On the number of large complete arcs in PG$(2,q)$, $23 \leq q \leq 32$

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Abstract

The classification of large complete arcs in PG$(2,q)$, $23 \leq q \leq 32$ by using superregular matrices is reported in the paper. The number of all different n-arcs and the number of all complete n-arcs are determined for sizes greater than or equal to $q-8$. Some particular results on $t(r,q)$ and $m'(r,q)$ – the smallest and the second largest size for complete arcs in PG(r,q) – are also reported, stating that $m'(2,31)=22$, $m'(2,32)=24$, $t(3,23)=10$ and $m'(3,23)=16$. All results of the paper are computational.

Keywords: complete n-arc, MDS code, superregular matrix.

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Introduction

An $[n, k, d]$ linear code over GF(q) is called MDS (maximum distance separable) if it attains the Singleton bound $d \le n - k + 1$ [10, Ch. 11]. A $k \times n$ systematic generator matrix $G = (I|A)$ generates an MDS code if and only if A is superregular, i.e., every square submatrix of A is nonsingular [13].

If q is a prime, then two MDS codes are considered identical if and only if they belong to the same equivalence class up to PGL equivalence. Otherwise, for a prime power q, both PGL and PΓL equivalence relations are taken into consideration.

Differently from the usual way of typifying the MDS codes according to the completeness or incompleteness of the associated n-arcs, 4 types of MDS codes will be distinguished here, so, that both the n-arc that belong to an MDS code and the n-arc that belong to its dual are examined.

Two slightly different methods are applied for obtaining the number of non-equivalent superregular $3 \times (n-3)$ matrices, i.e. the number of n-arcs in PG$(2, q)$ where $23 \le q \le 32$ and $q - 8 \le n \le q + 1$. Both methods are based on the ordering of finite field elements, which induces a lexicographic ordering of superregular matrices.

a) A direct procedure for exhaustive search is applied for the cases when $q \le 27$ or $n \le q - 1$. For more details about this method, see [8].

The direct exhaustive search for finding, e.g., all non-equivalent 22- or 21-arcs in PG$(2, 29)$, 25- or 24-arcs in PG$(2, 31)$ would have required too much CPU time on a typical personal computer (with a 2.4 GHz processor). My estimate was roughly 3 hours per input record for $q = 29$ and about 14 hours per input record for $q = 31$. The input records of these jobs are the non-equivalent 21-, 20-, 24- and 23-point sets in the appropriate projective lines, which are 629, 1339, 415 and 992 records, respectively, and so, the necessary processor time comes to 1887, 4017, 5810 and 13888 hours, respectively.

b) Fortunately, I could find a method, based on a kind of partitioning, which can solve the above-mentioned cases within a reasonable time, i.e. in about 9, 48, 14 and 70 hours, respectively. To go further, generating the non-equivalent 23- and 22-arcs in PG$(2, 31)$ required "only" 154 and 700 hours, respectively, by using this alternative method. As a matter of fact, first I tested this method for $q = 27$ for a few cases, which had been settled already by using the direct method.

The description of the applied method will be given in Section 3.

1 The four types of superregular matrices

The definitions and propositions of this section are repeated from [9] without proofs and comments.

Definition 1 *A rectangular matrix A with elements from* GF(q) *is called superregular if every submatrix of A is non-singular.*

Definition 2 *Let A be a $k \times l$ superregular matrix over* GF(q) *where $2 \leq k, l \leq q-1$. A is said to be of*

– type 1 if A can be extended to either a $(k+1) \times l$ or a $k \times (l+1)$ superregular matrix by adding a new row or column,

– type 2 if A can be extended to a $(k+1) \times l$ superregular matrix, but not to a $k \times (l+1)$ superregular matrix,

– type 3 if A can be extended to a $k \times (l+1)$ superregular matrix, but not to a $(k+1) \times l$ superregular matrix,

– type 4 if neither a $k \times (l+1)$, nor a $(k+1) \times l$ superregular extension exists.

