A Study on Iterative Learning Control with Adjustment of Learning Interval for Monotone Convergence in the Sense of Sup-norm

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Abstract

It has been found that some huge overshoot in the sense of sup-norm may be observed when typical iterative learning control (ILC) algorithms are applied to LTI systems, even though monotone convergence in the sense of λ-norm is guaranteed. In this paper, a new ILC algorithm with adjustment of learning interval is proposed to resolve such an undesirable phenomenon, and it is shown that the output error can be monotonically converged to zero in the sense of sup-norm when the proposed ILC algorithm is applied. A numerical example is given to show the effectiveness of the proposed algorithm.

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1 Introduction

There has been a number of studies in designing advanced control systems to guarantee that a given output trajectory will be executed by the system with acceptable accuracy from the demand for high precision control technique. As one of the alternatives, the iterative learning control (ILC) method has been developed by which complete tracking performance can be achieved as the given task is imposed iteratively[1, 2, 3, 4, 5].

The important task in designing of iterative learning controller is to find an algorithm for generating the next input in such a way that the output error is reduced on successive trials. This is usually quantified by showing that the error converges in the sense of some norm. In most of the investigations, the λ-norm is adopted as a measure of distance between two time functions to prove the convergence of the ILC algorithms.

For a vector function $h : [0, T] \to \mathcal{R}^n, h(t) = (h^1(t), \cdots h^n(t))^T$ and a real number $\lambda > 0$, the formal definition of the λ-norm[6] is given by

$$\|h(\cdot)\|_\lambda = \sup_{0 \le t \le T} e^{-\lambda t} \|h(t)\|_\infty.$$

where $\|h(t)\|_\infty = \sup_{1 \le i \le n} |h^i(t)|$. From the definition of λ-norm, it is easily shown that

$$\|h(\cdot)\|_\lambda \leq \sup_{0\leq t\leq T} \|h(t)\|_\infty \leq e^{\lambda T}\|h(\cdot)\|_\lambda$$

and thus the λ-norm is equivalent to the sup-norm defined by

$$\|h[0,T]\|_{sup} = \sup_{0\leq t\leq T} \|h(t)\|_\infty.$$

where $h[0,T]$ denotes a time function h defined over time interval $[0,T]$.

Thus, the convergence property in the sense of sup-norm seems to be equivalently obtained in the sense of λ-norm. We can, however, observe some huge overshoot in the sense of sup-norm even though the monotone convergence is guaranteed in the sense of λ-norm.

We remark that, in the real-world applications, the maximum absolute magnitude of the error signal may be of major concern, which can cause failure of hardware components. In analyzing the behavior of a system equiped with an ILC algorithm, it is more practical and sometimes necessary to investigate the routes of convergence in the sense of sup-norm.

Such undesirable phenomenon of the λ-norm was first observed by Lee and Bien[7], and it was reported in [8] that the pure error term of a PD-type ILC algorithm plays an important role in a bound of the interval where the monotone convergence is guaranteed in the sense of sup-norm. To be more specific, consider the linear system described by (1) and the PD-type ILC law described by (2).

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

$$u_{k+1}(t) = u_k(t) + \Gamma\left(\dot{e}_k(t) - Re_k(t)\right) \quad (2)$$

Here, $e_k(t) = y_d(t) - y_k(t)$ is the output error, and $x \in \mathcal{R}^n$, $u \in \mathcal{R}^r$ and $y \in \mathcal{R}^m$ denote the state, the input and the output, respectively. A, B and C are matrices with appropriate dimensions and it is assumed that CB is a full rank matrix. Let $y_d(\cdot)$ be the desired output trajectory, and $u_d(\cdot)$ and $x_d(\cdot)$ be the corresponding input trajectory and state trajectory, respectively. It is shown by Lee and Bien[8] that if the desired output trajectory is given on the interval $t \in [0, T_{sup}]$ where T_{sup} is bounded by

$$T_{sup} < \frac{1}{\|A\|_\infty} \, ln\left(1 + \frac{\|A\|_\infty(1-\rho)}{\|\Gamma(CA - RC)\|_\infty \|B\|_\infty}\right)$$

then, there exists $\rho_u < 1$ such that

$$\|\Delta u_{k+1}[0, T_{sup}]\|_{sup} \leq \rho_u \|\Delta u_k[0, T_{sup}]\|_{sup}$$

where $\Delta u_k(\cdot) = u_d(\cdot) - u_k(\cdot)$.

Thus, we find that the upper bound of T_{sup} depends on $\|CA - RC\|_\infty$: if R is chosen such that $\|CA - RC\|_\infty$ is arbitrarily small, then T_{sup} becomes very large, implying that the interval for monotone convergence in the sense of sup-norm becomes wider. If the given time interval T_{sup} is fixed, on the other hand, we have to obtain an accurate model of the plant in order to get a desired error convergence behavior. Furthermore, when the upper bound T of the given time interval $[0, T]$ is larger than T_{sup}, we can not guarantee the monotone convergence of the output error.

