A Mathematical Framework for Global Illumination Algorithms

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1 Abstract

This paper describes a mathematical framework for rendering algorithms. Starting from the rendering equation and the potential equation, we will introduce the Global Reflection Distribution Function (GRDF). By using the GRDF, we are able to compute the behaviour of light in an environment, independent of the initial lighting or viewpoint conditions. This framework is able to describe most existing rendering algorithms.

2 Introduction

The global illumination problem is formulated by the well known rendering equation [Kajiya86]. Different methods have been proposed to solve this equation: Monte Carlo Path Tracing, which is in fact an application of distributed ray tracing [Cook et al. 84, Shirley-Wang91, Shirley-Wang92]; various two-pass methods [Chen et al. 91, Sillion-Puech89, Wallace et al. 87], which combine a radiosity and a ray tracing pass; methods based on particle tracing [Pattanaik-Mudur92], which are related to solutions presented in recent heat transfer literature [Brewster92].

Algorithms which solve the global illumination problem can be subdivided into four different classes. A first group of methods is based upon gathering techniques: the illumination of a point or surface is computed by looking at its surroundings, and by taking into account possible contributions towards the illumination of the surface. A second group of methods simulates the propagation of light in an environment, starting from the light sources. Both these approaches can further be divided in deterministic and probabilistic algorithms. Gathering algorithms are described by the traditional rendering equation, but shooting algorithms are best described by the so called potential equation [Pattanaik-Mudur93, Pattanaik93].

3 The rendering equation

3.1 Exitant and incident radiance

Radiance is the basic quantity for describing light transport. It is expressed as power per unit surface area per unit solid angle. Exitant radiance ($L^{\rightarrow}$) is the radiance leaving a surface point in a given direction of the hemisphere. Incident radiance ($L^{\leftarrow}$) is radiance arriving at a surface point from a direction belonging to the hemisphere. Equation 1 gives the realtionship between exitant and incident radiance (figure 1).

$$L(x \rightarrow \theta) = L(p(x, \theta) \leftarrow \theta^{-1})$$
$$L(y \leftarrow \psi) = L(p(y, \psi) \rightarrow \psi^{-1}) \quad (1)$$

where:

- $L(x \rightarrow \theta)$: exitant radiance leaving x in direction θ [Watt / m^2 sr].
- $L(y \leftarrow \psi)$: incident radiance arriving at y from direction ψ [Watt / m^2 sr].

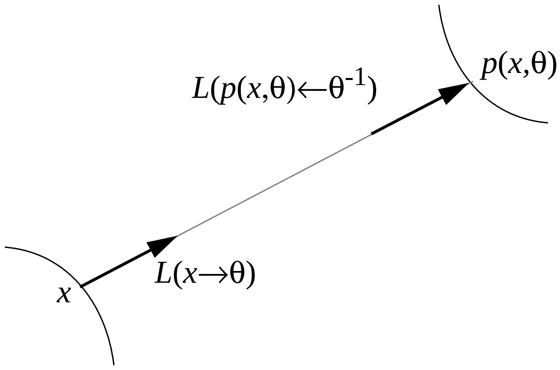

FIGURE 1. Exitant and incident radiance

Propagation of radiance in an environment is described by the well known rendering equation (figure 2):

$$L(x \rightarrow \theta) = L_e(x \rightarrow \theta) + \int_{\Omega_x} L(p(x, \varphi) \rightarrow \varphi^{-1}) f_r(\varphi, x, \theta) \cos(\varphi, n_x) \, d\omega_\varphi \quad (2)$$

