Selfish Unsplittable Flows*

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Abstract. What is the price of anarchy when unsplittable demands are routed selfishly in general networks with load-dependent edge delays? Motivated by this question we generalize the model of [14] to the case of *weighted congestion games*. We show that varying demands of users crucially affect the nature of these games, which are no longer isomorphic to exact potential games, even for very simple instances. Indeed we construct examples where even a single-commodity (weighted) network congestion game may have no pure Nash equilibrium.
On the other hand, we study a special family of networks (which we call the *ℓ-layered networks*) and we prove that any weighted congestion game on such a network with resource delays equal to the congestions, possesses a pure Nash Equilibrium. We also show how to construct one in pseudo-polynomial time. Finally, we give a surprising answer to the question above for such games: The price of anarchy of any weighted ℓ-layered network congestion game with m edges and edge delays equal to the loads, is $\Theta\left(\frac{\log m}{\log\log m}\right)$.

1 Introduction

Consider a model where selfish users having varying demands compete for some shared resources. The quality of service provided by a resource decreases with its *congestion*, ie, the amount of demands of the users willing to be served by it. Each user may reveal its actual (unique) choice (called a *pure strategy*) among the resources available to it, or it may reveal a probability distribution for choosing one of its candidate resources (a *mixed strategy*). The users determine their actual behavior based on other users' behavior, but they do not cooperate. We are interested in situations where the users have reached some kind of equilibrium. The most popular notion of equilibrium in noncooperative game theory is the *Nash equilibrium*: a "stable point" among the users, from which no user is willing to deviate unilaterally. In [14] the notion of the *coordination ratio* or *price of anarchy* was introduced, as a means for measuring the performance degradation due to lack of users' coordination when sharing common goods.

A realistic scenario for the above model is when unsplittable demands are routed selfishly in general networks with load-dependent edge delays. When the underlying

* This work was partially supported by the EU within the Future and Emerging Technologies Programme under contract IST-2001-33135 (CRESCCO) and within the 6th Framework Programme under contract 001907 (DELIS).

network consists of two nodes and parallel links between them, there has been an extensive study on the existence and computability of equilibria, as well as on the price of anarchy. Motivated by the work of [14], we generalize their concept to the *weighted congestion games* in a non-trivial way. When users have identical demands, such a game is indeed isomorphic to an *exact potential game* ([19]) and thus always possesses a pure Nash equilibrium, ie, an equilibrium where each user adopts a pure strategy. We show that varying demands of users crucially affect the nature of these games, which are no longer isomorphic to exact potential games. Indeed we construct examples where even a single-commodity (weighted) network congestion game may have no pure Nash equilibrium at all.

On the other hand, we explore weighted congestion games on a special family of networks, the *ℓ-layered networks*. We prove the existence of pure Nash equilibria for such games. We also propose a pseudo-polynomial time algorithm for constructing one. Finally, we study the price of anarchy for these networks and we come to a rather surprising conclusion: Within constant factors, the worst case instance (wrt the price of anarchy) among weighted ℓ-layered network congestion games with m edges and edge delays equal to the loads, is the parallel links game introduced in [14].

1.1 The Model

Consider having a set of resources E in a system. For each $e \in E$, let $d_e(\cdot)$ be the delay per user that requests its service, as a function of the total usage of this resource by all the users. Each such function is considered to be non-decreasing in the total usage of the corresponding resource. Each resource may be represented by a pair of points: an entry point to the resource and an exit point from it. So, we represent each resource by an arc from its entry point to its exit point and we associate with this arc the cost (eg, the delay as a function of the load of this resource) that each user has to pay if she is served by this resource. The entry/exit points of the resources need not be unique; they may coincide in order to express the possibility of offering joint service to users, that consists of a sequence of resources. We denote by V the set of all entry/exit points of the resources in the system. Any nonempty collection of resources corresponding to a directed path in $G \equiv (V, E)$ comprises an *action* in the system.

Let $N \equiv [n]$ be a set of users, each willing to adopt some action in the system. $\forall i \in N$, let w_i denote user i's *demand* (eg, the flow rate from a source node to a destination node), while $\Pi_i \subseteq 2^E \setminus \emptyset$ is the collection of actions, any of which would satisfy user i (eg, alternative routes from a source to a destination node, if G represents a communication network). The collection Π_i is called the *action set* of user i and each of its elements contains at least one resource. Any tuple $\varpi \in \Pi \equiv \times_{i=1}^{n} \Pi_i$ is a *pure strategies profile*, or a *configuration* of the users. Any real vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ s.t. $\forall i \in [n],\ p_i : \Pi_i \to [0, 1]$ is a probability distribution over the set of allowable actions for user i, is called a *mixed strategies profile* for the n users.

A congestion model typically deals with users of identical demands, and thus, resource delay functions depend on the *number* of users adopting each action ([21, 19, 7]). In this work we consider the more general case, where a *weighted congestion model* is the tuple $((w_i)_{i \in N}, (\Pi_i)_{i \in N}, (d_e)_{e \in E})$. That is, we allow the users to have different demands for service from the whole system, and thus affect the resource delay functions in a different way, depending on their own weights. The *weighted congestion game* associated with this model, is the game in strategic form with the set of users N and user demands $(w_i)_{i\in N}$, the action sets $(\Pi_i)_{i\in N}$ and cost functions $(\lambda^i_{\varpi_i})_{i\in N,\varpi_i\in\Pi_i}$ defined as follows: For any configuration $\varpi\in\Pi$ and $\forall e\in E$, let $\Lambda_e(\varpi)=\{i\in N : e\in\varpi_i\}$ be the set of users exploiting resource e according to ϖ. The *cost* $\lambda^i(\varpi)$ *of user* i *for adopting strategy* $\varpi_i\in\Pi_i$ in a given configuration ϖ is $\lambda^i(\varpi)=\lambda_{\varpi_i}(\varpi)=\sum_{e\in\varpi_i} d_e(\theta_e(\varpi))$ where, $\forall e\in E, \theta_e(\varpi)\equiv\sum_{i\in\Lambda_e(\varpi)} w_i$ is the load on resource e wrt the configuration ϖ. On the other hand, for a mixed strategies profile $\mathbf{p}$, the *expected cost of user* i *for adopting strategy* $\varpi_i\in\Pi_i$ is $\lambda^i_{\varpi_i}(\mathbf{p})=\sum_{\varpi^{-i}\in\Pi^{-i}} P(\mathbf{p}^{-i},\varpi^{-i})\cdot\sum_{e\in\varpi_i} d_e\left(\theta_e(\varpi^{-i}\oplus\varpi_i)\right)$ where, ϖ^{-i} is a configuration of all the users except for i, $\mathbf{p}^{-i}$ is the mixed strategies profile of all users except for i, $\varpi^{-i}\oplus\varpi_i$ is the new configuration with i choosing strategy ϖ_i, and $P(\mathbf{p}^{-i},\varpi^{-i})\equiv\prod_{j\in N\setminus\{i\}} p_j(\varpi_j)$ is the occurrence probability of ϖ^{-i}.