In this paper, a new ILC algorithm is proposed with adjustment of learning interval, which is found to be more robust against parameter uncertainty, and achieves monotone convergence of the output error in the sense of sup-norm.

2 Main Result

In this section, monotone convergence of output error in the sense of sup-norm is shown for LTI systems. First, we show that the convergence can be proved directly from the sup-norm, not by using λ-norm. For this end, consider LTI system described by (1) and the ILC algorithm described by (2).

Before showing convergence of the ILC law (2), we need the following

Lemma 1, whose result is utilized in the proof of the convergence in the sense of sup-norm.

Lemma 1 *Let* $a_{k,i}$ *be a nonnegative function for every nonnegative integers* $k, i \geq 0$*, and let* α *and* β *be nonnegative constants. Suppose* $a_{k,i} = 0, \forall i < 0$ *and* $0 \leq \rho < 1$*. Then, the inequality*

$$0 \leq a_{k+1,i} \leq \rho a_{k,i} + \beta \sum_{j=0}^{i-1} \alpha^{i-j} a_{k,j}, \forall k, i \geq 0$$

implies

$$\lim_{k \to \infty} a_{k,i} = 0, \forall i \geq 0. \tag{3}$$

Proof: For the proof, we employ the method of methematical induction. For each $i \geq 0$, let P_i be the statement that

$$\lim_{k \to \infty} a_{k,i} = 0.$$

From the conditions, we obtain that $0 \leq a_{k+1,0} \leq \rho a_{k,0}$ and $0 \leq \rho < 1$. So, the statement P_0 is true. That is

$$\lim_{k \to \infty} a_{k,0} = 0.$$

Now suppose that statement P_n is true for every interger n with $0 \leq n < m$. Then it is easily seen that

$$\lim_{k \to \infty} \beta \sum_{j=0}^{m-1} \alpha^{m-j} a_{k,j} = 0.$$

This implies that for any $\epsilon > 0$, there exists a positive K such that

$$\beta \sum_{j=0}^{m-1} \alpha^{m-j} a_{k,j} < \epsilon$$

for all $k \geq K$. This gives us

$$a_{k+1,m} < \rho a_{k,m} + \epsilon, \forall k \geq K.$$

We can choose ϵ' as follows:

$$a_{k,m} - \rho^k a_{0,m} + \frac{\rho^k}{1-\rho}\epsilon < \frac{\epsilon}{1-\rho} = \epsilon'.$$

This means that for any ϵ', there exists a positive K' such that

$$a_{k,m} - \rho^k a_{0,m} + \frac{\rho^k}{1-\rho}\epsilon < \epsilon'$$

for all $k \geq K'$. So, we can write

$$\lim_{k\to\infty} a_{k,m} = \lim_{k\to\infty} \rho^k a_{0,m} - \lim_{k\to\infty} \frac{\rho^k}{1-\rho}\epsilon.$$

Since $0 \leq \rho < 1$, we can conclude that

$$\lim_{k\to\infty} a_{k,m} = 0$$

which establishes the truth of the statement P_m. By mathematical induction, (3) is true. This completes the proof.

Now, the convergence of the ILC law (2) will be shown.

Theorem 1 *Suppose that the update law (2) is applied to the system (1) and that the initial state at each iteration is the same as the desired initial state, i.e.,* $x_k(0) = x_d(0)$ *for* $k = 0, 1, 2, \cdots$. *If*

$$\|I - CB\Gamma\|_\infty \leq \rho < 1$$

then,

$$\lim_{k \to \infty} \|e_k[0,T]\|_{sup} = 0. \tag{4}$$

Proof: It follows from (2) that

$$\begin{aligned} e_{k+1}(t) &= e_k(t) - C\int_0^t e^{A(t-\tau)} B\Gamma\left(\dot{e}_k(\tau) - Re_k(\tau)\right) d\tau \\ &= \left(I - CB\Gamma\right) e_k(t) - C\int_0^t e^{A(t-\tau)}\left(AB\Gamma - B\Gamma R\right) e_k(\tau) d\tau. \end{aligned} \tag{5}$$

Let ϵ be a real number satisfying the following inequality:

$$0 < \epsilon < \frac{1}{a} ln\left[1 + \frac{a(1-\rho)}{\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty}\right]$$

and

$$t_{i+1} = t_i + \epsilon, i = 0, 1, \cdots, N$$

$$t_0 = 0, \quad t_{N+1} = T,$$

where $a = \|A\|_\infty$. Then, from (5), we find that, for $t \in [i\epsilon, (i+1)\epsilon]$,

$$e_{k+1}(t) = \left(I - CB\Gamma\right) e_k(t) - C\int_0^{t_1} e^{A(t-\tau)}\left(AB\Gamma - B\Gamma R\right) e_k(\tau) d\tau$$

$$
\begin{aligned}
&-C\int_{t_1}^{t_2} e^{A(t-\tau)}\left(AB\Gamma - B\Gamma R\right) e_k(\tau) d\tau \\
&- \cdots \\
&-C\int_{t_i}^{t} e^{A(t-\tau)}\left(AB\Gamma - B\Gamma R\right) e_k(\tau) d\tau. \qquad (6)
\end{aligned}
$$

