorithm 0*: for any given initial point u_0, we update the current iteration u by the formula $u_+ := Mu$. In next theorem we show that such a sequence of u's converges to the solution V of (QVI) (or equivalently, (5)) and we establish its rate of convergence.

THEOREM 3.4. *Let V be the optimal cost in* (5)*, and let $u \in \mathcal{V}$ be an arbitrary initial point. There exist constants $C(u) \in]K(u), 2K(u)[$ and $\delta(u) \in]0,1[$ such that*

$$\forall m \in \mathbb{N} \qquad \|M^m(u) - V\| \leq C(u)(1 - \delta(u))^m\,,$$

where $K(u)$ and $\delta(u)$ have been defined in (10) *and* (11) *respectively.*

Proof. Given u, and $K(u)$, set $\underline{u} \equiv -K(u)$ and $\bar{u} \equiv K(u)$. Let $v\,,\, w \in \mathcal{V}$ such that $\underline{u} \leq v\,,\, w \leq \bar{u}$; there exist $\theta\,,\, \tau \in [0,1]$ such that

$$\tau(\underline{u} - w) \leq v - w \leq \theta(v - \underline{u})\,.$$

Consider now the convex sum $z := (1-\theta)v + \theta\underline{u}$, we have $z = v - \theta(v - \underline{u}) \leq w$. Apply succesively the results from Theorem 3.3, namely the concavity of $M(\cdot)$, (12) and (13). We get

$$\begin{array}{rcl} M(w) & \geq & (1-\theta)M(v) + \theta M(\underline{u}) \\ & = & M(v) - \theta(M(v) - M(\underline{u})) \\ & \geq & M(v) - \theta[M(v) - \delta\bar{u} - (1-\delta)(\underline{u})] \\ & \geq & M(v) - \theta[M(v) - \delta M(v) - (1-\delta)\underline{u}] \\ & \geq & [1 - \theta(1-\delta)]M(v) - (1-\delta)\underline{u}\,, \end{array}$$

that is,

$$M(v) - M(w) \leq \theta(1-\delta)(M(v) - \underline{u})\,. \tag{18}$$

Proceeding in a similar way, but interchanging v and w, we obtain

$$\tau(1-\delta)[\underline{v} - M(w)] \leq M(v) - M(w)\,. \tag{19}$$

Putting together (18) and (19) give

$$\tau(1-\delta)[\underline{u} - M(w)] \leq M(v) - M(w) \leq (1-\delta)\theta[M(v) - \underline{u}]\,,$$

for any $v\,,\, w$ such that $\underline{u} \leq v\,,\, w \leq \bar{u}$. In particular, for $w = u$, and $v = V$: there exists τ, θ and δ such that

$$\tau(1-\delta)(\underline{V} - M(u)) \leq V - M(u) \leq (1-\delta)\theta(V - \underline{V})\,,$$

now, by induction, the monotony of $M(\cdot)$

$$\tau(1-\delta)^m(\underline{V} - M^m(u)) \leq V - M^m(u) \leq (1-\delta)^m\theta(V - \underline{V})\,.$$

From these last inequalities, we get our bound, with $C(u) := 2K\max(\tau, \theta)$. □

4. Numerical solution. We have characterized V as the maximum subsolution of (QVI). To solve the problem numerically, we approximate it by the maximum subsolution of a *discrete* (QVI).

4.1. The discretized problem. At each installation i we discretize the stock in N_i points, therefore $h_i := \frac{\bar{x}_i - \underline{x}_i}{N_i - 1}$ is the stepsize in the direction i and

$$\Omega_h := \prod_{i=1}^{N} \{\underline{x}_i + jh_i, \quad j = 0, \ldots, N_i - 1\}$$

is the discrete state space. As for the control space, we have a similar discretization, but with Nq_i points at each level: $H_i := \frac{\bar{x}_i - \underline{x}_i}{Nq_i - 1}$ is the corresponding "ordering" stepsize. We work on $\mathcal{W} \subset \mathcal{V}$, the espace of finite element functions on Ω_h, with first degree polynomials as basis functions.

We assume the probability distribution $m_j(\cdot)$ is discrete for each $j \in \mathcal{J}$: there exists a finite family $\mathcal{Z}_j$ such that for any function $\psi \in \mathcal{V}$, $\int \psi(\Delta\xi_j) m_j(d\Delta\xi_j) = \sum_{zj \in \mathcal{Z}_j} \psi(\Delta\xi_{zj}) m_{zj}$ with $\sum_{zj \in \mathcal{Z}_j} m_{zj} = 1$.