A congestion game in which all users are indistinguishable (ie, they have the same user cost functions) and have the same action set, is called *symmetric*. When each user's action set Π_i consists of sets of resources that comprise (simple) paths between a unique origin-destination pair of nodes (s_i,t_i) in (V,E), we refer to a *network congestion game*. If additionally all origin-destination pairs of the users coincide with a unique pair (s,t) we have a *single commodity network congestion game* and then all users share exactly the same action set. Observe that a single-commodity network congestion game is not necessarily symmetric because the users may have different demands and thus their cost functions will also differ.

Selfish Behavior. Fix an arbitrary (mixed in general) strategies profile $\mathbf{p}$ for a congestion game $((w_i)_{i\in N},(\Pi_i)_{i\in N},(d_e)_{e\in E})$. We say that $\mathbf{p}$ is a *Nash Equilibrium (NE)* if and only if $\forall i\in N$, $\forall\varpi_i,\pi_i\in\Pi_i,\ \ p_i(\varpi_i)>0\Rightarrow\lambda^i_{\varpi_i}(\mathbf{p})\le\lambda^i_{\pi_i}(\mathbf{p})$. A configuration $\varpi\in\Pi$ is a *Pure Nash Equilibrium (PNE)* if and only if $\forall i\in N,\forall\pi_i\in\Pi_i,\ \lambda_{\varpi_i}(\varpi)\le\lambda_{\pi_i}(\varpi^{-i}\oplus\pi_i)$ where, $\varpi^{-i}\oplus\pi_i$ is the same configuration with ϖ except for user i that now chooses action π_i. The *social cost* $SC(\mathbf{p})$ in this congestion game is $SC(\mathbf{p})=\sum_{\varpi\in\Pi} P(\mathbf{p},\varpi)\cdot\max_{i\in N}\{\lambda_{\varpi_i}(\varpi)\}$, where $P(\mathbf{p},\varpi)\equiv\prod_{i=1}^{n} p_i(\varpi_i)$ is the probability of configuration ϖ occurring, wrt the mixed strategies profile $\mathbf{p}$. The *social optimum* of this game is defined as $OPT=\min_{\varpi\in\Pi}\{\max_{i\in N}[\lambda_{\varpi_i}(\varpi)]\}$. The *price of anarchy* for this game is then defined as $\mathcal{R}=\max_{\mathbf{p}\text{ is a NE}}\left\{\frac{SC(\mathbf{p})}{OPT}\right\}$.

Configuration Paths and Dynamics Graph. For a congestion game $\Gamma=((w_i)_{i\in N},(\Pi_i)_{i\in N},(d_e)_{e\in E})$, a *path* in $\Pi=\times_{i\in N}\Pi_i$ is a sequence of configurations $\gamma=(\varpi(0),\varpi(1),\ldots,\varpi(k))$ s.t. $\forall j\in[k],\ \varpi(j)=(\varpi(j-1))^{-i}\oplus\pi_i$, for some $i\in N$ and $\pi_i\in\Pi_i$. γ is a *closed path* if $\varpi(0)=\varpi(k)$. It is a *simple path* if no configuration is contained in it more than once. γ is an *improvement path* wrt Γ, if $\forall j\in[k],\ \lambda^{i_j}(\varpi(j))<\lambda^{i_j}(\varpi(j-1))$ where i_j is the unique user differing in its strategy between $\varpi(j)$ and $\varpi(j-1)$. The *Dynamics Graph* of Γ is a directed graph whose vertices are configurations and there is an arc from a configuration ϖ to a configuration $\varpi^{-i}\oplus\pi_i$ for some $\pi_i\in\Pi_i$ if and only if $\lambda^i(\varpi)>\lambda^i(\varpi^{-i}\oplus\pi_i)$.

Layered Networks. We now define a special family of networks whose behavior wrt the price of anarchy (we shall prove that) is asymptotically equivalent to that of the parallel links model of [14], which is actually a 1-layered network: Let $\ell\ge 1$ be an integer. A directed network $G = (V, E)$ with a distinguished source - destination pair (s, t), $s, t \in V$, is *ℓ-layered* if every directed $s - t$ path has length exactly ℓ and each node lies on a directed $s - t$ path. In a layered network there are no directed cycles and all directed paths are simple. In the following, we always use m to denote the number $|E|$ of edges in an ℓ-layered network $G = (V, E)$.

Atomic Assignments. We consider *atomic* assignments of users to actions, ie, each user $i \in N$ requires all its demand w_i from exactly one allowable action $\varpi_i \in \Pi_i$. Nevertheless, we allow users to adopt mixed strategies. Our focus in this paper is two-fold: We are interested in families of resource delay functions for which the weighted single-commodity network congestion game has a PNE, and we are also interested in the price of anarchy for a special case of this problem where G has the form of an ℓ-layered network (to be defined later) and the delay functions are identical to the loads of the resources.