Taking the sup-norm on both sides of (6), we find that

$$
\begin{aligned}
&\|e_{k+1}[t_i, t_{i+1}]\|_{sup} \le \rho \|e_k[t_i, t_{i+1}]\|_{sup} \\
&+\frac{1}{a}\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty \sum_{j=0}^{i} e^{a(i-j)\epsilon}\left(e^{a\epsilon}-1\right)\|e_k[t_j, t_{j+1}]\|_{sup}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
\rho_0 &= \rho + \frac{1}{a}\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty \left(e^{a\epsilon}-1\right) \\
&< \rho + \frac{1}{a}\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty \left(\left[1+\frac{a(1-\rho)}{\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty}\right]-1\right) \\
&= 1.
\end{aligned}
$$

Since $0 \le \rho_0 < 1$, from Lemma 1, we can conclude that

$$
\lim_{k\to\infty} \|e_k[t_i, t_{i+1}]\|_{sup} = 0, \forall i \in \{0, 1, \cdots, N\}.
$$

This completes the proof.

Theorem 1 shows that the convergence of the ILC algorithm can be directly proved from the sup-norm, which is different from conventional proof of the convergence using the λ-norm. From the proof, we can observe that the convergence in the sense of sup-norm in the ith subinterval, $[t_i, t_{i+1}]$, is guaranteed by the convergence in the prior subintervals, $[t_0, t_1], \cdots, [t_{i-1}, t_i]$. This means that the output trajectory converges one after another to the desired output trajectory from the prior time to the end of the given time interval while the ILC algorithm is applied repetitively.

Now, we propose a new type of ILC algorithm, which guarantees monotone convergence of the output error in the sense of sup-norm. Consider the following PD-type ILC algorithm with adjustment of learning interval.

$$u_{k+1}(t) = u_k(t) + \Gamma\left[\dot{e}_k(t) - Re_k(t)\right], 0 \leq t \leq t_k^{\lambda} \tag{7}$$

Here, t_k^{λ} is the maximum value among the time when $e^{-\lambda t}\|e_k(t)\|_{\infty}$ takes its maximum value over the given time interval $[0, T]$, that is,

$$t_k^{\lambda} = sup\left\{t' \,\middle|\, e^{-\lambda t'}\|e_k(t')\|_{\infty} = \sup_{0 \leq t \leq T} e^{-\lambda t}\|e_k(t)\|_{\infty}\right\}$$

where λ is a real number satisfying the following inequality:

$$\lambda > a + \frac{\|C\|_{\infty}\|AB\Gamma - B\Gamma R\|_{\infty}}{1-\rho}, \tag{8}$$

and

$$e_{k+1}(t) = e_k(t), t_k^{\lambda} \leq t \leq T.$$

Monotone convergence of the proposed ILC algorithm is presented in the following theorem.

Theorem 2 *Suppose that the update law (7) is applied to the system (1) and that the initial state at each iteration is same as the desired initial state, i.e.,* $x_k(0) = x_d(0)$ *for* $k = 0, 1, 2, \cdots$. *If*

$$\|I - CB\Gamma\|_\infty \leq \rho < 1$$

then, there exists a constant $\rho_0, 0 \leq \rho_0 < 1$, *such that*

$$\sup_{0 \leq t \leq t_k^\lambda} \|e_{k+1}(t)\|_\infty \leq \rho_0 \sup_{0 \leq t \leq t_k^\lambda} \|e_k(t)\|_\infty.$$

Proof: Similarly in the proof of the Theorem 1, we can obtain that

$$\begin{aligned} e_{k+1}(t) &= e_k(t) - C \int_0^t e^{A(t-\tau)} B\Gamma \left[\dot{e}_k(\tau) - Re_k(\tau)\right] d\tau \\ &= \left(I - CB\Gamma\right) e_k(t) + C \int_0^t e^{A(t-\tau)} \left(AB\Gamma - B\Gamma R\right) e_k(\tau) d\tau \qquad (9) \end{aligned}$$

Taking the λ-norm on both sides of (9), we find that

$$\begin{aligned} \|e_{k+1}(\cdot)\|_\lambda &\leq \rho \|e_k(\cdot)\|_\lambda \\ &\quad + \|C\|_\infty \int_0^t e^{-(a-\lambda)(t-\tau)} \|AB\Gamma - B\Gamma R\|_\infty d\tau \|e_k(\cdot)\|_\lambda \\ &\leq \left(\rho + \frac{1}{\lambda - a} \|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty\right) \|e_k(\cdot)\|_\lambda. \qquad (10) \end{aligned}$$

From (8) and (10), we further find that

$$\begin{aligned} \rho_0 &= \rho + \frac{1}{\lambda - a} \|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty \\ &< \rho + \frac{1 - \rho}{\|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty} \|C\|_\infty \|AB\Gamma - B\Gamma R\|_\infty \\ &= 1. \end{aligned}$$