Accordingly, given $x^k := (x_1^k, x_2^k, \ldots, x_N^k)$ an arbitrary meshpoint in Ω_h, the approximation of $I(\cdot)$ is

$$(I_h(w))(x^k) := \frac{1}{\alpha + \Lambda} f(x^k) + \sum_{j=1}^{N} \frac{\lambda_j}{\alpha + \Lambda} \left[w(\mathcal{P}_j(x^k + \Delta\xi_{zj} e_j)) + \Phi_j(x_j^k + \Delta\xi_{zj}) \right] m_{zj} \, , \Big]$$

for all $w \in \mathcal{W}$, and e_j the canonical jth vector. When $\mathcal{P}_j(x^k + \Delta\xi_{zj} e_j) \notin \Omega_h$, we perform a linear interpolation on the jth-coordinate.

The discretized equation for $\mathcal{M}(\cdot)$ is

$$\mathcal{M}_h(w)(x^k) = \min_{\sum_i q_i \in Q(x^k)} \{ k(\sum_i q_i e_i) + w(x^k + \sum_i q_i(e_i - e_I)) \} \, ,$$

where I is the predecessor of ordering indexes i involved in the sum and $Q(x^k) := \prod_{i=1}^{N} \left\{ m_i H_i : i = 0, \cdots, \left[(x_i^k - \underline{x}_i)/H_i \right] \right\}$, with $[y]$ the nearest integer less than or equal to y. Again, when $x^k + \sum_i q_i(e_i - e_I) \notin \Omega_h$, we interpolate. Altogether, the discrete problem is

$$w(x^k) \leq \min \left\{ \begin{array}{l} \frac{1}{\alpha+\Lambda} f(x^k) + \sum\limits_{1 \leq j \leq N} \dfrac{\lambda_j}{\alpha + \Lambda} [w(\mathcal{P}_j(x^k + \Delta\xi_{zj} e_j)) + \Phi_j(x_j^k + \Delta\xi_{zj})] m_{zj} \\ \\ \min_{\sum_i q_i \in Q(x^k)} \{ k(\sum_i q_i e_i) + w(x^k + \sum_i q_i(e_i - e_I)) \} \end{array} \right. \qquad (QVI_h)$$

The problem of finding the maximum subsolution of (QVI_h) is equivalent to solving a fixed-point problem. We denote this solution by V_h. Indeed, arguing exactly as we have done for the continuous case, but working in the subspace $\mathcal{W}$ of $\mathcal{V}$, all the results of § 2.2 can be reproduced. Thus we have a discrete contractive operator $M_h : \mathcal{W} \to \mathcal{W}$ defined by the righthand side in (QVI_h). The fixed point of M_h can be iteratively computed by the algorithm:

Algorithm 0

Step 0 : Give $v_h^0 \in \mathcal{W}$ and set $m = 0$.
Step 1 : Define $v_h^{m+1} = M_h(v_h^m)$
Step 2 : Set $m = m + 1$, and go to Step 1.

Moreover, we can prove

THEOREM 4.1. *Let V_h be the discrete optimal cost in $(QVI)_h$, and let $u_h \in \mathcal{W}$ be an arbitrary initial point. There exist constants $C_h(u_h) \in]K_h(u_h), 2K_h(u_h)[$ and $\delta_h(u_h) \in]0,1[$ such that*

$$\forall m \in \mathbb{N} \qquad \|M_h^m(u_h) - V_h\| \leq C_h(u_h)(1 - \delta_h(u_h))^m \qquad \square$$

4.2. Convergence rates. We study now the speed of convergence of V_h to V when the discretization step tends to zero. We consider a family of associated stopping-time problems $(STP)^m$, $m = 1, \cdots$. Namely, for $u^0 \in \mathcal{V}$ an arbitrary initial point, define a sequence $\{u^m\}_m$ the maximum subsolutions of

$$u \leq \begin{cases} I(u^0) & \text{if } m = 1 \\ \min[I(u), \mathcal{M}(u^{m-1}] & \text{otherwise.} \end{cases} \qquad (STP)^m$$

Consider also the discrete sequence $\{u_h^m\}_m$, the maximum subsolutions of

$$u_h \leq \begin{cases} I_h(u_h^0) & \text{if } m = 1 \\ \min[I_h(u), \mathcal{M}_h(u_h^{m-1})] & \text{otherwise.} \end{cases} \qquad (STP)_h^m$$

where u_h^0 is an initial point in $\mathcal{W}$.

We establish a bound for the gap between both sequences. Following the notation of (6), $\psi := \mathcal{M}^{m-1}$ gives $u^m = M^m(u^0)$; similarly, $u_h^m = M_h^m(u_h^0)$.