1.2 Related Work

Existence and Tractability of PNE. It is already known that the class of unweighted (atomic) congestion games (ie, users have the same demands and thus, the same affection on the resource delay functions) is guaranteed to have at least one PNE: actually, Rosenthal ([21]) proved that any potential game has at least one PNE and it is easy to write any unweighted congestion game as an exact potential game using Rosenthal's potential function[1] (eg, [7, Thm1]). In [7] it is proved that a PNE for any unweighted single-commodity network congestion game[2] (no matter what resource delay functions are considered, so long as they are non-decreasing with loads) can be constructed in polynomial time, by computing the optimum of Rosenthal's potential function, through a nice reduction to min-cost flow. On the other hand, it is shown that even for a symmetric congestion game or an unweighted multicommodity network congestion game, it is PLS-complete to find a PNE (though it certainly exists).

The special case of single-commodity, parallel-edges network congestion game where the resources are considered to behave as parallel machines, has been extensively studied in recent literature. In [9] it was shown that for the case of users with varying demands and uniformly related parallel machines, there is always a PNE which can be constructed in polynomial time. It was also shown that it is NP-hard to construct the best or the worst PNE. In [10] it was proved that the fully mixed NE (FMNE), introduced and thoroughly studied in [17], is worse than any PNE, and any NE is at most $(6 + \epsilon)$ times worse than the FMNE, for varying users and identical parallel machines. In [16] it was shown that the FMNE is the worst possible for the case of two related machines and tasks of the same size. In [15] it was proved that the FMNE is the worst possible when the global objective is the sum of squares of loads.

[8] studies the problem of constructing a PNE from any initial configuration, of social cost at most equal to that of the initial configuration. This immediately implies the existence of a PTAS for computing a PNE of minimum social cost: first compute a configuration of social cost at most $(1 + \epsilon)$ times the social optimum ([11]), and

[1] For more details on Potential Games, see [19].

[2] Since [7] only considers unit-demand users, this is also a symmetric network congestion game.

consequently transform it into a PNE of at most the same social cost. In [6] it is also shown that even for the unrelated parallel machines case a PNE always exists, and a potential-based argument proves a convergence time (in case of integer demands) from arbitrary initial configuration to a PNE in time $\mathrm{O}\left(mW_{\mathrm{tot}} + 4^{W_{\mathrm{tot}}/m + w_{\max}}\right)$ where $W_{\mathrm{tot}} = \sum_{i \in N} w_i$ and $w_{\max} = \max_{i \in N}\{w_i\}$.

[18] studies the problem of weighted parallel-edges network congestion games with user-specific costs: each allowable action of a user consists of a single resource and each user has its own private cost function for each resource. It is shown that: (1) weighted (parallel-edges network) congestion games involving only two users, or only two possible actions for all the users, or equal delay functions (and thus, equal weights), always possess a PNE; (2) even a single-commodity, 3-user, 3-actions, weighted (parallel-edges network) congestion game may not possess a PNE (using 3-wise linear delay functions).

Price of Anarchy in Congestion Games. In the seminal paper [14] the notion of coordination ratio, or price of anarchy, was introduced as a means for measuring the performance degradation due to lack of users' coordination when sharing common resources. In this work it was proved that the price of anarchy is $3/2$ for two related parallel machines, while for m machines and users of varying demands, $\mathcal{R} = \Omega\left(\log m / \log\log m\right)$ and $\mathcal{R} = \mathrm{O}\left(\sqrt{m \log m}\right)$. For m identical parallel machines, [17] proved that $\mathcal{R} = \Theta\left(\log m / \log\log m\right)$ for the FMNE, while for the case of m identical parallel machines and users of varying demands it was shown in [13] that $\mathcal{R} = \Theta\left(\log m / \log\log m\right)$. In [4] it was finally shown that $\mathcal{R} = \Theta\left(\log m / \log\log\log m\right)$ for the general case of related machines and users of varying demands. [3] presents a thorough study of the case of general, monotone delay functions on parallel machines, with emphasis on delay functions from queuing theory. Unlike the case of linear delays, they show that the price of anarchy for non-linear delays is in general far worse and often even unbounded.

In [22] the price of anarchy in a multicommodity network congestion game among infinitely many users, each of negligible demand, is studied. The social cost in this case is expressed by the total delay paid by the whole flow in the system. For linear resource delays, the price of anarchy is at most $4/3$. For general, continuous, non-decreasing resource delay functions, the total delay of any Nash flow is at most equal to the total delay of an optimal flow for double flow demands. [23] proves that for this setting, it is actually the class of allowable latency functions and not the specific topology of a network that determines the price of anarchy.

1.3 Our Contribution.

In this paper, we generalize the model of [14] (KP-model) to the weighted congestion games. We also define a special class of networks, the ℓ-layered networks, which demonstrate a rather surprising behavior: their worst instance wrt the price of anarchy is (within constant factors) the parallel links network introduced in [14]. More specifically, we prove that: (I) Weighted congestion games are *not* isomorphic to potential games. We show the existence of weighted single-commodity network congestion games with resource delays being either linear or 2-wise linear functions of the loads, for which there PNE cannot exist (lemma 1). (II) There exist weighted single-commodity network congestion games which admit no exact potential function, even when the resource delays are identical to their loads (lemma 2). (III) For a weighted ℓ-layered network congestion game with resource delays equal to their loads, at least one

PNE exists and can be constructed in pseudo-polynomial time (theorem 1). (IV) The price of anarchy of any weighted ℓ-layered network congestion game with m resources (edges) and resource delays equal to their loads, is at most $8\,\mathrm{e}\,(\frac{\log m}{\log\log m}+1)$, where e is the basis of the natural logarithm (theorem 2). To our knowledge this is the first time that the KP-model is studied in non-trivial networks (other than the parallel links).

2 Pure Nash Equilibria

In this section we deal with the existence and tractability of PNE in the weighted single-commodity network congestion games. First we show that it is not always the case that a PNE exists for such a congestion game, even when we allow only linear and 2-wise linear (ie, the maximum of two linear functions) resource delays. In contrast, it is well known ([21, 7]) that any unweighted (not necessarily single-commodity, or even network) congestion game has a PNE, for any kind of non-decreasing delays.