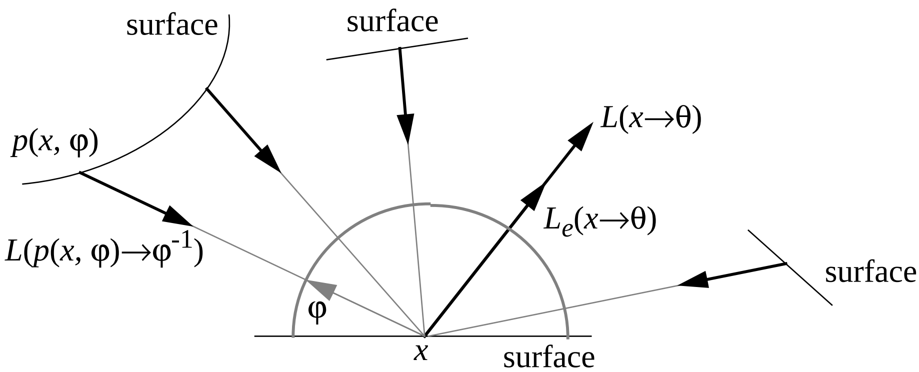

FIGURE 2. Transport of exitant radiance

where:

- $L_e(x \rightarrow \theta)$: self-emitted excitant radiance leaving x in direction θ [Watt / m^2 sr].
- Ω_x: hemisphere around x.
- $d\omega_\varphi$: differential solid angle around direction φ.
- $p(x, \varphi)$: the closest point seen from x in direction φ.
- φ^{-1}: direction opposite to φ.
- $f_r(\varphi^{-1}, x, \theta)$: Bidirectional Reflection Distribution Function (BRDF) evaluated at x, with incoming direction φ^{-1} and outgoing direction θ.

- $\cos(\varphi, n_x)$: the absolute value of the cosine of the angle between direction φ and the normal direction at x.

The integral over the hemisphere around x can be written as a transport operator T working on $L(x \to \theta)$:

$$L(x \to \theta) = L_e(x \to \theta) + TL(x \to \theta)$$
$$TL(x \to \theta) = \int_{\Omega_x} L(p(x, \varphi) \to \varphi^{-1}) f_r(\varphi, x, \theta) \cos(\varphi, n_x) \, d\omega_\varphi \qquad (3)$$

Substituting equation 1 in equation 2, we derive a similar transport equation for incident radiance (figure 3):

$$L(y \leftarrow \psi) = L_e(y \leftarrow \psi) + \int_{\Omega_{p(y, \psi)}} L(p(y, \psi) \leftarrow \varphi) f_r(\varphi, p(y, \psi), \psi^{-1}) \cos(\varphi, n_{p(y, \psi)}) \, d\omega_\varphi \qquad (4)$$

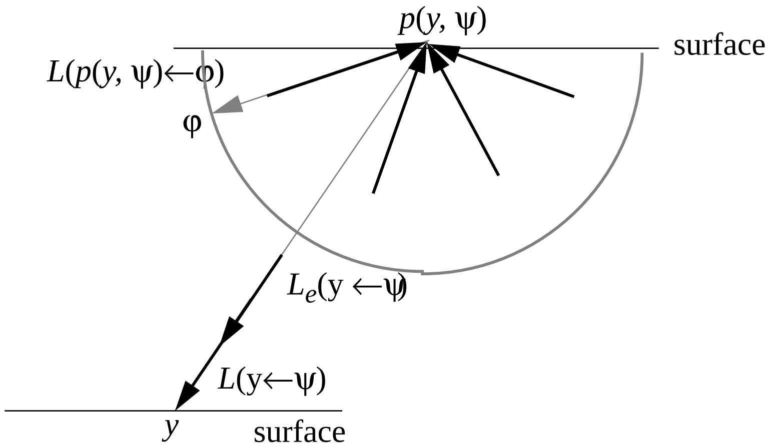

FIGURE 3. Transport of incident radiance

In analogy with the transport equation for exitant radiance, we define an operator Q such that:

$$L(y \leftarrow \psi) = L_e(y \leftarrow \psi) + QL(y \leftarrow \psi)$$
$$QL(y \leftarrow \psi) = \int_{\Omega_{p(y, \psi)}} L(p(y, \psi) \leftarrow \varphi) f_r(\varphi, p(y, \psi), \psi^{-1}) \cos(\varphi, n_{p(y, \psi)}) \, d\omega_\varphi \qquad (5)$$