Definition 3 *A linear code over a finite field* GF(q) *of length n, dimension k and minimum distance d is called* MDS *(maximum distance separable) if $d = n - k + 1$.*

Definition 4 *An n-*arc *in* PG($k-1, q$) *is a set of n points with at most $k-1$ in any hyperplane.*

Definition 5 *An n-arc in* PG($k-1, q$) *is complete if it is not contained in an $(n+1)$-arc.*

Definition 6 *An $[n, k, d = n - k + 1]$* MDS *code is complete if it is not a projection of an $[n+1, k, d+1 = n-k+2]$* MDS *code.*

Proposition 1 *An* MDS *code belonging to the systematic generator matrix $G = (I|A)$ is complete if and only if A is of type 2 or 4.*

Proposition 2 *A complete n-arc exists in* PG($k-1, q$) *if and only if a $k \times (n-k)$ superregular matrix of type 2 or 4 can be found.*

2 The main results of the paper

For the range $q \leq 19$, the number of complete arcs of different lengths in PG$(2, q)$ has been tabulated in [7]. The spectrum of the sizes of complete arcs is known ([5, 11, 12]) also for $q = 23$, 25 and 27. The length $m'(2, q)$ of the second largest complete arc in PG$(2, q)$ is known for $q \leq 29$ – and is monotonously non-decreasing until this range – unknown for $q \geq 31$. As regards the case $q = 29$, it has been revealed ([3]) that there is no complete $(m'(2, 29) - 1 = 23)$-arc in the corresponding space, but nothing has been known so far about the existence of 21- or 22-arcs in the same space. It is also contained in [3] that the complete $m'(2, q)$-arc is unique for $q = 25$, 27 and 29.

In the present paper, computational results are provided for the number of different types of superregular $k \times l$ matrices for $23 \leq q \leq 31$, $k = 3$ and $k + l \geq q - 8$. (In the case $q = 32$ the superregular $3 \times l$ matrices are classified only for $l \geq q - 3$, but the question concerning the existence and number of superregular $3 \times l$ matrices is answered for $l \geq q - 11$.) The interpretation of this classification for arcs in projective spaces provides the number of all different n-arcs as well as the number of all different *complete* n-arcs in PG$(2, q)$ for $n \geq q - 8$. In particular, we have the following new results:

1. The monotonicity of $m'(2, q)$ is disproved by the following assertion, which derives from the non-existence of superregular $3 \times l$ matrices of type 2 or 4 for $20 \leq l \leq 28$.

Theorem 1 $m'(2, 31) = 22$ *and the number of non-equivalent complete* $m'(2, 31)$*-arcs in* PG$(2, 31)$ *is* 11.

2. For $q = 32$, we have an analogous result as follows:

Theorem 2 $m'(2, 32) = 24$ *and the number of non-equivalent complete* $m'(2, 32)$*-arcs in* PG$(2, 32)$ *is* 19.

3. The spectrum of the sizes of complete arcs in PG$(2, 29)$ is completely described as the set of lengths

$$\{13, 14, 15, 16, 17, 18, 19, 20, 21, 24, 30\}.$$

4. The number of non-equivalent complete n-arcs in PG$(2, q)$, $q = 25$, 27 and 29, belonging to the third largest possible size is established according to the following small table:

q	25	27	29
n	18	19	21
number of arcs	41	5	2

(See also Table 4 and Example 4 for $q = 25$ and 27, Table 2 and Example 5 for $q = 29$.)

5. We have a computational result also for the 3-dimensional projective space PG$(3, 23)$ as follows:

Theorem 3 *The size of the smallest and the second largest complete arc in* $PG(3, 23)$ *is*

$$t(3, 23) = 10 \text{ and } m'(3, 23) = 16.$$

In connection with an earlier work [8] we determined and filed, among others, the non-equivalent n-arcs in PG$(2, 23)$ for $n \leq 9$ and $n \geq 15$. There were found 257, 7613, 172416, 2235523 non-equivalent n-arcs for $n = 6, 7, 8, 9$ and there were found 1, 1, 4, 6, 22, 41, 692, 64773 non-equivalent n-arcs for $n = 22, 21, 20, 19, 18, 17, 16, 15$. The collection of these data can be used to seek for complete arcs in PG$(3, 23)$ by adding a fourth column to the corresponding $3 \times (n - 3)$ superregular matrices in any possible way so that the $(n + 1)$-arcs formed from the four unit vectors and the column vectors of the augmented $4 \times (n - 3)$ matrices will be complete.