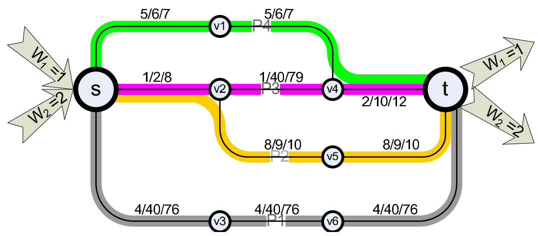

Fig. 1. A weighted single-commodity network congestion game that has no PNE, for two players with demands $w_1 = 1$ and $w_2 = 2$. The notation $a/b/c$ means that a load of 1 has delay a, a load of 2 has delay b and a load of 3 has delay c.

Lemma 1. *There exist instances of weighted single-commodity network congestion games with resource delays being either linear or 2-wise linear functions of the loads, for which there is no PNE.*

Proof. We demonstrate this by the example shown in figure 1. In this example there are exactly two users of demands $w_1 = 1$ and $w_2 = 2$, from node s to node t. The possible paths that the two users may follow are labeled in the figure. The resource delay functions are indicated by the 3 possible values they may take given the two users. Observe now that this example has no PNE: there is a simple closed path $\gamma = ((P3, P2), (P3, P4), (P1, P4), (P1, P2), (P3, P2))$ of length 4 that is an improvement path (actually, each defecting user moves to its new best choice) and additionally, any other configuration not belonging in γ is either one, or two best-choice moves away from some of these nodes. Therefore there is no sink in the Dynamics Graph of the game and thus there exists no PNE. Observe that the delay functions are not user-specific in our example, as was the case in [18]. □

Consequently we show that there may exist no exact potential function[3] for a weighted single-commodity network congestion game, even when the resource delays are identical to their loads. The next argument shows that theorem 3.1 of [19] does not hold anymore even in this simplest case of weighted congestion games.

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[3] Fix a vector $\mathbf{b} \in \mathbb{R}^n_{>0}$. $F : \times_{i\in N}\Pi_i \to \mathbb{R}$ is a $\mathbf{b}$-*potential* for a weighted congestion game $\Gamma = ((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$, if $\forall \varpi \in \times_{i\in N}\Pi_i,\ \forall i \in N, \forall \pi_i \in \Pi_i,\ \lambda^i(\varpi) - \lambda^i(\varpi^{-i} \oplus \pi_i) = b_i \cdot [F(\varpi) - F(\varpi^{-i} \oplus \pi_i)]$. It is an *exact potential* for Γ, if $\mathbf{b} = \mathbf{1}$.

Lemma 2. *There exist weighted single-commodity network congestion games which are not exact potential games, even for resource delays identical to their loads.*

Proof. Let $\Gamma = ((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$ denote a weighted single commodity network congestion game with $d_e(x) = x,\ \forall e \in E$. Let's define the quantity $I(\gamma, \lambda) = \sum_{k=1}^{r} \left[\lambda^{i_k}(\varpi(k)) - \lambda^{i_k}(\varpi(k-1))\right]$, where i_k is the unique user in which the configurations $\varpi(k)$ and $\varpi(k-1)$ differ. Our proof is based on the fact that Γ is an (exact) potential game if and only if every simple closed path γ of length 4 has $I(\gamma, \lambda) = 0$ ([19, Thm2.8]). Indeed, for an arbitrary initial configuration ϖ and any $\pi_1 \in \Pi_1 \setminus \{\varpi_1\},\ \pi_2 \in \Pi_2 \setminus \{\varpi_2\}$, we consider the closed, simple 4-path $\gamma = (\varpi,\ \varpi^{-1} \oplus \pi_1,\ \varpi^{-(1,2)} \oplus (\pi_1, \pi_2),\ \varpi^{-2} \oplus \pi_2),\ \varpi$. We then prove (see full paper) that $I = (w_1 - w_2) \cdot [|(\pi_1 \setminus \varpi_1) \cap (\pi_2 \setminus \varpi_2)| + |(\varpi_1 \setminus \pi_1) \cap (\varpi_2 \setminus \pi_2)| - |(\varpi_1 \setminus \pi_1) \cap (\pi_2 \setminus \varpi_2)| - - |(\pi_1 \setminus \varpi_1) \cap (\varpi_2 \setminus \pi_2)|]$, which is typically not equal to zero for a single-commodity network. It should be noted that the second parameter, which is network dependent, can be non-zero even for some cycle of a very simple network. For example, in the network of figure 1 (which is a simple 2-layered network) the simple closed path $\gamma = (\varpi(0) = (P1, P3), \varpi(1) = (P2, P3), \varpi(2) = (P2, P1), \varpi(3) = (P1, P1), \varpi(4) = (P1, P3))$ has this quantity equal to -4 and thus no weighted single commodity network congestion game on this network can admit an exact potential. □

Our next step is to focus our interest on the ℓ-layered networks with resource delays identical to their loads. We shall prove that any weighted ℓ-layered network congestion game with these delays admits at least one PNE, which can be computed in pseudo-polynomial time. Although we already know that even the case of weighted ℓ-layered network congestion games with delays equal to the loads cannot have any exact potential[4], we will next show that $\Phi(\varpi) \equiv \sum_{e\in E}[\theta_e(\varpi)]^2$ is a $\mathbf{b}$-potential for such a game and some positive n-vector $\mathbf{b}$, assuring the existence of a PNE.