3.2 Adjoint equations

Two operators O_1 and O_2, operating on elements of the same vectorspace V, are said to be adjoint with respect to an inner product $<A, B>$ if:

$$\forall A, B \in V: \langle O_1 A, B \rangle = \langle A, O_2 B \rangle \qquad (6)$$

O_2 is called the adjoint operator of O_1 and is denoted as O_1^*. It is easy to prove that the above defined operators T and Q are adjoint to each other for the following inner product, defined in the space of all functions operating on arguments $(z, \theta) \in A \mathrm{x} \Omega$:

$$\langle F_1, F_2 \rangle = \int_A \int_{\Omega_z} F_1(z \to \varphi) F_2(z \leftarrow \varphi) \cos(\varphi, n_z) \, d\omega_\varphi d\mu_z \qquad (7)$$

We will refer to the above defined transport operators as T and T^*, corresponding to exitant and incident radiance respectively.

4 Flux

The flux is the total amount of power emitted by a set of points and a set of directions around these points. This total set is described by a function $g(x, \theta)$. g equals 1 if (x, θ) belongs to the set, and equals 0 if (x, θ) does not belong to the set. The flux, associated with a set S defined by g, can then be written as:

$$\Phi(S) = \int_A \int_{\Omega_z} L(z \to \varphi)\, g(z, \varphi) \cos(\varphi, n_z)\, d\omega_\varphi d\mu_z \tag{8}$$

or, using the inner product defined above:

$$\Phi(S) = \langle L^{\to}, g \rangle \tag{9}$$

5 The potential equation

The potential equation (referenties) describes the global illumination problem from a different point of view. The advantage of the potential equation is that shooting algorithms are better described using this equation. Instead of computing the radiance of all pairs (x, θ) we are interested in, shooting algorithms compute the potential $W(x, \theta)$ for each pair (x, θ), w.r.t. a given set of which we want to compute the flux. $W(x, \theta)$ describes the fraction of the radiance $L(x \to \theta)$ which contributes to the flux of the set S. $W_e(x, \theta)$ equals 1 for points belonging to the set and equals 0 for all other points. The potential $W(x, \theta)$ can be described by the following transport equation (figure 3):

$$W(x, \theta) = W_e(x, \theta) + \int_{\Omega_{p(y, \psi)}} W(p(x, \theta), \varphi)\, f_r(\varphi, p(x, \theta), \theta^{-1}) \cos(\varphi, n_{p(x, \theta)})\, d\omega_\varphi \tag{10}$$

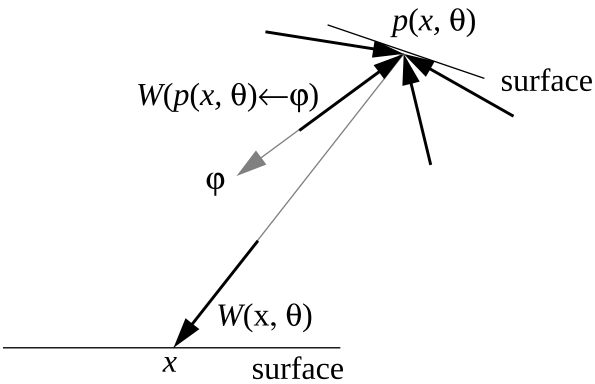

FIGURE 4. Transport of potential

This equation is the same transport equation as was used to describe the transport of incident radiance. Therefore, we will apply the same notion of “incidence” to potential. $W(x,\theta)$ as described here is thus incident potential and we will use the notation $W(x \leftarrow \theta)$. Thus the transport equation of $W(x \leftarrow \theta)$ can be written as:

$$W(x \leftarrow \theta) = W_e(x \leftarrow \theta) + T^* W(x \leftarrow \theta) \tag{11}$$