LEMMA 4.2. *Assume k and V are Lipschitzian with Lipschitz constants L_k and L_V. Then*

$$\|M^m(u) - M_h^m(u_h)\| \leq \frac{\alpha + \Lambda}{\alpha} L_V h + m(L_k + L_V)h\,,$$

where $h := \max_i\{h_i, H_i\}$.

Proof. We proceed by induction on m. Given $x \in \Omega_h$, take $m = 0$, we have

$$u^0(x) - u_h^0(x) = \sum_j \frac{\lambda_j}{\alpha + \Lambda} \int [u^0(\mathcal{P}_j(x_j + \Delta\xi_j) - u_h^0(\mathcal{P}_j(x_j + \Delta\xi_j))] m_j(d\Delta\xi_j)\,.$$

Since $u_h^0 \in \mathcal{W}$ is a finite element, for every j there exist μ_{kj} such that $\sum_k \mu_{kj} = 1$ and

$$u_h^0(\mathcal{P}_j(x_j + \Delta\xi_j)) = \sum_k \mu_{kj} u_h^0(x_j^k)\,.$$

Thus we can write

$$\begin{array}{rcl} u^0(x) - u_h^0(x) & = & \sum_j \frac{\lambda_j}{\alpha+\Lambda} \int [u^0(\mathcal{P}_j(x_j + \Delta\xi_j) - \sum_k \mu_{kj} u_h^0(x_j^k)] m_j(d\Delta\xi_j) \\ & = & \sum_j \frac{\lambda_j}{\alpha+\Lambda} \int \{[u^0(\mathcal{P}_j(x_j + \Delta\xi_j) - \sum_k \mu_{kj} u^0(x_j^k)] m_j(d\Delta\xi_j) \\ & & \quad + \sum_k \mu_{kj}[u^0(x_j^k) - u_h^0(x_j^k)] m_j(d\Delta\xi_j)\}\,. \end{array}$$

The function V is Lipschitz-continuous (this can be easily proved, since all the functions involved: f, Φ_j and k are Lipschitzian) The functions u^m are also Lipschitz with the same Lipschitz constant, L_V. Taking norms, and using $\|\mathcal{P}_j(x_j + \Delta\xi_j) - x_j^k\| \le h$ for all k and j, we obtain

$$\|u^0 - u_h^0\| \le \sum_j \frac{\lambda_j}{\alpha + \Lambda}(L_V h + \|u^0 - u_h^0\|)\,,$$

that is,

$$\|u^0 - u_h^0\| \le \frac{\alpha + \Lambda}{\alpha} L_V h\,.$$

Let us now consider $m = 1$. Given a meshpoint $x \in \Omega_h$ and proceeding as for the case $m = 0$, when $u_h^1(x) = (I_h(u_h^0))(x)$ we get

$$u^1(x) - u_h^1(x) \le \sum_j \frac{\lambda_j}{\alpha + \Lambda}(L_V h + \|u^1 - u_h^1\|)\,.$$

The same bound is obtained for $u_h^1(x) - u^1(x)$ when $u^1(x) = (I(u^0))(x)$. We have to analyze the remaining possibilities, i.e., either $u_h^1(x)$ reaches the obstacle $(\mathcal{M}_h u_h^0))(x)$ or $u^1(x)$ reaches the obstacle $(\mathcal{M}(u^0))(x)$.

In the first case, there exists a discrete control ν_h such that

$$\begin{array}{rcll} u_h^1(x) & = & k(\nu_h) + u_h^0(x + \nu_h) & \quad \text{and} \\ u^1(x) & \le & k(\nu_h) + u^0(x + \nu_h)\,. & \end{array}$$

Writing as before $u_h^0(x + \nu_h) = \sum_k \mu_k u_h^0(x)$, with $\sum_k \mu_k = 1$, we obtain

$$u^0(x + \nu_h) - u_h^0(x + \nu_h) \le L_V h + \sum_k \mu_k (u^0(x) - u_h^0(x))\,.$$

Altogether, we conclude $u^1(x) - u_h^1(x) \le L_V h + \|u^0 - u_h^0\|\,.$

In the second case, $u_h^1(x) - u^1(x)$ and $u^1(x) = \mathcal{M}u^0(x)$. There exist a control ν and a discrete control ν_h such that $\|\nu - \nu_h\| \leq h$ and

$$\begin{array}{rcll} u^1(x) & = & k(\nu) + u^0(x+\nu) & \quad \text{and} \\ u_h^1(x) & \leq & k(\nu_h) + u_h^0(x+\nu_h)\,. & \end{array}$$