Theorem 1. *For any weighted ℓ-layered network congestion game with resource delays equal to their loads, at least one PNE exists and can be computed in pseudo-polynomial time.*

Proof. Fix an arbitrary ℓ-layered network (V, E) and denote by $\mathcal{P}$ all the $s - t$ paths in it from the unique source s to the unique destination t. Let $\varpi \in \mathcal{P}^n$ be an arbitrary configuration of the users for the corresponding congestion game on (V, E). Also, let i be a user of demand w_i and fix some path $\pi_i \in \mathcal{P}$. Denote $\varpi' \equiv \varpi^{-i} \oplus \pi_i$. Observe that $\Phi(\varpi) - \Phi(\varpi') = \sum_{e\in\varpi_i\setminus\pi_i} \left(\theta_e^2(\varpi) - \theta_e^2(\varpi')\right) + \sum_{e\in\pi_i\setminus\varpi_i} \left(\theta_e^2(\varpi) - \theta_e^2(\varpi')\right) = \sum_{e\in\varpi_i\setminus\pi_i} \left([\theta_e(\varpi^{-i}) + w_i]^2 - \theta_e^2(\varpi^{-i})\right) + \sum_{e\in\pi_i\setminus\varpi_i} \left(\theta_e^2(\varpi^{-i}) - [\theta_e(\varpi^{-i}) + w_i]^2\right) = 2w_i \cdot \left(\sum_{e\in\varpi_i\setminus\pi_i} \theta_e(\varpi^{-i}) - \sum_{e\in\pi_i\setminus\varpi_i} \theta_e(\varpi^{-i})\right) = 2w_i \cdot \left[\lambda^i(\varpi) - \lambda^i(\varpi')\right]$, since, $\forall e \in \varpi_i \cap \pi_i\ \theta_e(\varpi) = \theta_e(\varpi')$, in ℓ-layered networks $|\varpi_i \setminus \pi_i| = |\pi_i \setminus \varpi_i|$, $\lambda^i(\varpi) = \sum_{e\in\varpi_i} \theta_e(\varpi) = \sum_{e\in\varpi_i\setminus\pi_i} \theta_e(\varpi^{-i}) + w_i|\varpi_i \setminus \pi_i| + \sum_{e\in\varpi_i\cap\pi_i} \theta_e(\varpi)$ and $\lambda^i(\varpi') = \sum_{e\in\pi_i} \theta_e(\varpi') = \sum_{e\in\pi_i\setminus\varpi_i} \theta_e(\varpi^{-i}) + w_i|\pi_i \setminus \varpi_i| + \sum_{e\in\varpi_i\cap\pi_i} \theta_e(\varpi)$. Thus, Φ is a $\mathbf{b}$-potential for our game, where $\mathbf{p} = (1/(2w_i))_{i\in N} > \mathbf{0}$, assuring the existence of at least one PNE.

[4] The example at the end of the proof of lemma 2 involves the 2-layered network of figure 1.

Wlog assume that the users have integer weights. Then each user performing any improving defection, must reduce its cost by at least 1 and thus the potential function decreases by at least $2w_{\min} \geq 2$ along each arc of the Dynamics Graph of the game. Consequently, the naïve algorithm that, starting from an arbitrary initial configuration $\varpi \in \mathcal{P}$, follows any improvement path that leads to a sink (ie, a PNE) of the Dynamics Graph, cannot move more than $\frac{1}{2}|E|W_{\text{tot}}^2$ times, since $\forall \varpi \in \mathcal{P}, \Phi(\varpi) \leq |E|W_{\text{tot}}^2$. $\square$

3 The price of anarchy in ℓ-layered networks

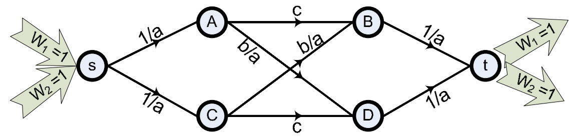

Fig. 2. Example of an ℓ-layered network with linear resource delays and unbounded anarchy.

In this section we focus our interest on weighted ℓ-layered network congestion games where the resource delays are identical to their loads. The main reason why we focus on this specific category of resource delays is that selfish unsplittable flows have usually unbounded price of anarchy. In [22, p. 256] an example is given where the price of anarchy is unbounded. This example is easily converted in an ℓ-layered network. The resource delay functions used are either constant or M/M/1-like delay functions. But we can be equally bad even with linear resource delay functions: Observe the following example of figure 2. Two users, each of unit demand, want to move selfishly from s to t. The edge delays are shown above them. We assume that $a \gg b \gg 1 \geq c$). It is easy to see that the configuration (sCBt,sADt) is a PNE of social cost $2 + b$ while the optimum configuration is (sABt,sCDt) whose social optimum is $2 + c$. Thus, $\mathcal{R} = \frac{b+2}{c+2}$.

So in this section we study weighted ℓ-layered networks whose resource delays equal their loads. Our main tool is to interpret mixed (in general) strategies profiles into some sort of (splittable) flows in this network.

Flows and Mixed Strategies Profiles. Fix an arbitrary ℓ-layered network $G = (V, E)$ and n distinct users willing to satisfy their own traffic demands from the unique source $s \in V$ to the unique destination $t \in V$. Again, $\mathbf{w} = (w_i)_{i \in [n]}$ denotes the varying demands of the users. Fix an arbitrary mixed strategies profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. A *feasible flow* for the n users is a function $\rho : \mathcal{P} \mapsto \mathbb{R}_{\geq 0}$, s.t. $\sum_{\pi \in \mathcal{P}} \rho(\pi) = W_{\text{tot}} \equiv \sum_{i \in [n]} w_i$, ie, all users' demands are actually met. We distinguish between *unsplittable* and *splittable* (feasible) flows. A flow is unsplittable if each user's traffic demand is satisfied by a unique path of $\mathcal{P}$. A flow is splittable if the traffic demand of each user is divided into infinitesimally small parts which are then routed over several paths of $\mathcal{P}$.