Let t_{k+1}^s be the time when $\|e_{k+1}(t)\|_\infty$ takes its maximum value on $[0, t_k^\lambda]$. Then

$$\begin{aligned}
e^{-\lambda t_{k+1}^s}\|e_{k+1}(t_{k+1}^s)\|_\infty &\leq \|e_{k+1}(\cdot)\|_\lambda \leq \rho_0\|e_k(\cdot)\|_\lambda \\
&= \rho_0 e^{-\lambda t_k^\lambda}\|e_k(t_k^\lambda)\|_\infty \\
&\leq \rho_0 e^{-\lambda t_k^\lambda} \sup_{0\leq t\leq t_k^\lambda} \|e_k(t)\|_\infty. \qquad (11)
\end{aligned}$$

From (11), we can find that

$$\begin{aligned}
\sup_{0\leq t\leq t_k^\lambda} \|e_{k+1}(t)\|_\infty &= \|e_{k+1}(t_{k+1}^s)\|_\infty \\
&\leq \rho_0 e^{-\lambda(t_k^\lambda - t_{k+1}^s)} \sup_{0\leq t\leq t_k^\lambda} \|e_k(t)\|_\infty. \qquad (12)
\end{aligned}$$

Since $t_k^\lambda \geq t_{k+1}^s$, we can conclude that

$$\sup_{0\leq t\leq t_k^\lambda} \|e_{k+1}(t)\|_\infty \leq \rho_0 \sup_{0\leq t\leq t_k^\lambda} \|e_k(t)\|_\infty$$

This completes the proof.

A simple method that ensures the monotone convergence is to divide the given time interval $[0, T]$ into subintervals by T_{sup} and to apply the ILC algorithm (2) in a subinterval after the output error converges to zero in prior time interval. Theorem 2 shows, however, that if the proposed ILC algorithm (7) is applied, then the control input can be updated ensuring the monotone convergence, even though the output error in the time interval $[0, T_{sup}]$ is not converged to zero. We remark that t_k^λ approaches to T as $k \to \infty$, since $\|e_k(\cdot)\|_\lambda$ converges to zero.

The upper bound of the learning interval, t_k^λ depends on the parameters of the plant and the learning gain R, and we can choose R as follow as commented in [8]:

$$R = \left(\left(\tilde{B}\Gamma\right)^T \tilde{B}\Gamma\right)^{-1} \left(\tilde{B}\Gamma\right)^T \tilde{A}\tilde{B}\Gamma,$$

where $\tilde{A}$ and $\tilde{B}$ are models of A and B of the system (1), respectively.

3 Numerical Example

The following example is given to illustrate the effectiveness of the proposed algorithm.

Example 1:

Consider the following linear time-invariant system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0.1 \\ 0.02 & -0.03 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -0.1 & 0.2 \end{bmatrix} x(t). \end{aligned}$$

Let the desired output trajectory be given as follows.

$$y_d(t) = t(6 - t), 0 \leq t \leq 6$$

Based on the result in [8], suppose that the following ILC algorithm is applied:

$$u_{k+1}(t) = u_k(t) + 3\left(\dot{e}_k(t) + 0.7 e_k(t)\right). \tag{13}$$

Here, Γ is chosen as 3 under the assumption of 10% uncertainty of the system parameters, and R is chosen as -0.7. It is already known that the ILC algorithm (13) makes the output error monotonically decrease in the sense of λ-norm as shown in Figure 1. In Figure 2 (a), (b), (c) and (d), the output trajectories at the 2nd, 4th, 5th and 7th iteration are shown, respectively. We can easily observe that the output trajectory converges to the desired output trajectory from the forepart of the time interval to the end part of the time interval. Figure 3 shows, however, that there is a huge overshoot in the sense of supnorm, even though the output error monotonically decrease in the sense of λ-norm as shown in Figure 1.

Now, consider the proposed ILC algorithm with the same learning gains with the ILC algorithm (13) with $\lambda = 0.2$:

$$u_{k+1}(t) = u_k(t) + 3\left(\dot{e}_k(t) + 0.7 e_k(t)\right), 0 \leq t \leq t_k^{\lambda}.$$

Figure 4 (a), (b), (c) and (d) show the output trajectories at 2nd, 4th, 6th and 8th iteration. In Figure 5, we can easily observe that there is no overshoot and the output error converges to zero monotonically in the sense of sup-norm for the proposed ILC algorithm, while the conventional PD-type ILC algorithm causes a huge overshoot. Since the control input is not updated over time interval $[t_k^\lambda, T]$ during the prior some iterations, the sup-norm of the output error in whole time interval $[0, T]$ may not decrease where the sup-norm over $[t_k^\lambda, T]$ is larger than the sup-norm over $[0, t_k^\lambda]$.

4 Concluding Remark

In this paper, we first presented a new proof of the convergence by direct use of the sup-norm, which is different from the conventional proof using the λ-norm. Then, we proposed a new type of ILC algorithm with adjustment of learning interval to ensure the monotone convergence of the output error and investigated the relation between the upper bound of the learning interval, t_k^λ, and the learning gain.