Adding $\pm u^0(x+\nu_h)$ to the first equation we obtain

$$\begin{array}{rcl} u_h^1(x) - u^1(x) & \leq & L_k h + [u^0(x+\nu) - u^0(x+\nu_h)] + [u^0(x+\nu_h) - u_h^0(x+\nu_h)] \\ & \leq & (L_k + L_V)h + \|u^0 - u_h^0\| \end{array}$$

In all cases we get

$$\begin{array}{rcl} \|u^1 - u_h^1\| & \leq & \|u^0 - u_h^0\| + (L_k + L_V)h \\ & \leq & \dfrac{\alpha+\Lambda}{\alpha} L_V h + (L_k + L_V)h\,, \end{array}$$

and by induction the conclusion follows. □

THEOREM 4.3. *Assume f, $\{\varphi_j\}_j$ and k are Lipschitz continuous, and* (1) *holds. Let V and V_h solve (QVI) and (QVI_h) respectively. Then, for h small enough,*

$$\|V - V_h\| \leq o(h\, ln\frac{1}{h})\,.$$

Proof. Clearly

$$\|V - V_h\| \leq \|V - M^m(u)\| + \|M^m(u) - M_h^m(u_h)\| + \|M_h^m(u_h) - V_h\|\,.$$

Theorems 3.4, Lemma 4.2 and 4.1 give upper bounds for each term. Namely, for a fixed m,

$$\begin{array}{rcl} \|V - V_h\| & \leq & C(u)(1-\delta(u))^m + \frac{\alpha+\Lambda}{\alpha} L_V h + m(L_k + L_V)h + C(u_h)(1-\delta_h(u_h))^m \\ & \leq & C\Delta^m + \frac{\alpha+\Lambda}{\alpha} L_V h + m(L_k + L_V)h\,, \end{array} \tag{20}$$

with $C := C(u) + C_h(u_h)$ and $\Delta := \max\{1-\delta(u), 1-\delta_h(u_h)\}$. Since the bound holds for any $m > 0$, it holds in particular for the (integer part of) the minimand,

$$[\bar{m}] := \frac{ln\left(\frac{(Lk+LV)h}{C|ln\Delta|}\right)}{ln\Delta} = \frac{ln(\mu h)}{ln\Delta}\,,$$

provided $h < 1/\mu$, with $\mu := \frac{L_k + L_V}{C|ln\Delta|}$.

Plugging this value in (20), we obtain

$$\begin{aligned}\|V - V_h\| &\le \mu\, h + \tfrac{\alpha+\Lambda}{\alpha} L_V h - C\mu\, hln(\mu\, h) \\ &= \mu\, h\,(1 - Cln(\mu\, h)) + \tfrac{\alpha+\Lambda}{\alpha} L_V h \\ &= C\mu\, hln\left(\frac{e^{1+\frac{\alpha+\Lambda}{\alpha\mu}L_V}}{C\mu h}\right) \\ &= o(hln\tfrac{1}{h})\,,\end{aligned}$$

for all $h < \dfrac{e^{1+\frac{\alpha+\Lambda}{\alpha\mu}L_V}}{C\mu}$. □

4.3. Accelerated Algorithm. We concentrate now on the algorithmic pattern of resolution. We have to solve $(QVI)_h$, or equivalently, we need the fixed-point of $M_h := \min[I_h, \mathcal{M}_h]$. We have frequently observed that Algorithm 0 converges dismally slowly when the contraction factor $\eta = \Lambda/(\alpha+\Lambda)$ gets closer to the unity.

To improve computer times we apply a modified algorithm, introduced in [9]. This fast algorithm performs standard iterations of Algorithm 0, memorizing at each step if a control has been applied and identifying it. Then, if the same control has been applied repeteadly, Algorithm 0 quits the standard iteration to solve an associated linear system; its solution gives a new (better) starting point for Algorithm 0. More formally,

Algorithm 1

Step 0 : Give $v_h^0 \in \mathcal{W}$, $p_{\max} \ge 1$. Set $m = 0$, $p = 0$, $\nu_r = 0$.

Step 1: Compute $v_h^{m+1}(x^k) = (M_h(v_h^m))(x^k)$. If

$(M_h(v_h^m))(x^k) = (\mathcal{M}_h(v_h^m))(x^k) = v_h^m(x^k+\nu) + k(\nu)$. then:

If $\nu = \nu_r$, set $p = p+1$.

Otherwise, if $\nu \ne \nu_r$, set $\nu = \nu_r$ and $p = 0$.

Step 2: If $v_h^{m+1}(x^k) = v_h^m(x^k)$ for all x^k, then stop; else go to Step 3.

Step 3: If $p \le p_{\max}$ then set $m = m+1$ and go to Step 1; else go to Step 4.