We map any profile $\mathbf{p}$ to a flow $\rho_{\mathbf{p}}$ as follows: $\forall \pi \in \mathcal{P},\ \rho_{\mathbf{p}}(\pi) \equiv \sum_{i \in [n]} w_i \cdot p_i(\pi)$. That is, we handle the *expected load traveling along* π *according to* $\mathbf{p}$ as a splittable flow created by all the users, where $\forall i \in [n]$, i routes a fraction $p_i(\pi)$ of its total demand w_i along π. Observe that for the special case where $\mathbf{p}$ is a pure strategies profile, the corresponding flow is then unsplittable. Recall now that $\forall e \in E,\ \theta_e(\mathbf{p}) \equiv \sum_{i=1}^n w_i \sum_{\pi \ni e} p_i(\pi) = \sum_{\pi \ni e} \rho_{\mathbf{p}}(\pi) \equiv \theta_e(\rho_{\mathbf{p}})$ is the expected load (and in our case, also the expected delay) of e wrt $\mathbf{p}$. As for the expected delay along a path $\pi \in \mathcal{P}$ according to $\mathbf{p}$, this is $\theta_\pi(\mathbf{p}) \equiv \sum_{e \in \pi} \theta_e(\mathbf{p}) = \sum_{e \in \pi} \sum_{\pi' \ni e} \rho_{\mathbf{p}}(\pi') =$

$\sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| \rho_{\mathbf{p}}(\pi') \equiv \theta_\pi(\rho_{\mathbf{p}})$. Let $\theta^{\min}(\rho) = \min_{\pi \in \mathcal{P}}\{\theta_\pi(\rho)\}$ be the minimum expected delay among all $s-t$ paths. From now on for simplicity we drop the subscript of $\mathbf{p}$ from its corresponding flow $\rho_{\mathbf{p}}$, when this is clear by the context. We evaluate flow ρ using the objective of *maximum latency among used paths*: $L(\rho) \equiv \max_{\pi:\rho(\pi)>0}\{\theta_\pi(\rho)\} = \max_{\pi:\exists i,\, p_i(\pi)>0}\{\theta_\pi(\mathbf{p})\} \equiv L(\mathbf{p})$. This is nothing more than the *maximum expected delay paid by the users*, wrt $\mathbf{p}$. Sometimes we also evaluate flow ρ using the objective of *total latency*: $C(\rho) \equiv \sum_{\pi \in \mathcal{P}} \rho(\pi)\theta_\pi(\rho) = \sum_{e \in E} \theta_e^2(\rho) = \sum_{e \in E} \theta_e^2(\mathbf{p}) \equiv C(\mathbf{p})$. We get the second equality by summing over the edges of π and reversing the order of the summation. From now on we denote by ρ^* and ρ_f^* the optimal unsplittable and splittable flows respectively.

Flows at Nash Equilibrium. Let $\mathbf{p}$ be a mixed strategies profile and let ρ be the corresponding flow. The cost of user i on path π is $\lambda_\pi^i(\mathbf{p}) = \ell w_i + \theta_\pi^{-i}(\mathbf{p})$ (G is an ℓ-layered network with resource delays equal to the loads), where $\theta_\pi^{-i}(\mathbf{p})$ is the expected delay along path π if the demand of user i was removed from the system: $\theta_\pi^{-i}(\mathbf{p}) = \sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| \sum_{j \neq i} w_j p_j(\pi') = \theta_\pi(\mathbf{p}) - w_i \sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| p_i(\pi')$ and thus, $\lambda_\pi^i(\mathbf{p}) = \theta_\pi(\mathbf{p}) + \left[\ell - \sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| p_i(\pi')\right] w_i$. Observe now that, if $\mathbf{p}$ is a NE, then $L(\mathbf{p}) = L(\rho) \leq \theta^{\min}(\rho) + \ell\, w_{\max}$. Otherwise, the users routing their traffic on a path of expected latency greater than $\theta^{\min}(\rho) + \ell\, w_{\max}$ could improve their latency by defecting to a path of expected latency $\theta^{\min}(\rho)$. When we say that a flow ρ corresponding to a mixed strategies profile $\mathbf{p}$ is a NE, we imply that it is actually $\mathbf{p}$ which is a NE.

Maximum Latency versus Total Latency. We show that a splittable flow is optimal wrt the objective of maximum latency if and only if it is optimal wrt the objective of total latency. As a corollary, we obtain that the optimal splittable flow defines a NE where all users adopt the same mixed strategy for their demands. Consider the $s-t$ flow polytope (FP): $\left\{\sum_{\pi \in \mathcal{P}} \rho(\pi) = W_{\text{tot}};\ \ \rho(\pi) \geq 0,\ \forall \pi \in \mathcal{P}\right\}$. One can ask for the flow that minimizes either $L(\rho) = \max_{\pi:\rho(\pi)>0}\{\theta_\pi(\rho)\}$, or $C(\rho) = \sum_{e \in E} \theta_e^2(\rho)$. For general resource delay functions the two objectives are different. However, in the special case that the delay of an edge is equal to the load routed through it, we prove that the two objectives are equivalent.

Lemma 3. *There is a unique splittable flow ρ which minimizes both $L(\rho)$ and $C(\rho)$.*

Proof. For every flow ρ, the average latency of ρ cannot exceed the maximum latency induced by ρ: $C(\rho) = \sum_{\pi \in \mathcal{P}} \rho(\pi)\theta_\pi(\rho) = \sum_{\pi:\rho(\pi)>0} \rho(\pi)\theta_\pi(\rho) \leq L(\rho) W_{\text{tot}}$. A (splittable) flow ρ minimizes $C(\rho) = \sum_{e \in E} \theta_e^2(\rho)$ if and only if for every $\pi_1, \pi_2 \in \mathcal{P}$: $\rho(\pi_1) > 0,\ \theta_{\pi_1}(\rho) \leq \theta_{\pi_2}(\rho)$ (eg, [2], [20, Section 7.2], [22, Corollary 4.2]). Hence, if ρ is optimal wrt the total latency, then $\forall\ \pi_1, \pi_2 \in \mathcal{P} : \rho(\pi_1) \cdot \rho(\pi_2) > 0,\ \theta_{\pi_1}(\rho) = \theta_{\pi_2}(\rho) = L(\rho)$, implying that $C(\rho) = \sum_{\pi \in \mathcal{P}:\rho(\pi)>0} \rho(\pi)\theta_\pi(\rho) = L(\rho) W_{\text{tot}}$.