The result of the search is that *in* PG$(3, 23)$ *the smallest size for complete arcs is* 10*, the second largest size for complete arcs is* 16 *and there are exactly* 12 *non-equivalent complete* 16*-arcs.* (In view of the huge number of 9-arcs in PG$(2, 23)$ we failed to determine the set of all non-equivalent 10-arcs in the 3-dimensional projective space.) We checked also the type of the corresponding 4×6 and 4×12 superregular matrices and found that some superregular 4×6 matrices are of type 2, others are of type 4, while all of the superregular 4×12 matrices are of type 4 (see Examples 1-3 in Section 4).

3 A more effective method for the search of large arcs in finite projective planes

Let $S_{k,l,q}$ denote the set of lexicographically ordered superregular $k \times l$ matrices over GF(q). When using the direct method for the construction of $S_{3,l,q}$, we start from $S_{2,l,q}$, then add a third row to each matrix in $S_{2,l,q}$ in any possible way to get superregular $3 \times l$ matrices. Finally we drop the equivalent matrices and leave only one of them for each class of equivalence.

For larger values of l, it is worth applying the following method instead of the direct search: For an appropriate value of λ where $\lambda < l$ and for any $A \in S_{2,l,q}$, consider the $2 \times \lambda$ matrix A' built from the first λ columns of A. Let

$$S'_{2,\lambda,q} = S'_{2,\lambda,q}(l) = \{A' : A \in S_{2,l,q}\} \text{ (without multiple elements).}$$

Note that the set $S'_{2,\lambda,q}$ is not independent of l and its dependence is reverse: for a greater value of l, the cardinality of this set becomes smaller.

By adding a third row to each matrix $A' \in S'_{2,\lambda,q}$ in any possible way that leads to a superregular $3 \times \lambda$ matrix, we obtain the set that we denote by $S'_{3,\lambda,q}$.

The cardinality of this set can be reduced by dropping all complete superregular matrices. This reduction leads to the set

$$S^{\star}_{3,\lambda,q} = \left\{B : B \in S'_{3,\lambda,q}, B \text{ is incomplete}\right\}.$$

We have to gather all such superregular $3 \times l$ matrices, the upper $2 \times l$ submatrices of which are contained in $S_{2,l,q}$, while the leftmost $3 \times \lambda$ submatrices of which are contained in $S^{\star}_{3,\lambda,q}$. Finally, the resulted set of superregular $3 \times l$ matrices is scanned for equivalence and only one matrix is left from among the equivalent ones. After this whole procedure the set $S_{3,l,q}$ is obtained.

The main features of the procedure during its application to the particular cases are as follows. The choice of the actual values of λ (namely $\lambda = 13$, 14, 15, 17 for $q = 27$, 29, 31, 32, respectively) was the result of a calibration towards the optimum value of this parameter, carried out empirically. As regards the actual rate of reduction in the cardinality of the set $S'_{3,\lambda,q}$, it was found that the ratio of complete superregular matrices in them is as much as 82% for $q = 27$, 87% for $q = 29$ and 90% for $q = 31$. Thus, the set $S^{\star}_{3,\lambda,q}$ contains only the amount between one tenth and one fifth of the elements of $S'_{3,\lambda,q}$, for the above-mentioned values of q. The case that belongs to $q = 32$ is somewhat different. Due to the existence of several $(q + 2)$-arcs and a lot of non-classical $(q + 1)$-arcs, the number of n-arcs soon becomes very huge when the length n is further decreased. In the same time, the ratio of complete superregular 3×17 matrices under consideration was found very small (under 0.1%). The check for equivalence of the large number of 32-arcs (3×29 superregular matrices) required about 120 hours CPU time, so, we stopped the classification at this point.