Step 4: Solve the linear system

$$w(x^k) = (M_h(w))(x^k) = (\mathcal{M}_h(w))(x^k) = w(x^k+\nu_r) + k(\nu_r)\,.$$

Set $m = 0$, $v_h^0(x^k) = w(x^k)$ and loop to 1. □

The convergence of Algorithm 1 can be proved following essentially the proof presented in [9] and using the convergence stated above for Algorithm 0 as a fundamental tool.

We finish this section with some comparisons between computing times of Algorithm 0 and Algorithm 1. Remark the strong dependence of the acceleration phenomenum on the contraction factor η: the closer to 1 η is, the better is Algorithm 1. The results shown have been produced in a VAX 720, for the solution of a problem where card$(\Omega_h) = 1024$.

η	CPU time Alg. 0 (in sec.)	CPU time Alg. 1 (in sec.)	% Reduction
0.50	22.33	17.76	23.13
0.86	70.58	20.00	71.66
0.91	108.71	21.01	81.68
0.96	300.73	22.09	92.65
0.99	991.02	22.43	97.74

5. Comparison of two systems with three hierarchical levels.

5.1. Description of the examples. In this section we present some numerical results obtained with the methodology proposed, considering two different possibilities for the number of installations, namely we have optimized the following systems:

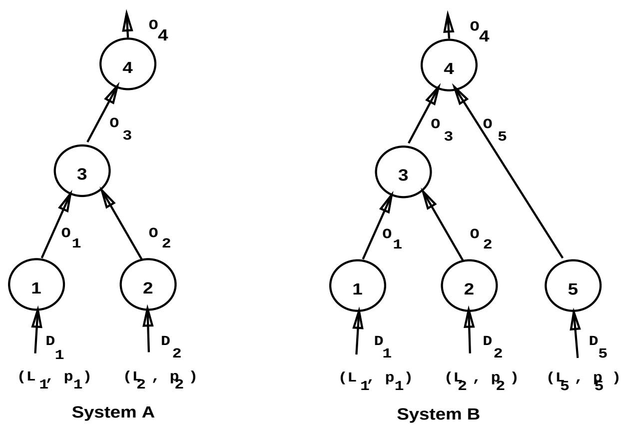

FIG. 2. *Optimized systems*

where $\lambda_1 = 1 =: L_1$, $\lambda_2 = 1 =: L_2$, $\lambda_5 = 2 =: L_5$ represent the rate of arriving demands in each installation that is receiving an outer demand, therefore the probability of one arrival at node i is approximately given by $\lambda_i \Delta t$. Demand distributions are independent for each node. For our model input of different amounts of demand are allowed; these amounts are given by a random variable taking the following values:

$$D_1 = 1.1 \quad \text{(with probability } p_1 = 0.9\text{)}$$
$$D_2 = 1.7 \quad \text{(with probability } p_2 = 0.1\text{)}$$

This distribution has been assumed identical in nodes 1, 2 and 5. Each node has a maximum and a minimum stock, the last one may be negative if we consider backlogging (in such case there is a maximum backlogging $|\underline{x}_i|$). We have discretized each continuous interval of stock $[\underline{x}_i, \bar{x}_i]$ into a set of N_i points. Each node can place an order of an arbitrary amount, provided it can be stocked and supplied. The continuous interval of orders $[0, \bar{x}_i - \underline{x}_i]$ is discretized into a set of N_{q_i} points. These values are:

Node i	$\underline{x}_i$	$\bar{x}_i$	N_i	N_{q_i}
1	-1	3	5	5
2	-1	3	5	5
3	0	10	5	5
4	0	60	3	3
5	-1	9	5	5

Clearly, nodes 1, 2, 3 and 4 are the same in both systems: both systems are identical, except for the addition of an "extra" node # 5 at System B.

The ordering cost has the following expression:

$$k(q) = \sum_i k_{0i} + k_{1i}\, q_i \qquad \text{with } q = \sum_i q_i e_i\,.$$

For our examples we have:

Node i	k_{0i}	k_{1i}
1	6.0	0.08
2	0.6	0.08
3	10	0.01
4	60	0.005
5	3	0.04

Finally, the stocking cost f has a linear additive structure. Each installation varies its cost according to the following criteria:

If $S \in [0, \bar{x}_i]$, then $f_i(S) = f_i^+ .S$ represents the real stocking cost.

If $S \in [\underline{x}_i, 0]$ then $f_i(S) = f_i^- .S$ measures the cost related to the backlogging phenomenum.

When the entered demand "D" is so big that x_i reaches $\underline{x}_i$, the rupture of the maximum backlogging has a cost equal to $\varphi_i(x_i - D - \underline{x}_i)$, the system stays at $\underline{x}_i$ (the demand accepted will just be $x_i - \underline{x}_i$).