In analogy with the relation between exitant and incident radiance, we can also define exitant potential:

$$W(y \to \psi) = W(p(y, \psi) \leftarrow \psi^{-1}) \quad \text{or} \quad W(x \leftarrow \theta) = W(p(x, \theta) \to \theta^{-1}) \tag{12}$$

Substituting equation 12 in equation 10:

$$W(y \to \psi) = W_e(y \to \psi) + \int_{\Omega_{p(y,\psi)}} W(p(y, \psi) \to \varphi) f_r(\varphi, p(y, \psi), \psi^{-1}) \cos(\varphi, n_{p(y,\psi)}) d\omega_\varphi \tag{13}$$

or

$$W(y \to \psi) = W_e(y \to \psi) + TW(y \to \psi) \tag{14}$$

Thus, we have derived two different sets of transport equations which describe the global illumination problem. On the one hand we have excitant radiance and incident potential, described by adjoint transport equations. On the other hand, we have incident radiance and excitant potential, also described by a set of adjoint transport equations.

From the observations above, we can derive an alternate formula for the flux. The function g can be replaced by W_e, since these functions describe the same property of a point.

$$\begin{aligned} \Phi(S) &= \langle L^{\to}, g \rangle \\ &= \langle L^{\to}, W_e^{\leftarrow} \rangle \\ &= \langle L^{\to}, W^{\leftarrow} - T^* W^{\leftarrow} \rangle \\ &= \langle L^{\to}, W^{\leftarrow} \rangle - \langle L^{\to}, T^* W^{\leftarrow} \rangle \\ &= \langle L^{\to}, W^{\leftarrow} \rangle - \langle TL^{\to}, W^{\leftarrow} \rangle \\ &= \langle L^{\to} - TL^{\to}, W^{\leftarrow} \rangle \\ &= \langle L_e^{\to}, W^{\leftarrow} \rangle \end{aligned} \tag{15}$$

In order to solve the global illumination problem, we need to compute the flux for a number of sets. If a ray tracing approach is used, one set consist of the points and associated directions visible to a pixel; if a radiosity algorithm is used, a set consists of a single patch with an associated hemisphere for each point. We have two sets of equations at our disposal to compute the fluxes:

$$\begin{aligned} \Phi(S) &= \langle L^{\to}, W_e^{\leftarrow} \rangle \qquad & \Phi(S) &= \langle L_e^{\to}, W^{\leftarrow} \rangle \\ L^{\to} &= L_e^{\to} + TL^{\to} \qquad & W^{\leftarrow} &= W_e^{\leftarrow} + T^* W^{\leftarrow} \end{aligned} \tag{16}$$

The set of equations on the left corresponds to gathering algorithms. Given a set S defined by $W_e^{\leftarrow}$, we have to compute the corresponding $L^{\to}$. $L^{\to}$ is computed by gathering all incoming radiances at the point of interest. Ray tracing algorithms are a typical example of gathering algorithms.

The set of equations on the right corresponds to shooting algorithms. With respect to a given set S, a potential is computed for each point belonging to a light source. $W^{\leftarrow}$ is computed by "shooting" light into the environment, until we eventually reach the set S. Progressive radiosity is a typical example.

Both approaches have distinct advantages. Some two-pass methods use the advantages of both radiosity and ray tracing algorithms.