Further acceleration of the procedure can be achieved by some considerations about the nature of lexicographical ordering of superregular matrices. As a superregular $k \times (n-k)$ matrix is geometrically equivalent to an n-arc in the projective space, therefore, to generate the non-equivalent $k \times (n-k)$ superregular matrices it is enough to use the set of non-equivalent superregular $(k-1) \times (n-k+1)$ matrices. Consequently, it is allowed to use the smaller set $S'_{2,\lambda,q}(l+1)$ instead of $S'_{2,\lambda,q}(l)$ when applying the method for the particular cases.

4 Tables and examples

The spectra of the sizes of complete arcs in $\mathrm{PG}(2,q), 23 \leq q \leq 32$ are summarized in Table 1. The size of the smallest complete arc, $t(2,q)$, and that of the second largest complete arc, $m'(2,q)$ are also placed here. Parts of this table contain known results (when $q = 23, 25$ and 27). The spectra that belongs to $q = 31$ and $q = 32$ are not yet completely determined; there are 2 open cases for lengths 12 and 13. For the non-existence of complete arcs for lengths less than 12, see [4, 7, 8].

In Tables 2 and 3 the number of non-equivalent superregular matrices is listed according to their types for $23 \leq q \leq 31$, $k = 3$ and $l \geq q - 11$. However, we consider also $l = q - 12$ for $q = 31$, which is a case of special interest. In the last two columns of Table 2 the number of input records are given for the direct search (recno1) and for the longest procedure of the method that was described in Section 3 and applied for $q = 27$, 29, 31 (recno2). They are the cardinalities of sets $S_{2,l,q}$ and $S_{2,\lambda,q}$ for the given cases. The value of recno2 applies to the described method without the trick for shortening that was mentioned at the end of the previous section. If this shortening is also taken into consideration, then each item of this column skips up with one place, for a given q.

The classification of superregular matrices proves also the following fact: If $23 \leq q \leq 31$, then a $3 \times l$ superregular matrix is unique for $l \geq q - 5$, i.e., the arcs of the highest four lengths in $\mathrm{PG}(2,q)$ are unique. (The same property is known also for $q = 17$ and $q = 19$.) This is why the listing in Table 2 is finished at $l = q - 6$ for the particular values of q, and then it continues in a shortened form in Table 3 for the remaining values. In Table 4 the results of classification regarding PΓL equivalence are presented. Table 5 applies for $q = 32$, both PGL and PΓL equivalence. The results that belong to PΓL are given in parentheses.

One of the 2 different superregular 3×29 matrices of type 1 is classical, the other is non-classical. This means that there exist both classical and non-classical q-arcs of type 1 in $\mathrm{PG}(2,32)$, $(q+1)$-arcs in $\mathrm{PG}(3,32)$. The latter are, however, of type 4; a computer search proved that their number is exactly two. At this point we have to note that the classification of $(q+1)$-arcs in $\mathrm{PG}(3,q)$, q even, in general, was given in [2].

For the numerical examples, shown in this section, the elements of a finite field $\mathrm{GF}(q)$ are denoted by $\{0,1,\ldots,q-1\}$ for a prime or prime power q. If q is a prime ($q=23$, 29 or 31), we operate on these elements modulo q. For the prime power $q=27$, we operate modulo $\alpha^3+2\alpha+1$ where α is a primitive element in GF(27). In this case an integer $9c_0+3c_1+c_2$ is assigned to the field element $c_0\alpha^2+c_1\alpha+c_2$ $(0\le c_0,c_1,c_2\le 2)$. Similarly, for $q=32$, we operate modulo $\alpha^5+\alpha^2+1$ where α is a primitive element in GF(32) and assign $16c_0+8c_1+4c_2+2c_3+c_4$ to the field element $c_0\alpha^4+c_1\alpha^3+c_2\alpha^2+c_3\alpha+c_4$ $(0\le c_0,c_1,c_2,c_3,c_4\le 1)$.