When the initial state at each iteration can be different from the desired initial state, i.e., $x_k(0) = x_0 \neq x_d(0)$ for $k = 0, 1, 2, \cdots$, then we can easily show that, based on the result in [9], there exists a constant $\rho_1, 0 \leq \rho_1 < 1$, such that

$$\sup_{0 \leq t \leq t_k^\lambda} \|\tilde{e}_{k+1}(t)\|_{sup} \leq \rho_1 \sup_{0 \leq t \leq t_k^\lambda} \|\tilde{e}_k(t)\|_{sup}$$

where

$$\begin{aligned} y_a(t) &= y_d(t) + e^{Rt} C\left(x_0 - x_d(0)\right) \\ \tilde{e}_k(t) &= y_a(t) - y_k(t). \end{aligned}$$

It is remarked that the monotone convergence in the sense of sup-norm for nonlinear systems is open to further investigation.

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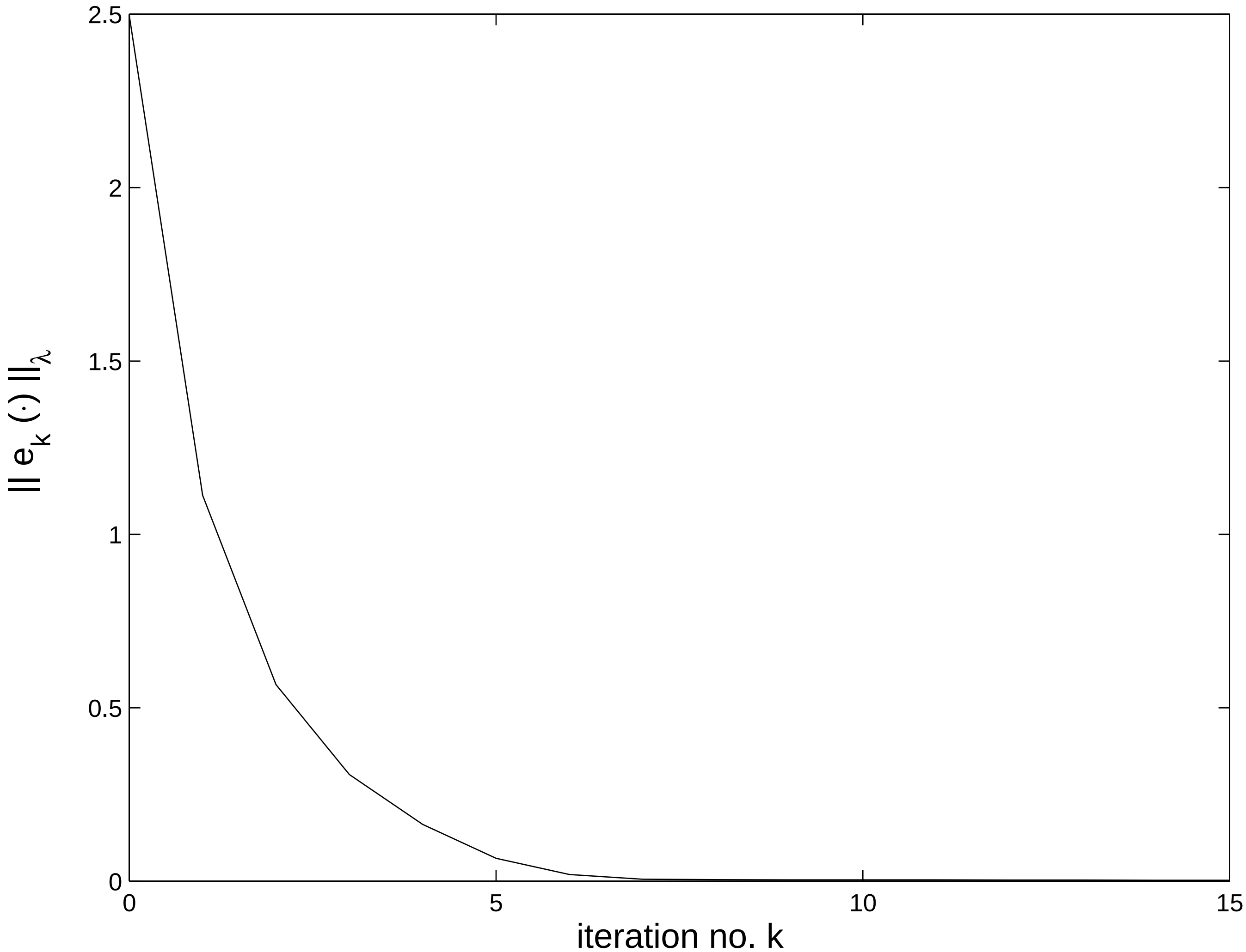
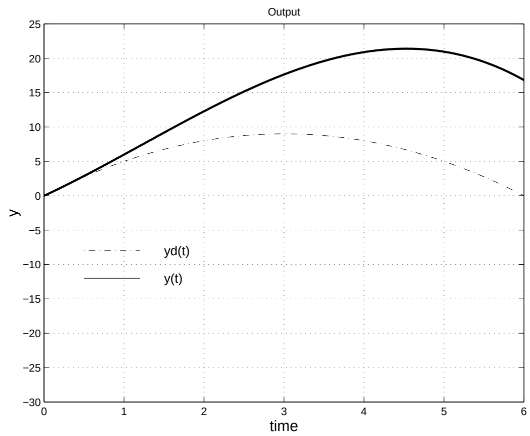
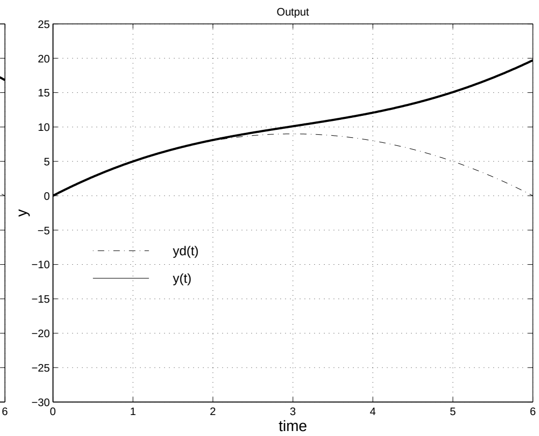


