Tight Bounds on the Mutual Information of the Binary Input AWGN Channel

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Abstract — **We derive tight upper and lower bounds on the mutual information of the binary input AWGN channel with power constraint P. These bounds, while more complex than previous bounds, have the advantage of being very tight for all values of SNR, and are thus suitable for general purpose use.**

I. Introduction

To determine the capacity of many communication channels of interest, numerical integration is required. However, analytic bounds may be useful if they are sufficiently tight. For the binary input AWGN channel, several bounds exist (see [1] and the references therein), but they are only accurate for limited ranges in SNR.

II. AWGN Channel Capacity and Shaping Gain

Consider the AWGN channel with power-constrained input X, noise $Z \sim \mathcal{N}(0, N)$, and output $Y = X + Z$. The capacity of this channel is achieved when the distribution of the input X is also Gaussian, and is given by (1) in nats where P is the power constraint on the Gaussian input $X_g \sim \mathcal{N}(0, P)$.

$$C = I(X_g; Y_g) = \frac{1}{2} \ln \left(1 + \frac{P}{N}\right) \tag{1}$$

An input X_{ng} with non-Gaussian distribution is suboptimal from a capacity point of view. We can write the mutual information of the AWGN channel with input X_{ng} as the difference between (1) and its maximum shaping gain $I(X_g; Y_g) - I(X_{ng}; Y_{ng})$. For X_g and X_{ng} transmitting at the power constraint, the shaping gain can be expressed as the relative entropy between f_g and f_{ng}, which are the distributions of outputs Y_g and Y_{ng}, respectively. Thus the mutual information of the channel with input X_{ng} is given by $I(X_{ng}; Y_{ng}) = \frac{1}{2} \ln \left(1 + \frac{P}{N}\right) - D(f_{ng} || f_g)$.

III. Binary Input Mutual Information Bounds

Let X_b represent the binary input with iid symbols $\pm\sqrt{P}$. It can be shown that the relative entropy between the two output distributions $f_g(y)$ and $f_b(y)$ eventually yields:

$$\begin{aligned} D(f_b || f_g) = & \int_{-\infty}^{\infty} f_b(y) \left[\ln \frac{1}{2} + \frac{1}{2} \ln \left(1 + \frac{P}{N}\right) - \frac{P}{2N}\right] dy \\ & + 2 \int_{0}^{\infty} f_b(y) \frac{\sqrt{P}}{N} y dy \\ & + \int_{-\infty}^{\infty} f_b(y) \frac{-P}{2N(P+N)} y^2 dy \\ & + 2 \int_{0}^{\infty} f_b(y) \ln(1 + \exp(-2\sqrt{P}y/N)) dy. \end{aligned} \tag{2}$$

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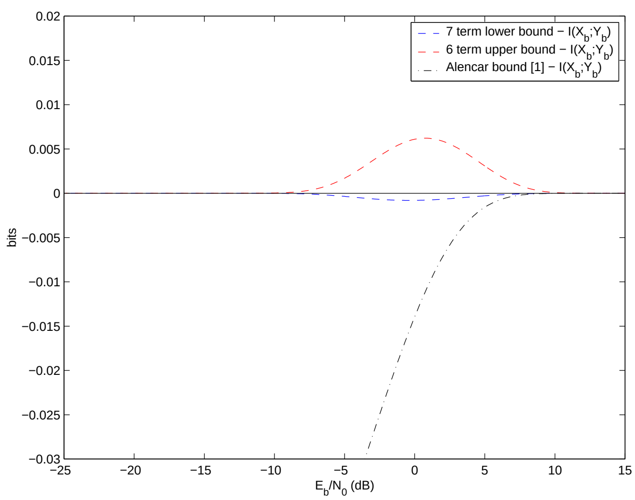

Fig. 1: Error in bounds on $I(X_b; Y_b)$.

The first three integrals are easily computed. There is no analytic solution for the last integral, so bounds are used.

Let $g(x) = \ln(1 + e^{-2x})$, where $x = \sqrt{P}y/N$. For $x > 1$, $g(x)$ is tightly bounded by:

$$\ln(2) e^{-2x} \leq \ln(1 + e^{-2x}) \leq e^{-2x}. \tag{3}$$

For $x < 1$, $g(x)$ is tightly bounded by its truncated Taylor series expansion around $x = 0$:

$$\ln(1 + e^{-2x}) = \ln 2 - x + \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \cdots \tag{4}$$

Truncating the series to an odd number of terms gives an upper bound on $g(x)$; an even number of terms gives a lower bound. The series bound can be made arbitrarily tight by using more terms in the expansion.

Replacing $g(x)$ in last integral in (2) with a combination of the bounds given by (3) and (4) gives tight bounds on $D(f_b || f_g)$, and thus $I(X_b; Y_b)$.

We have derived tight bounds using 6 and 7 terms from the Taylor series expansion. Interested readers can download the MATLAB code for these bounds from our website[2]. While these bounds are complicated, they give nearly exact results. Fig. 1 plots the error in bits as the difference between these bounds and $I(X_b; Y_b)$, which was obtained via numerical integration. Also shown for comparison is the error of the bound by Alencar [1].

References

[1] M.S. Alencar, "A comparison of bounds on the capacity of a binary channel," in *Proceedings of GLOBECOM'96*, London, UK, Nov. 1996, pp. 1273–1275.

[2]http://www.ee.ucla.edu/~wesel/publications.html