The data we have set are

Node i	f_i^+	$\lvert f_i^- \rvert$	φ_i
1	0.1	80	45
2	0.1	80	45
3	0.007	-	-
4	0.0008	-	-
5	0.1	80	45

Remark:. Nodes 3 and 4 being "interior", they do not have values for negative stocks, since they do not operate with backlogging.

We have obtained by simulation of the system operation, the evolution of stocks, the orders placed by each installation as well as the demands received. We have assumed demands are the same for both examples.

5.2. Some remarks concerning cooperation.

5.2.1. Node 1 behaviour. For our comparisons we have performed the global optimization for five different systems, all having installation 1 as a basic node. Thus we have started with a one installation system (namely node 1) and gone on adding a new node at each time:

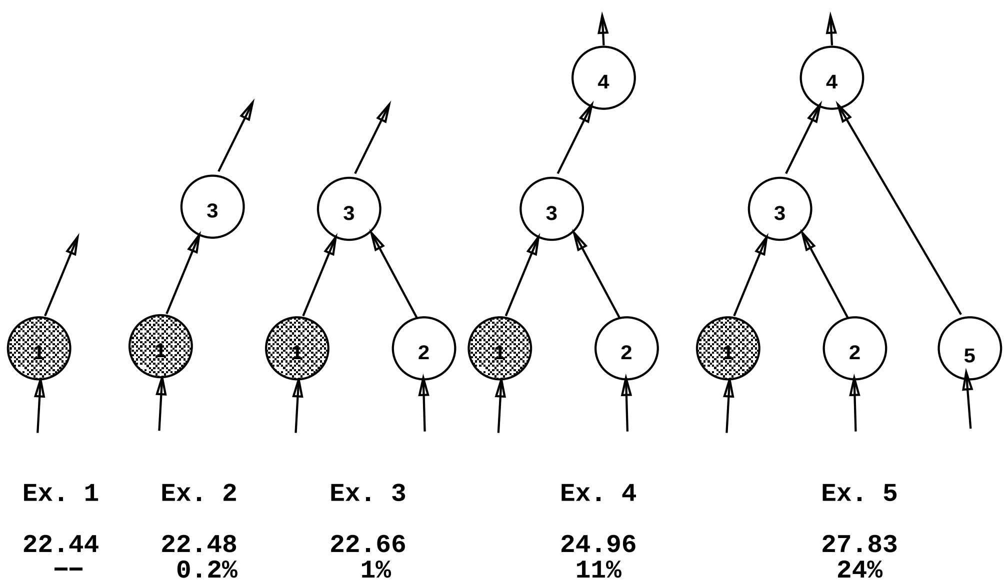

FIG. 3. *Operating cost of NODE 1*

Clearly, each centralized optimal policy will have a different strategy for node 1. The table at the bottom of 3 shows how the addition of a new node makes node 1 adopt policies involving individual higher costs. Such augmentations represent the amount of cooperation node 1 is offering to the system in order to achieve an optimal global cost. Let us point out that a decentralized approach would allow node to 1 keep the lowest cost (22.44). In section 5.2.3 we will show the negative consequences of this selfish action.

5.2.2. Subsystem (1-2-3). As we have done with one node, we analyze the evolution of the operating costs of a subsystems when more nodes are added. Figure 4 shows increasing costs of subsystem (1-2-3) due to the successive introduction of installations 4 (denoted by System A) and 5 (System B).