6 Global Reflection Distribution Function

Given the transport equation of $L^{\rightarrow}$, it is clear that each single value $L^{\rightarrow}(x\rightarrow\theta)$ can be written as a linear combination of all possible values $L_e^{\rightarrow}(z\rightarrow\varphi)$. Indeed:

$$\begin{aligned} L^{\rightarrow} &= L_e^{\rightarrow} + TL^{\rightarrow} \\ &= L_e^{\rightarrow} + T(L_e^{\rightarrow} + T(L_e^{\rightarrow} + T(L_e^{\rightarrow} + T(L_e^{\rightarrow} + \ldots)))) \\ &= (I + T + T^2 + T^3 + T^4 + \ldots)\, L_e^{\rightarrow} \end{aligned} \tag{17}$$

The fraction of each $L_e^{\rightarrow}(z\rightarrow\varphi)$ that counts towards $L^{\rightarrow}(x\rightarrow\theta)$ can be considered as a function $F_{x,\theta}$ over all (z, φ). Thus:

$$L(x \rightarrow \theta) = \langle L_e^{\rightarrow}, F_{x,\theta} \rangle \tag{18}$$

For each point (x, θ), there is a corresponding function $F_{x,\theta}$. We can express the single value $L(x\rightarrow\theta)$ as an inner product by using a suitable Dirac impulse:

$$\begin{gathered} \delta_{x,\theta}(z,\varphi) = 0 \quad \text{if} \quad (z,\varphi) \neq (x,\theta) \\ \int_A \int_{\Omega_z} f(x,\theta)\, \delta_{x,\theta}(z,\varphi) \cos(\varphi, n_z)\, d\omega_\varphi d\mu_z = f(x,\theta) \end{gathered} \tag{19}$$

This leads to the following observations:

$$\begin{aligned} L^{\rightarrow}(x \rightarrow \theta) = \langle L^{\rightarrow}, \delta_{x,\theta} \rangle &= \langle L_e^{\rightarrow}, F_{x,\theta} \rangle \\ &= \langle L_e^{\rightarrow}, F_{x,\theta} \rangle \\ &= \langle L^{\rightarrow} - TL^{\rightarrow}, F_{x,\theta} \rangle \\ &= \langle L^{\rightarrow}, F_{x,\theta} \rangle - \langle TL^{\rightarrow}, F_{x,\theta} \rangle \\ &= \langle L^{\rightarrow}, F_{x,\theta} \rangle - \langle L^{\rightarrow}, T^* F_{x,\theta} \rangle \\ &= \langle L^{\rightarrow}, F_{x,\theta} - T^* F_{x,\theta} \rangle \end{aligned} \tag{20}$$

This holds for all possible functions $L^{\rightarrow}$ and all values of x and θ. Therefore, we can say that:

$$\begin{gathered} \delta_{x,\theta} = F_{x,\theta} - T^* F_{x,\theta} \\ \text{or} \quad F_{x,\theta} = \delta_{x,\theta} + T^* F_{x,\theta} \end{gathered} \tag{21}$$

$F_{x,\theta}$ can be expressed by the same transport equation that was used for incident potential. We can also apply the notion of “incidence” to $F_{x,\theta}$. The full transport equation then reads:

$$F_{x,\theta}(y \leftarrow \psi) = \delta_{x,\theta}(y \leftarrow \psi) + T^* F_{x,\theta}(y \leftarrow \psi) \tag{22}$$

We can make an analogue reasoning for the potential. We can derive a fuction $G_{y,\psi}$, such that:

$$
\begin{aligned}
W(y \leftarrow \psi) &= \langle W_e^{\leftarrow}, G_{y,\psi} \rangle \\
G_{y,\psi}(x \rightarrow \theta) &= \delta_{y,\psi}(x \rightarrow \theta) + TG_{y,\psi}(x \rightarrow \theta)
\end{aligned}
\tag{23}
$$

It is clear that there must be some relationship between $F_{x,\theta}$ and $G_{y,\psi}$. Based on the two different expressions for the flux of a given set, we can derive this relationship.