Let ρ be the flow that minimizes the total latency and let ρ' be the flow that minimizes the maximum latency. We prove the lemma by establishing that the two flows are identical. Observe that $L(\rho') \geq \frac{C(\rho')}{W_{\text{tot}}} \geq \frac{C(\rho)}{W_{\text{tot}}} = L(\rho)$. The first inequality follows from the general bound on $C(\rho')$, while the rest comes from the assumption that ρ minimizes the total latency. On the other hand, $L(\rho') \leq L(\rho)$ due to the assumption that the flow ρ' minimizes the maximum latency. Hence, $L(\rho') = L(\rho)$ and $C(\rho') = C(\rho)$. Since the function $C(\rho)$ is strictly convex and the $s-t$ flow polytope (FP) is also convex, there is a unique flow which minimizes the total latency. □

Lemma 3 implies that the optimal splittable flow can be computed in polynomial time, since it is the solution of a convex program. The following corollary states that the optimal splittable flow defines a NE where all users follow exactly the same strategy.

Corollary 1. *Let ρ_f^* be the optimal splittable flow and $\mathbf{p}$ the mixed strategies profile where $\forall i \in N$ and $\forall \pi \in \mathcal{P}$, $p_i(\pi) = \rho_f^*(\pi)/W_{\mathrm{tot}}$. Then, $\mathbf{p}$ is a NE.*

Proof. See full paper. □

An Upper Bound on the Social Cost. We derive an upper bound on the social cost of any strategy profile whose maximum expected delay (ie, the maximum latency of its flow) is within a constant factor from the maximum latency of an optimal flow.

Lemma 4. *Let ρ^* be the optimal unsplittable flow, and let $\mathbf{p}$ be a mixed strategies profile and ρ its corresponding flow. If $L(\mathbf{p}) = L(\rho) \leq \alpha\, L(\rho^*)$, for some $\alpha \geq 1$, then, if $m = |E|$ is the number of edges in the network, $\mathrm{SC}(\mathbf{p}) \leq (\alpha+1)\,\mathrm{O}(\frac{\log m}{\log\log m})\, L(\rho^*)$.*

Proof. $\forall e \in E$ and $\forall i \in [n]$, let the r.v. describing the actual load routed through e by i be $X_{e,i} = w_i \cdot \mathbb{I}_{[i\text{'s demand is routed through a path } \pi \ni e]}$. Then, $\mathbb{E}[X_{e,i}] = \sum_{\pi \ni e} w_i p_i(\pi)$. Since each user selects its path independently, for each fixed edge e, the r.v.s of $\{X_{e,i}\}_{i\in[n]}$ are independent of each other. $\forall e \in E$, let $X_e = \sum_{i=1}^n X_{e,i}$ describe the actual load routed through e, and thus, the actual delay paid by any user traversing e. By linearity of expectation, $\mathbb{E}[X_e] = \theta_e(\rho)$. By applying the Hoeffding bound[5] with $w = w_{\max}$ and $t = \mathrm{e}\,\kappa \max\{\theta_e(\rho), w_{\max}\}$, we obtain that $\forall \kappa \geq 1$, $\mathbb{P}[X_e \geq \mathrm{e}\,\kappa \max\{\theta_e(\rho), w_{\max}\}] \leq \kappa^{-\mathrm{e}\,\kappa}$. By the union bound we conclude that $\mathbb{P}[\exists e \in E : X_e \geq \mathrm{e}\,\kappa \max\{\theta_e(\rho), w_{\max}\}] \leq m\kappa^{-\mathrm{e}\,\kappa}$. Now, $\forall \pi \in \mathcal{P} : \rho(\pi) > 0$, we define the r.v. $X_\pi = \sum_{e\in\pi} X_e$ describing the actual delay along π. The social cost of $\mathbf{p}$, which is equal to the expected maximum delay experienced by some user, cannot exceed the expected maximum delay among paths π with $\rho(\pi) > 0$. Formally, $\mathrm{SC}(\mathbf{p}) \leq \mathbb{E}[\max_{\pi:\rho(\pi)>0}\{X_\pi\}]$. If $\forall e \in E,\ X_e \leq \mathrm{e}\,\kappa \max\{\theta_e(\rho), w_{\max}\}$, then $\forall \pi \in \mathcal{P} : \rho(\pi) > 0$, $X_\pi \leq \mathrm{e}\,\kappa \sum_{e\in\pi} \max\{\theta_e(\rho), w_{\max}\} \leq \mathrm{e}\,\kappa \sum_{e\in\pi}(\theta_e(\rho) + w_{\max}) = \mathrm{e}\,\kappa\,(\theta_\pi(\rho) + \ell w_{\max}) \leq \mathrm{e}\,\kappa\,(L(\rho) + \ell w_{\max}) \leq \mathrm{e}\,(\alpha+1)\kappa\, L(\rho^*)$. The third equality follows from $\theta_\pi(\rho) = \sum_{e\in\pi} \theta_e(\rho)$, the fourth inequality from $\theta_\pi(\rho) \leq L(\rho)$ since $\rho(\pi) > 0$, and the last inequality from the hypothesis that $L(\rho) \leq \alpha\, L(\rho^*)$ and the fact that $\ell w_{\max} \leq L(\rho^*)$ because ρ^* is an unsplittable flow. Therefore, we conclude that $\mathbb{P}[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geq \mathrm{e}\,(\alpha+1)\kappa\, L(\rho^*)] \leq m\kappa^{-\mathrm{e}\,\kappa}$. In other words, the probability that the actual maximum delay caused by $\mathbf{p}$ exceeds the optimal maximum delay by a factor greater than $2\mathrm{e}\,(\alpha+1)\kappa$ is at most $m\kappa^{-\mathrm{e}\,\kappa}$. Therefore, for every $\kappa_0 \geq 2$, $\mathrm{SC}(\mathbf{p}) \leq \mathrm{e}\,(\alpha+1)L(\rho^*)\big(\kappa_0 + \sum_{k=\kappa_0}^{\infty} kmk^{-\mathrm{e}\,k}\big) \leq \mathrm{e}\,(\alpha+1)L(\rho^*)\big(\kappa_0 + 2m\kappa_0^{-\mathrm{e}\,\kappa_0+1}\big)$. If $\kappa_0 = \frac{2\log m}{\log\log m}$, then $\kappa_0^{-\mathrm{e}\,\kappa_0+1} \leq m^{-1}$, $\forall m \geq 4$. Thus, $\mathrm{SC}(\mathbf{p}) \leq 2\mathrm{e}\,(\alpha+1)(\frac{\log m}{\log\log m} + 1)L(\rho^*)$. □

Bounding the Coordination Ratio. We finally show that the maximum expected delay of every NE is a good approximation of the optimal maximum latency. Then, we can apply Lemma 4 to bound the price of anarchy for our selfish routing game.