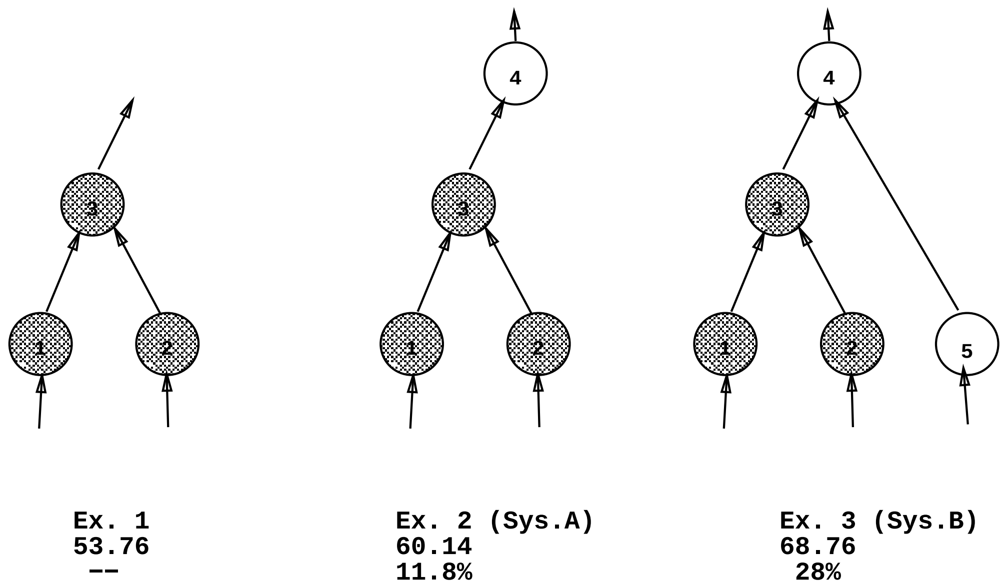

FIG. 4. *Operating cost of subsystem (1-2-3)*

5.2.3. Subsystems "selfishness" versus cooperation. Let us consider subsystem (1-2-3). We already know that centralized optimization produces an optimal global cost of 139.99 for System B. This centralized strategy imposes a cost equal to 68.76 for the subsystem (1-2-3), see fig. 4.

Assume now subsystem (1-2-3) conditions the entrance of node 5 to the system because of the higher costs du to such incorporation. This condition could be to give (1-2-3) a "priority" for being supplied. In this way its privilege over node 5 would let (1-2-3) get at least a cost closer to its former one (in System A, i.e. 60.14), even when belonging to System B. Table 1 shows how (1-2-3) gets decreasing costs as it asks for more and more privileges. But on the other side these "priorities" make System B increase its global cost with an amount much more significant than the individual gain obtained by (1-2-3). System B increasing costs are originated by node 5's rejected demand.

	Operation costs of subsystem (1-2-3) in B	Operation costs of System B
(no privileges)	68.76	139.09
	62.50	156.61
	60.20	160.28
	59.60	162.30
	59.40	162.68

TABLE 1
(1-2-3) decreasing costs

Summing up:. if (1-2-3) intends to conserve its former cost (the one achieved in System A namely, 60.14) then System B has to increase its global cost in 15 %.

Suppose now (1-2-3) refuses node 5 incorporation. The global cost of System A results from the addition (1-2-3) cost and node 4 cost (60.14 and 18.58 respectively). In order to let node 5 operate, it should put its orders to a new "extra" node 4*, with identical characteristics as node 4. We would get then an auxiliary system (4*-5) which operates under global optimization with a cost of 29.37 for node 5 and 38.13 for 4*. Figure 5 shows the pernicious effect of considering two separated systems: System A and system (4*-5). Total costs are higher than System B global cost, although (1-2-3) and 5 obtain both better costs when operating separatedly.

Summing up:. A centralized optimization (System B operating costs) asks from:

Subsystem (1-2-3) to pay a cooperative cost of
68.76 - 60.14 = 8.62
and from
Node 5 to pay a cooperative cost of
30.77 - 29.37 = 1.44
Total cooperative cost = 10.02

But this cooperation allows the existence of just one node "4" at the maximum level of hierarchy. This fact reduces global cost in

$$(18.58 + 38.13) - 39.56 = 17.13$$

Hence in System B we have

17.13 - 10.02 = 7.11

That is, we get a reduction of approximatedly a 5 % over the cost obtained when operating in a decentralized way (System A + (4*-5)).