Figure 1: Trend of convergence of the output error in the sense of λ-norm



(a)



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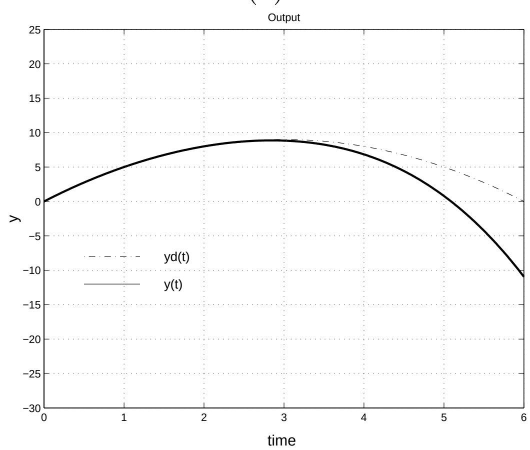

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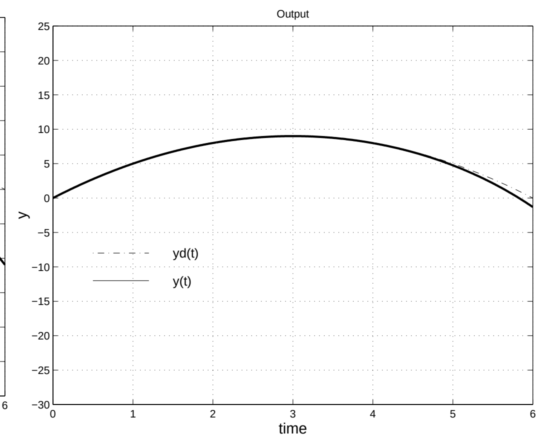


(d)