[5] We use the following version of the Hoeffding bound ([12]): Let $X_1, X_2, \ldots, X_n$ be independent r.v.s with values in $[0, w]$. Let $X = \sum_{i=1}^n X_i$. Then, $\forall t > 0$, $\mathbb{P}[X \geq t] \leq \big(\frac{\mathrm{e}\,\mathbb{E}[X]}{t}\big)^{t/w}$.

Lemma 5. *For any flow ρ corresponding to a NE* $\mathbf{p}$, $L(\rho) \leq 3L(\rho^*)$.

Proof. We actually show that $L(\rho) \leq L(\rho^*_f) + 2\,\ell\,w_{\max}$, where ρ^*_f is the optimal splittable flow. This implies the lemma because $L(\rho^*) \geq \max\{L(\rho^*_f), \ell\,w_{\max}\}$. The proof is based on Dorn's Theorem [5] establishing strong duality in quadratic programming.

Let Q be the square matrix describing the number of edges shared by pairs of paths. Ie, $\forall \pi, \pi' \in \mathcal{P}$, $Q[\pi, \pi'] = |\pi \cap \pi'|$. Clearly Q is symmetric. We prove that it is also positive semi-definite (see full paper). $\forall \rho \in \mathbb{R}^{|\mathcal{P}|}_{\geq 0}$, the total latency of flow ρ is $C(\rho) = \rho^T Q \rho$. In addition, $(Q\rho)_\pi = \theta_\pi(\rho)$. Thus, the problem of computing a flow of value $W_{\rm tot}$ and minimum total latency is equivalent to computing the optimal solution of the following quadratic program (CP): $\min\{\rho^T Q \rho : \mathbf{1}^T \rho \geq W_{\rm tot}, \rho \geq \mathbf{0}\}$. Notice that no flow of value greater than $W_{\rm tot}$ can be optimal for CP. The Dorn's dual of (CP) is (DP): $\max\{z W_{\rm tot} - \rho^T Q \rho : 2Q\rho \geq \mathbf{1} z, z \geq 0\}$ (see, [5], [1, Chapter 6]). We observe that any flow ρ which is feasible for (CP) can be regarded as a feasible solution for (DP) if we set $z = 2\,\theta^{\min}(\rho)$. The objective value of the solution $(\rho, 2\theta^{\min}(\rho))$ in (DP) is $2\,\theta^{\min}(\rho)\,W_{\rm tot} - C(\rho)$. Hence, an intuitive way of thinking about the dual program is that it asks for the flow ρ that maximizes the difference $2\,\theta^{\min}(\rho)\,W_{\rm tot} - C(\rho)$.

By Dorn's Theorem [5], since$Q$ is symmetric and positive semi-definite and both (CP) and (DP) are feasible, they both have optimal solutions of the same objective value. In our case, the optimal splittable flow ρ^*_f, which is the optimal solution for (CP), corresponds to the solution $(\rho^*_f, 2\,\theta^{\min}(\rho^*_f))$, which is feasible for (DP). Moreover, for $\rho^*_f,\ L(\rho^*_f) = \theta^{\min}(\rho^*_f)$ and $C(\rho^*_f) = W_{\rm tot} L(\rho^*_f) = W_{\rm tot}\theta^{\min}(\rho^*_f)$ (see also the proof of Lemma 3). Thus, the objective value of the solution $(\rho^*_f, 2\,\theta^{\min}(\rho^*_f))$ in (DP) is exactly $C(\rho^*_f)$, and thus by Dorn's Theorem [5], $(\rho^*_f, 2\,\theta^{\min}(\rho^*_f))$ is optimal for (DP). For every feasible flow ρ for (CP), $(\rho, 2\,\theta^{\min}(\rho))$ is feasible for (DP). Since the optimal solution for (DP) has objective value $C(\rho^*_f)$, it must be $2\,\theta^{\min}(\rho)\,W_{\rm tot} - C(\rho) \leq C(\rho^*_f)$. If the flow ρ is a NE, then $L(\rho) \leq \theta^{\min}(\rho) + \ell\,w_{\max}$. Hence, it suffices to prove that $\theta^{\min}(\rho) \leq L(\rho^*_f) + \ell\,w_{\max}$. Since the average latency of ρ cannot exceed its maximum latency (see also the proof of Lemma 3), it is the case that $C(\rho) \leq L(\rho)\,W_{\rm tot} \leq \theta^{\min}(\rho)\,W_{\rm tot} + \ell\,w_{\max}\,W_{\rm tot}$. Combining this with the last inequality, we obtain that $\theta^{\min}(\rho)\,W_{\rm tot} \leq C(\rho^*_f) + \ell\,w_{\max}\,W_{\rm tot}$. Using $C(\rho^*_f) = L(\rho^*_f)\,W_{\rm tot}$, we conclude that $\theta^{\min}(\rho) \leq L(\rho^*_f) + \ell\,w_{\max}$. □

The following theorem is an immediate consequence of Lemma 5 and Lemma 4.

Theorem 2. *The price of anarchy of any ℓ-layered network congestion game with resource delays equal to their loads, is at most* $8\,\mathrm{e}\,(\frac{\log m}{\log\log m} + 1)$.

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