$$
\begin{aligned}
\Phi(S) &= \langle L^{\rightarrow}, W_e^{\leftarrow} \rangle \\
&= \int_A \int_{\Omega_x} L^{\rightarrow}(x \rightarrow \theta) W_e^{\leftarrow}(x \leftarrow \theta) \cos(\theta, n_x)\, d\omega_\theta d\mu_x \\
&= \int_A \int_{\Omega_x} \langle L_e^{\rightarrow}, F_{x,\theta} \rangle W_e^{\leftarrow}(x \leftarrow \theta) \cos(\theta, n_x)\, d\omega_\theta d\mu_x \\
&= \int_A \int_{\Omega_x} \int_A \int_{\Omega_y} L_e^{\rightarrow}(y \rightarrow \psi) W_e^{\leftarrow}(x \leftarrow \theta) F_{x,\theta}(y \leftarrow \psi) \cos(\psi, n_y) \cos(\theta, n_x)\, d\omega_\psi d\mu_y d\omega_\theta d\mu_x
\end{aligned}
\tag{24}
$$

and also

$$
\begin{aligned}
\Phi(S) &= \langle L_e^{\rightarrow}, W^{\leftarrow} \rangle \\
&= \int_A \int_{\Omega_y} L_e^{\rightarrow}(y \rightarrow \psi) W^{\leftarrow}(y \leftarrow \psi) \cos(\psi, n_y)\, d\omega_\psi d\mu_y \\
&= \int_A \int_{\Omega_y} L_e^{\rightarrow}(y \rightarrow \psi) \langle W_e^{\leftarrow}, G_{y,\psi} \rangle \cos(\psi, n_y)\, d\omega_\psi d\mu_y \\
&= \int_A \int_{\Omega_y} \int_A \int_{\Omega_x} L_e^{\rightarrow}(y \rightarrow \psi) W_e^{\leftarrow}(x \leftarrow \theta) G_{y,\psi}(x \rightarrow \theta) \cos(\psi, n_y) \cos(\theta, n_x)\, d\omega_\theta d\mu_x d\omega_\psi d\mu_y
\end{aligned}
\tag{25}
$$

Since equation 24 and equation 25 have to be equal, and since this equality holds for all possible functions $L_e^{\rightarrow}$ and $W_e^{\leftarrow}$, the relation between F and G is found:

$$
F_{x,\theta}(y \leftarrow \psi) = G_{y,\psi}(x \rightarrow \theta) \tag{26}
$$

Because of this relationship, the functions $F_{x,\theta}$ and $G_{y,\psi}$ can be described by a single function F_r:

$$
F_{x,\theta}(y \leftarrow \psi) = G_{y,\psi}(x \rightarrow \theta) = F_r(y \leftarrow \psi, x \rightarrow \theta) \tag{27}
$$

F_r is the global reflection distribution function (GRDF), as introduced by Lafortune [Lafortune93b].

7 Properties of the GRDF

7.1 Transport equations

Since the GRDF is defined through the definitions of F and G, the following adjoint transport equations describe the behaviour of the GRDF:

$$
\begin{aligned}
F_r(y \leftarrow \psi, x \rightarrow \theta) &= \delta(y \leftarrow \psi, x \rightarrow \theta) + T^* F_r(y \leftarrow \psi, x \rightarrow \theta) \\
F_r(y \leftarrow \psi, x \rightarrow \theta) &= \delta(y \leftarrow \psi, x \rightarrow \theta) + TF_r(y \leftarrow \psi, x \rightarrow \theta)
\end{aligned}
\tag{28}
$$

This double formulation implies that there are two different ways to compute specific values of the GRDF: a gathering approach and a shooting approach.

7.2 Physical interpretation

From the above observations, the following interpretation can be given to the GRDF:

- $F_r(y{\leftarrow}\psi, x{\rightarrow}\theta)$ is the differential fraction of $L^{\rightarrow}(y{\rightarrow}\psi)\cos(\psi, n_y)d\omega_\psi d\mu_y$ which contributes to the value of $L^{\rightarrow}(x{\rightarrow}\theta)$.
- $F_r(y{\leftarrow}\psi, x{\rightarrow}\theta)$ is the differential fraction of $W^{\leftarrow}(x{\leftarrow}\theta)\cos(\theta, n_x)d\omega_\theta d\mu_x$ which contributes to the value of $W^{\leftarrow}(y{\leftarrow}\psi)$.