For each matrix that appears in the examples, their column vectors supplemented with the column vectors of the identity matrix provide the points of complete arcs with the appropriate parameters.

q	r	sizes of complete arcs	$t(2,q)$	$m'(2,q)$
23	2	10,12,13,14,15,16,17,24	10	17
25	2	12,13,14,15,16,17,18,21,26	12	21
27	2	12,13,14,15,16,17,18,19,22,28	12	22
29	2	13,14,15,16,17,18,19,20,21,24,30	13	24
31	2	14,15,16,17,18,19,20,21,22,32 and possibly 12,13	?	22
32	2	14,15,16,17,18,19,20,21,22,23,24,34 and possibly 12,13	?	24

Table 1. The sizes of complete n-arcs in $\mathrm{PG}(2,q), 23\le q\le 32$

q	k	l	type 1	type 2	type 3	type 4	all types	recno1	recno2
23	3	12	159	-	6253	58361	64773	196	
23	3	13	83	-	45	564	692	125	
23	3	14	36	-	-	5	41	83	
23	3	15	22	-	-	-	22	36	
23	3	16	6	-	-	-	6	22	
23	3	17	4	-	-	-	4	6	
25	3	14	235	-	415	843	1493	398	
25	3	15	131	-	26	65	222	225	
25	3	16	54	-	4	-	58	131	
25	3	17	28	-	1	-	29	54	
25	3	18	8	-	-	1	9	28	
25	3	19	5	-	-	-	5	8	
27	3	16	382	-	120	13	515	745	53
27	3	17	196	-	22	-	218	382	14
27	3	18	73	-	3	-	76	196	8
27	3	19	34	-	-	1	35	73	2
27	3	20	8	-	-	-	8	34	1
27	3	21	5	-	-	-	5	8	1
29	3	18	629	-	15	2	646	1339	65
29	3	19	289	-	4	-	293	629	16
29	3	20	97	-	1	-	98	289	3
29	3	21	42	-	-	1	43	97	1
29	3	22	10	-	-	-	10	42	1
29	3	23	5	-	-	-	5	10	1
31	3	19	2318	-	-	11	2329	4442	204
31	3	20	992	-	-	-	992	2318	50
31	3	21	415	-	-	-	415	992	11
31	3	22	132	-	-	-	132	415	5
31	3	23	51	-	-	-	51	132	1
31	3	24	11	-	-	-	11	51	1
31	3	25	6	-	-	-	6	11	1

Table 2. Possible types and the number of superregular matrices up to PGL orbits for $23 \le q \le 31$, $k = 3$, $q - 11$ (or $q - 12$) $\le l \le q - 6$

q	k	l	type 1	type 2	type 3	type 4	all types
q	3	$q-5$	1	-	-	-	1
q	3	$q-4$	1	-	-	-	1
q	3	$q-3$	1	-	-	-	1
q	3	$q-2$	-	-	-	1	1

Table 3. Possible types and the number of superregular matrices up to PGL orbits for $23 \le q \le 31$, $k = 3$, $q - 5 \le l \le q - 2$

q	k	l	type 1	type 2	type 3	type 4	all types
25	3	14	137	-	218	434	789
25	3	15	79	-	15	41	135
25	3	16	34	-	4	-	38
25	3	17	19	-	1	-	20
25	3	18	7	-	-	1	8
25	3	19	4	-	-	-	4
27	3	16	134	-	44	5	183
27	3	17	72	-	10	-	82
27	3	18	29	-	3	-	32
27	3	19	14	-	-	1	15
27	3	20	4	-	-	-	4
27	3	21	3	-	-	-	3