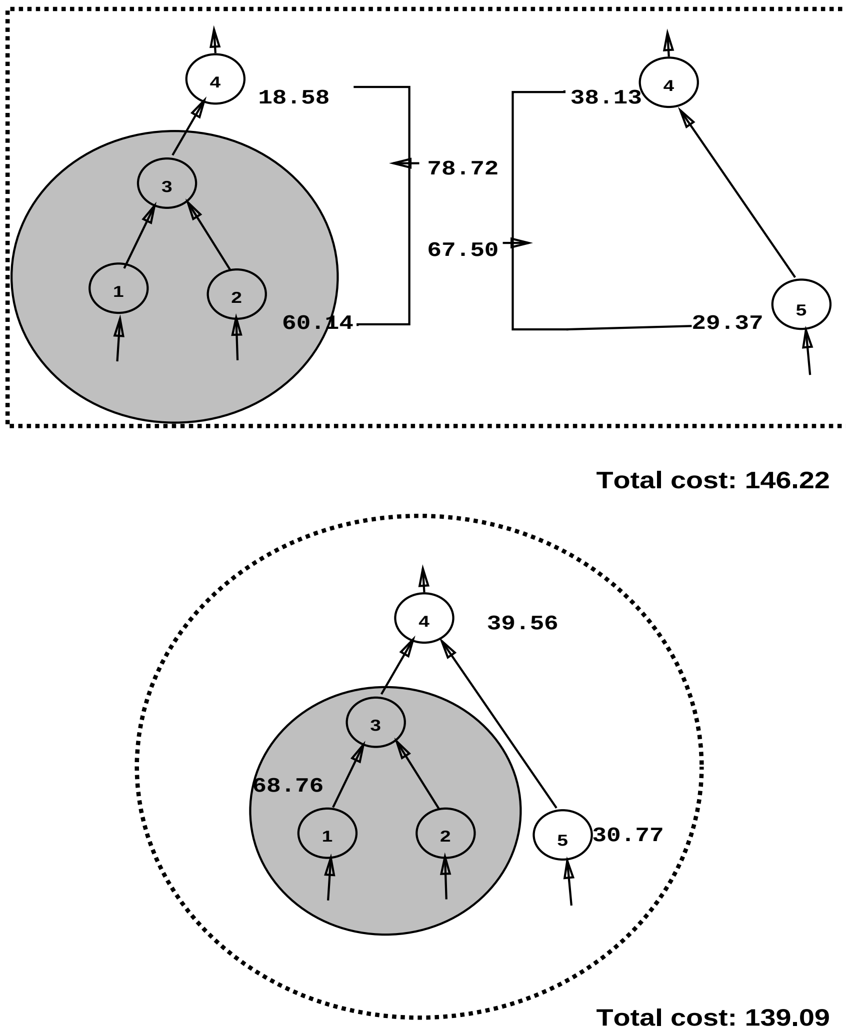

FIG. 5. *Comparison of costs*

REFERENCES

[1] A. Bensoussan, M. Crouhy, and J.M. Proth. *Mathematical theory of production planning*, volume 3. North Holland, 1983.

[2] A. Bensoussan and J.L. Lions. *Contrôle impulsionnel et inéquations quasi-variationnelles*. Dunod, 1982.

[3] A. Clark. An informal survey of multi-echelon inventory theory. *Naval Research Logistics Quarterly*, (14), 1972.

[4] A. Clark and H. Scarf. Optimal Policies for a Multi-echelon Inventory Problem. *Management Science*, 6:475–490, 1960.

[5] A. Clark and H. Scarf. Approximate Solution to a simple Multi-Echelon Inventory Problem. In K. Arrow, S. Karlin and H. Scarf, editor, *Studies in Applied Probability and Management Science*, pages 161–184. Stanford University Press, 1962.

[6] M. Crouhy. *Gestion Informatique de la production industrielle*. Editions de l'Usine Nouvelle, 1983.

[7] W. Fleming and R. Rishel. *Deterministic and stochastic optimal control*. Springer Verlag, 1975.

[8] R. González and E. Rofman. Deterministic control problems: an approximation procedure for the optimal cost. *SIAM Journal on Control and Optimization*, 23(2):242–285, 1985.

[9] R. González and C. Sagastizábal. Un algorithme pour la résolution rapide d'équations discrètes de Hamilton-Jacobi-Bellman. *Comptes Rendus de l'Académie des Sciences - Série I*, 311:45–50, 1990.

[10] S. Graves. Multistage lot sizing: an iterative approach. In L. Schwarz, editor, *Multi-level production/inventory control systems: theory and practice*, pages 95–110. TIMS Studies in Managament Science - 16, 1981.

[11] B. Hanouzet and J.L. Joly. Convergence uniforme des itérés définissant la solution d'une inéquation variationnelle abstraite. *Comptes Rendus de l'Académie des Sciences - Série I*, 286:735–738, 1978.

[12] F. Kabbaj, J.L. Menaldi, and E. Rofman. Variational approach of serial multi-level production/inventory systems. *Journal of Optimization Theory and Applications*, 3(65):447–483, 1990.

[13] R. Peterson and E. Silver. *Decision systems for inventory management and production planning*. Wiley, 1979.

[14] M. Robin. *Contrôle impulsionel des processus de Markov*. PhD thesis, Université de Paris IX-Dauphine, 1978. Thèse d'Etat.

[15] H. Scarf. A Survey of analytic techniques in inventory theory. pages 185–225. H. Scarf, D. Gilford and M. Shelly, 1963.

[16] H. Soner. Optimal Control with State-Space Constraint - Part I. *SIAM Journal on Control and Optimization*, 24(2):552–561, 1986.

[17] H. Soner. Optimal Control with State-Space Constraint - Part II. *SIAM Journal on Control and Optimization*, 24(6):1110–1122, 1986.

[18] A. Veinott. The status of mathematical inventory theory. *Management Science*, 12:745–777, 1966.

[19] W. Zangwill. A Backlogging Model and a Multi-Echelon Model of a Dynamical Lot Size Production System: A Network Approach. *Management Science*, 15:506–627, 1969.