Constellation Labeling for Error-Resilient Source Coding of PAM

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Abstract — **This paper considers PAM source data under the distortion criterion of maximum Euclidean distance. Under the assumption that bit errors are sparse, we seek a labeling that minimizes the maximum Euclidean distance associated with a single bit error. We call this quantity $d_{max}^{(1)}$. This paper lower bounds $d_{max}^{(1)}$ and proposes a labeling which approaches that lower bound.**

When lossy compression is followed by transmission over a noisy channel, the best combination of source and channel coding can be simply a properly labeled source.

In [1], Zeger and Gersho used a training set to design a pseudo-Gray code labeling that reduced the mean squared error of vector quantized data transmitted over a noisy channel.

This paper considers PAM source data under the distortion criterion of maximum Euclidean distance. The edge label is the exclusive-or of the labels of two points connected by the edge. The edge weight equals the Euclidean distance between the two corresponding points. These definitions are the same as used in [2]. We only consider the edges with Unit-Hamming-Weight (UHW) in this paper. The Euclidean distance between any two adjacent points is assumed to be 1.

The $d_{pmax}^{(1)}(i)$ of constellation point i is defined as the maximum Euclidean distance from the constellation point i to any of the n constellation points which have UHW edges connecting to point i. $d_{max}^{(1)}$ of a labeling is defined as the maximum of $d_{pmax}^{(1)}(i)$ of all 2^n constellation points.

$$d_{max}^{(1)} = \max_i d_{pmax}^{(1)}(i), \qquad i = 1, 2, \cdots, 2^n. \tag{1}$$

The widely used Gray-code (GC) labeling [3] turns out to be the worst possible labeling in this case with $d_{max}^{(1)} = 2^n - 1$, since the labels of the two end points differ by only one bit.

The "natural" labeling uses binary labels in the order of the integers they represent. For this labeling, $d_{pmax}^{(1)}(i) = 2^{n-1}$ for all $i = 1, 2, \cdots, 2^n$. Hence $d_{max}^{(1)} = 2^{n-1}$.

Before the design of an effective labeling, we present two lower bounds of the $d_{max}^{(1)}$. For 2^n-PAM,

$$d_{max}^{(1)} \geq 2n - 2 \tag{2}$$

$$d_{max}^{(1)} > \frac{2^n - 1}{n} \tag{3}$$

Without loss of generality, label the left end point with all zeros. Eq. (2) is true because the two leftmost points have at least $2n-2$ UHW neighbors, not including each other. Any placement of these neighbors forces $d_{max}^{(1)} \geq 2n-2$. Eq. (3) is true because there exists at least one path from the all-zeros point to the all-ones point passing through the right end point traveling along exactly n UHW edges. The minimum distance of such a path is $2^n - 1$.

The best labeling structure for 2^n-PAM we have found, in terms of minimizing $d_{max}^{(1)}$, is the Monotonic Hamming Weight with Decreasing Numerical Value (MHW-DNV) labeling: For 2^n-PAM, with 2^n n-bit unsigned binary numbers as point labels, label points according to the following two rules,

(i) Binary numbers with lower Hamming weight should be chosen first.

(ii) Under (i), the binary numbers should be selected in numerical order, with the largest first.

For example, MHW-DNV labeling for 8-PAM is:

f000, 100, 010, 001, 110, 101, 011, 111g

Equation (4) shows the performance of MHW-DNV labeling for 2^n-PAM in terms of $d_{max}^{(1)}$:

$$d_{max}^{(1)}(MHW - DNV) = 1 + \sum_{i=1}^{n-1} \binom{i}{\lfloor \frac{i}{2} \rfloor}, \qquad n \geq 2. \tag{4}$$

We now sketch The proof of (4). MHW-DNV-labeled 2^n-PAM can be divided into $n+1$ sections, each section having a constant Hamming weight. It turns out that the largest value of $d_{pmax}^{(1)}(i)$ occurs for a point in the center section(s). The additional observation that $d_{pmax}^{(1)}(i)$ is always achieved by an edge connecting to a point in an adjacent section yields (4) through induction.

Figure 1 shows $d_{max}^{(1)}$ of GC labeling, natural labeling, MHW-DNV labeling, and the lower bounds of (2) and (3).

Note that a 2^{2n}-QAM constellation labeled with the direct product of two MHW-DNV 2^n-PAM labelings will achieve the same $d_{max}^{(1)}$ as for 2^n-PAM.

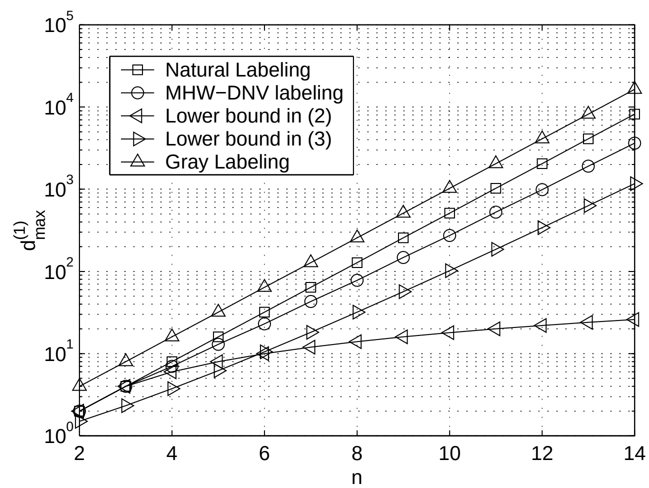

Figure 1: $d_{max}^{(1)}$ of 2^n-PAM vs. n

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