This is very similar to the definition of the common BRDF, which describes the same property for exitant and incident radiance in a single point. The GRDF expands on this concept and describes the relationship between any two radiance or potential values, taking into account all possible reflections. The BRDF can be considered as a special case of the GRDF. The name Global Reflection Distribution Function is therefore quite appropriate.

7.3 Transforming arguments

Since the values of $F_r(y{\leftarrow}\psi, x{\rightarrow}\theta)$ need to be the same, no matter which transport equations we use to compute the values, $T^*F_r(y{\leftarrow}\psi, x{\rightarrow}\theta)$ and $TF_r(y{\leftarrow}\psi, x{\rightarrow}\theta)$ should be equal:

$$
\begin{aligned}
T^* F_r(y \leftarrow \psi, x \rightarrow \theta) &= \int_{\Omega_{p(y,\psi)}} F_r(p(y,\psi) \leftarrow \varphi, x \rightarrow \theta) f_r(\varphi, p(y,\psi), \psi^{-1}) \cos(\varphi, n_{p(y,\psi)})\, d\omega_\varphi \\
TF_r(y \leftarrow \psi, x \rightarrow \theta) &= \int_{\Omega_x} F_r(y \leftarrow \psi, p(x,\varphi) \rightarrow \varphi^{-1}) f_r(\varphi, x, \theta) \cos(\varphi, n_x)\, d\omega_\varphi
\end{aligned}
\tag{29}
$$

From this equality, the following property of the GRDF can be derived:

$$
F_r(y \leftarrow \psi, x \rightarrow \theta) = F_r(p(x,\theta) \leftarrow \theta^{-1}, p(y,\psi) \rightarrow \psi^{-1}) \tag{30}
$$

This relationship is the generalisation of the property of the BRDF, in which the incoming and outgoing direction can be switched, resulting in the same value of the BRDF.

8 Practical applications

The GRDF is independent of both $L_e^{\rightarrow}$ and $W_e^{\leftarrow}$, it only depends on the geometry of the scene and the surface characteristics of the objects. If one is able to compute the GRDF in advance, applying different values of $L_e^{\rightarrow}$ or $W_e^{\leftarrow}$ is straightforward. Changing $L_e^{\rightarrow}$ means that the initial lighting conditions are changed; changing $W_e^{\leftarrow}$ means we want to compute the flux for a different set. This latter option also encompasses the change of viewpoint.

Since the GRDF is described by two transport equations, we have a choice of what equation to use in order to compute different values of the GRDF:

- If we use the T^* equation, we are actually using a gathering approach, leading to algorithms as stochastic ray tracing [Cook et al. 84], path tracing [Kajiya86] or Gauss-Seidel radiosity.
- If we use the T equation, a shooting approach is used, leading to algorithms such as particle tracing [Pattanaik92, Dutré et al. 93] or progressive radiosity.
- A simultaneous use of both transport equations has been described by [Lafortune93a]. This dual path tracing algoritm involves elements of both ray tracing and particle tracing, and uses the advantages of both.

9 Conclusion

We have described a powerful mathematical framework in which all rendering algorithms can be defined. The Global Reflection Distribution Function (GRDF) plays a major role in this framework. The advantages of the GRDF are:

- It allows us to compute the behaviour of light in a given environment, independent of initial lighting conditions and independent of the final choice of viewpoint.
- All rendering algorithms can be described as different ways of solving the GRDF equations.
- The GRDF is described by two adjoint transport equations. Combining both these equations in a single algorithm combines the advantages of both shooting and gathering algorithms.

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