Table 4. Possible types and the number of superregular matrices up to PΓL orbits for $q = 25, 27$, $k = 3$, $q - 11 \leq l \leq q - 6$

q	k	l	type 1	type 2	type 3	type 4	all types
32	3	29	2(2)	-	1796(372)	-	1798(374)
32	3	30	-	-	119(35)	-	119(35)
32	3	31	-	-		10(6)	10(6)
32	4	29	-	-		2(2)	2(2)

Table 5. Possible types and the number of superregular matrices up to PGL (PΓL) orbits for $q = 32$, $3 \leq k \leq 4$, $l \geq 29$

Example 1. Superregular 4×6 matrices of type 2 in $\mathrm{PG}(3, 23)$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 4 & 17 \\ 1 & 4 & 10 & 13 & 2 & 11 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 4 & 20 \\ 1 & 4 & 5 & 19 & 21 & 14 \end{pmatrix}$$

Example 2. Superregular 4×6 matrices of type 4 in $\mathrm{PG}(3, 23)$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 4 & 8 \\ 1 & 4 & 5 & 12 & 14 & 17 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 7 & 10 \\ 1 & 4 & 5 & 13 & 8 & 9 \end{pmatrix}$$

Example 3. Superregular 4×12 matrices of type 4 in $\mathrm{PG}(3, 23)$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 13 \\ 1 & 4 & 13 & 7 & 17 & 9 & 8 & 10 & 15 & 22 & 12 & 19 \\ 1 & 19 & 7 & 11 & 16 & 4 & 10 & 15 & 18 & 14 & 17 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 14 & 17 \\ 1 & 6 & 15 & 2 & 19 & 11 & 14 & 5 & 12 & 8 & 17 & 13 \\ 1 & 14 & 10 & 12 & 13 & 8 & 4 & 15 & 21 & 16 & 20 & 19 \end{pmatrix}$$

Example 4. Superregular 3×16 matrices of type 4 in $\mathrm{PG}(2, 27)$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 20 & 22 & 26 \\ 1 & 3 & 12 & 11 & 24 & 15 & 5 & 25 & 26 & 17 & 7 & 9 & 14 & 6 & 18 & 16 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 20 & 22 & 26 \\ 1 & 5 & 18 & 25 & 8 & 7 & 9 & 3 & 14 & 20 & 12 & 16 & 23 & 15 & 21 & 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 23 & 26 \\ 1 & 6 & 2 & 14 & 11 & 21 & 8 & 12 & 13 & 22 & 9 & 17 & 10 & 19 & 24 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 16 & 24 \\ 1 & 23 & 8 & 9 & 21 & 19 & 24 & 10 & 6 & 14 & 5 & 2 & 15 & 13 & 3 & 22 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 15 & 16 & 19 & 20 & 23 \\ 1 & 6 & 8 & 19 & 4 & 16 & 24 & 17 & 11 & 5 & 15 & 25 & 13 & 9 & 3 & 12 \end{pmatrix}$$

Example 5. Superregular 3×18 matrices of type 4 in $\mathrm{PG}(2, 29)$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 17 & 21 & 26 \\ 1 & 12 & 10 & 26 & 14 & 28 & 24 & 6 & 4 & 19 & 16 & 18 & 15 & 13 & 17 & 5 & 27 & 25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 14 & 18 & 20 & 21 & 22 & 26 \\ 1 & 3 & 15 & 10 & 16 & 14 & 18 & 2 & 22 & 11 & 7 & 17 & 20 & 12 & 25 & 4 & 8 & 23 \end{pmatrix}$$

Example 6. Superregular 3×19 matrices of type 4 in $\mathrm{PG}(2, 31)$:

The 11 PGL-nonequivalent matrices of this example – without their first row and first column, which consist of all 1-s as in the previous examples – are given in the 11 lines of the following table:

row 2	row 3
2,3,...,13,14,15,16,17,21,27	23,15,9,17,27,11,2,6,28,12,22,30,24,16,7,8,20,19
2,3,...,13,14,15,16,18,23,26	17,4,23,8,3,6,18,26,19,10,15,28,9,27,25,13,20,2
2,3,...,13,14,15,16,19,21,28	25,14,2,20,29,13,22,19,6,9,24,3,15,4,11,23,27,7
2,3,...,13,14,15,18,23,24,26	24,15,25,22,13,27,5,29,3,23,17,8,19,2,9,20,14,4
2,3,...,13,14,15,19,20,23,26	15,8,17,2,12,24,18,5,25,26,9,22,10,21,29,27,4,30
2,3,...,13,14,16,17,19,23,27	7,21,23,27,4,25,14,15,30,22,19,12,2,5,10,3,18,20
2,3,...,13,14,16,17,21,24,26	28,15,7,27,4,2,16,13,9,8,6,18,12,17,11,24,5,14
2,3,...,13,14,16,18,21,23,25	25,8,16,26,27,14,4,11,19,15,9,17,10,13,7,20,3,5
2,3,...,13,14,16,18,23,24,28	8,10,2,6,22,16,12,27,25,17,19,26,29,20,21,24,11,13
2,3,...,13,15,16,17,23,24,28	9,18,20,24,26,27,10,22,23,25,3,14,17,30,21,11,5,19
2,3,...,13,15,17,22,23,26,27	24,20,9,3,21,18,19,4,13,27,25,12,2,16,17,15,10,5

Example 7. Superregular 3×21 matrices of type 4 in $\mathrm{PG}(2, 32)$:

The 19 PΓL-nonequivalent matrices of this example – also without their first row and first column, – are given as follows:

row 2	row 3
2,3,...,18,19,20,21	12,13,26,27,17,16,3,2,23,22,31,30,10,11,21,20,7,6,19,18
2,3,...,18,19,20,21	18,31,2,22,13,28,11,27,24,10,7,26,6,23,15,21,17,25,12,20
2,3,...,18,19,20,28	18,14,30,22,15,12,16,26,21,17,4,8,7,23,29,10,3,2,31,25
2,3,...,18,19,20,28	27,28,23,22,24,30,10,4,20,16,5,12,15,7,25,26,13,8,14,17
2,3,...,18,19,24,25	14,24,19,20,27,3,28,6,18,29,7,11,16,10,26,21,25,8,9,15
2,3,...,18,19,24,25	16,20,19,9,10,14,17,4,2,24,21,30,26,7,29,3,22,23,12,5
2,3,...,18,19,24,25	22,28,10,7,17,12,29,3,16,31,4,15,6,23,25,26,20,9,13,19
2,3,...,18,19,24,28	8,6,24,23,3,20,15,30,9,28,25,18,31,16,17,22,14,2,10,7
2,3,...,18,19,28,29	9,20,27,31,29,21,22,2,11,5,28,14,19,17,6,23,24,8,12,10
2,3,...,18,19,28,29	31,30,2,3,22,23,12,13,20,21,15,14,18,19,25,24,7,6,5,4
2,3,...,18,20,21,24	5,4,8,24,28,25,27,31,14,21,19,23,12,16,7,2,26,17,29,15
2,3,...,18,20,21,24	9,14,7,30,13,10,18,12,24,26,22,17,31,28,11,19,16,6,2,20
2,3,...,18,20,21,24	29,24,6,15,7,20,19,26,18,2,21,31,28,3,25,27,22,11,17,14
2,3,...,18,20,21,26	3,21,8,18,31,27,5,16,26,6,22,23,24,14,7,13,12,9,17,30
2,3,...,18,20,21,26	4,20,27,23,17,8,11,22,15,30,21,3,13,9,6,31,7,2,16,14
2,3,...,18,20,21,26	16,17,8,27,29,13,7,6,25,26,23,4,2,18,3,14,19,9,11,24
2,3,...,18,20,23,25	4,24,20,21,15,10,22,5,6,16,13,19,8,12,25,18,31,26,2,14
2,3,...,18,20,23,27	12,2,15,18,25,14,6,11,17,20,9,7,30,29,21,8,16,10,22,13
2,3,...,18,20,23,28	9,13,7,28,18,30,6,8,25,12,11,15,16,3,10,27,24,5,21,14

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