Figure 2: The output trajectory at each iteration

(a) The output at 2nd iteration

(b) The output at 4th iteration

(c) The output at 5th iteration

(d) The output at 7th iteration

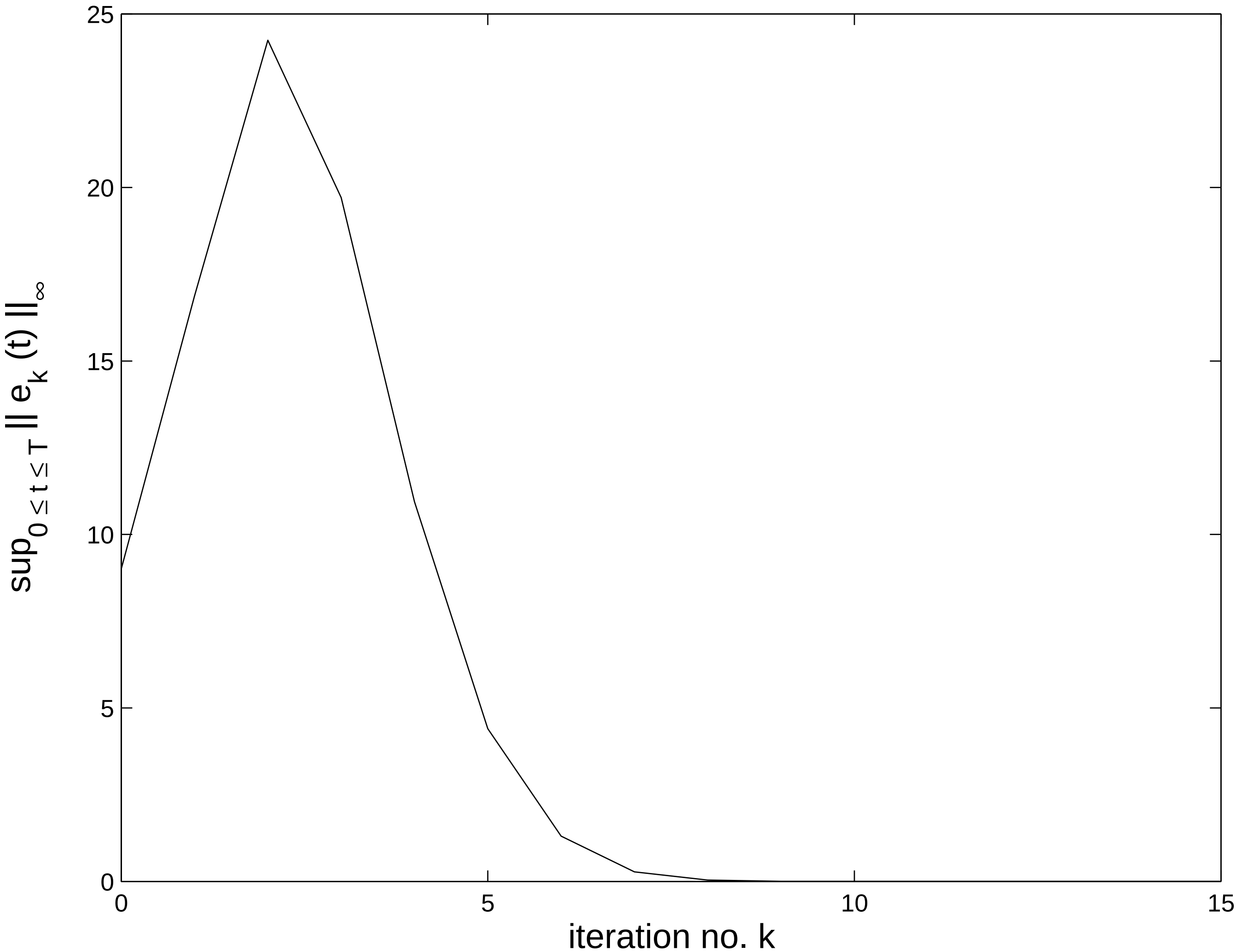
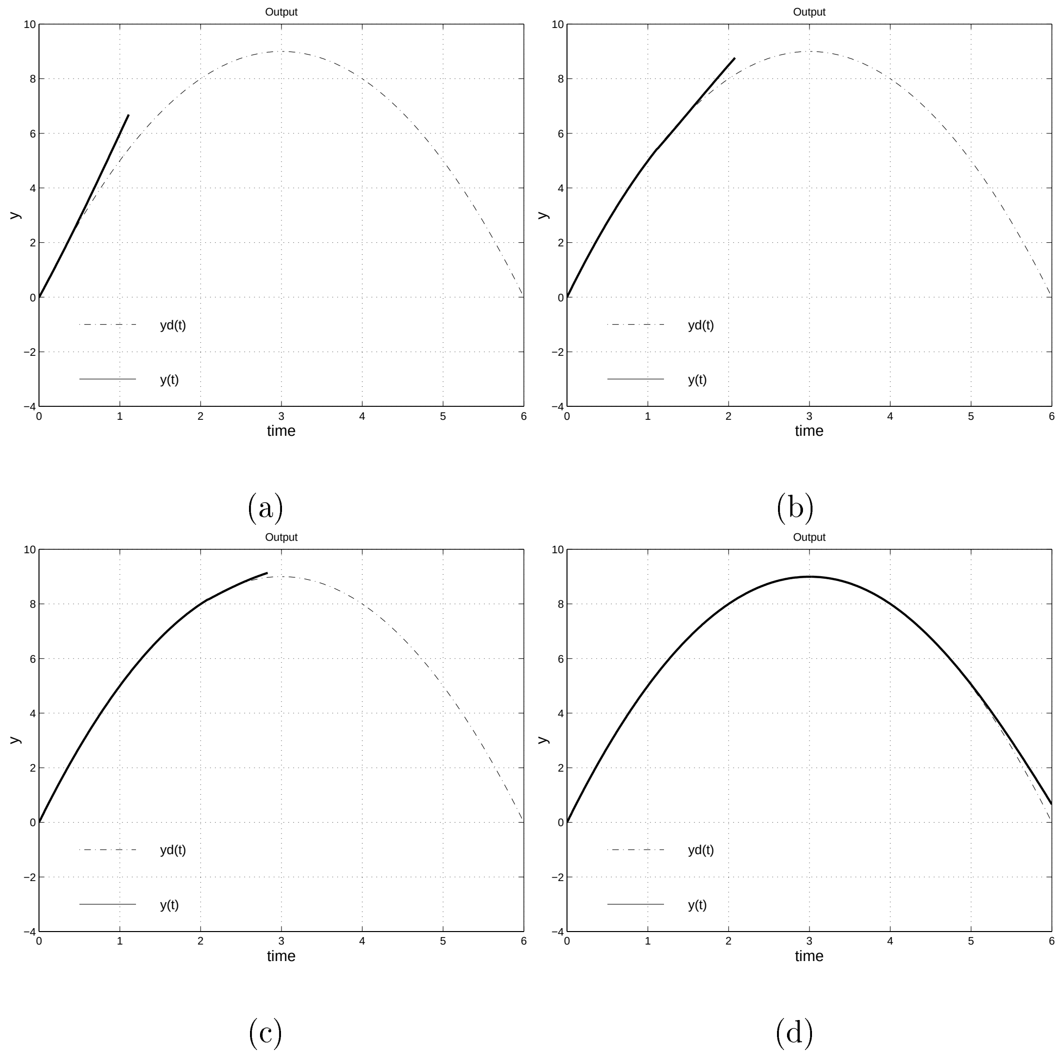


Figure 3: Trend of convergence in the sense of sup-norm for the conventional ILC algorithm



(a) (b) (c) (d)

Figure 4: The output trajectory at each iteration

(a) The output at 2nd iteration

(b) The output at 4th iteration

(c) The output at 6th iteration

(d) The output at 8th iteration

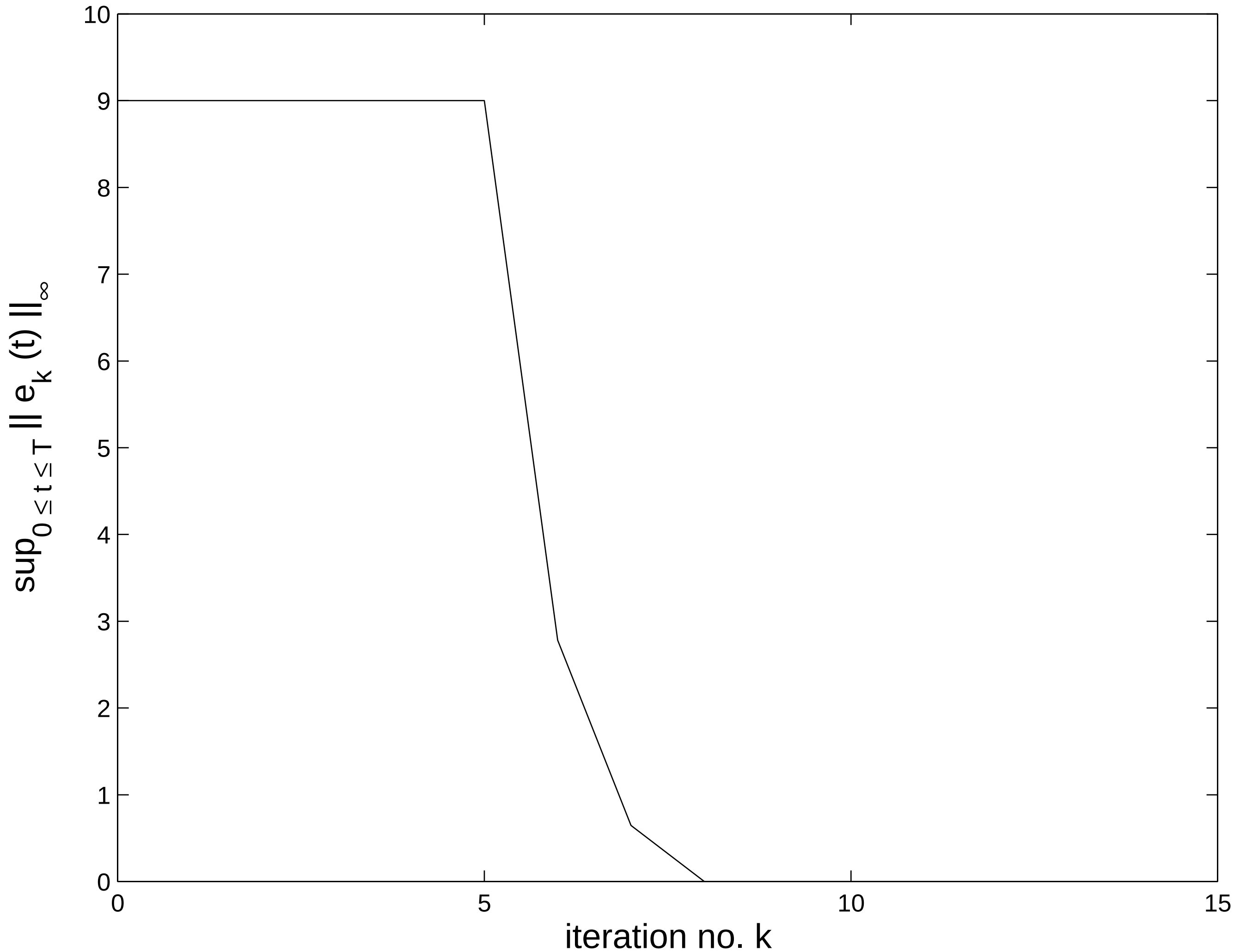


Figure 5: Trend of convergence int the sense of sup-norm for the proposed ILC algorithm