A Note on the Calculation of a Discrete-Event-System's Transfer Function

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Abstract

In recent research the relevancy of discrete event systems (DES) steadily increases. In order to handle DES the introduction of a DES-system theory is useful. Hereby the systems are characterised and described by the impulse response and the transfer function. Therefore knowledge about the calculation and the properties of these functions will be important. This article outlines the DES-system theory and shows an algorithm for an efficient calculation of the transfer function by means of straightforward algebraic simplifications.

1 Introduction

Due to the fact that many interesting results in common system theory are based on adapted mathematical description of the system, the development of a DES-system theory was mandatory. After the definition of the corresponding terms the problem of an efficient function calculation arises. This article presents an effective way for the calculation of the star-operator. As already mentioned, an effective computation of the star-operator may be necessary in order to determine the transfer function of a given DES.

The basics of dioid theory are given and applied to DES-system theory. The need of an effective calculation of the star-operator is motivated. As central result a theorem is proofed which allows the solution of an affine equation being slightly less elaborate than matrix multiplication. Finally the relative effort in comparison to the multiplication is visualised.

2 Basics of Dioid-Theory

This section will introduce the definition of a dioid. The considerations in this section will not present proofs but restrict to simple properties. This is done due to the fact that only basic algebraic knowledge is necessary in order to understand the following simplifications. Detailed examinations and characterisations can be found in [1], [2], [3] and [4]. First the algebraic properties of a dioid are defined:

Definition 2.1 ***(Dioid)***
A dioid is a set $\mathcal{D}$ endowed with two inner operations $\oplus$ and $\otimes$, called addition and multiplication, such that $\oplus$ is associative, commutative, idempotent and has a zero (denoted as ε); $\otimes$ is associative and has a unit (denoted as e); ε is absorbing for $\otimes$ ($\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$) and $\otimes$ distributes over $\oplus$. In most cases the multiplication sign is omitted.

A matrix dioid $\mathcal{D}^{N\times N}$ is defined on grounds of a dioid $\mathcal{D}$ by:

$$(\mathbf{A} \oplus \mathbf{B})_{i,k} = \mathbf{A}_{i,k} \oplus \mathbf{B}_{i,k} \quad , i,k = 1,\ldots,N \tag{1}$$

$$(\mathbf{A} \otimes \mathbf{B})_{i,k} = \bigoplus_{\nu=1}^{N} \mathbf{A}_{i,\nu} \otimes \mathbf{B}_{\nu,k} \quad , i,k = 1,\ldots,N \tag{2}$$

In [4], [5] it is shown that an algebraic structure possessing an idempotent, associative and commutative inner operation can be furnished with an order relation. The induced order $\preceq$ is given by $a \preceq b \iff a \oplus b = b \quad , a,b \in \mathcal{D}$. Such an ordered set may be completed adding the upper bound of each subset, resulting in an complete dioid:

Definition 2.2 *A dioid is complete, if it is closed for upper bounds*

$$a_i \in \mathcal{D}, i \in I \Rightarrow \bigoplus_{i\in I} a_i \in \mathcal{D}, \tag{3}$$

and the distributivity extends to infinite sums. Sums with an uncountable amount of elements are well defined as upper bounds of certain subsets.

In complete dioids the solution of an affine equation can be obtained by algebraic calculations. For this purpose the star-operation $*$ will be useful. Therefore an element of $\mathcal{D}$ to the power of n is recursively defined by

$$a^0 = e \quad , \quad a^1 = a \quad , \quad a^n = a^{n-1} \otimes a. \tag{4}$$

Through this the star-operation

$$a^* = \bigoplus_{n=0}^{\infty} a^n \tag{5}$$

is defined, which allows the characterisation of solutions of affine equations:

Theorem 2.1 *In a complete dioid the following statements concerning affine equation $x = ax \oplus b$ hold true:*

1. *$a^* \otimes b$ is the least solution of $x = ax \oplus b$.*
2. *For each solution $\tilde{x}$, the property $\tilde{x} = a^*\tilde{x}$ is valid.*

3 Affine Equations in the DES-Theory

This section justifies the necessity of an efficient calculation of the star-operator. For this purpose, results are presented demonstrating the description of DES by means of algebraic methods. For the sake of simplicity these results are not derived in a closed manner, but merely stated as facts. The justification can be found in [1], [5] or [4].

In [4] it has been shown, that synchronisation-graphs, a special kind of petri-nets, can be described by state space equations:

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{A}\mathbf{x}(k-1) \oplus \mathbf{B}\mathbf{u}(k)\\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \oplus \mathbf{D}\mathbf{u}(k)\end{aligned}$$

Hereby the state variables $\mathbf{x}_i(k)$ represent the instant in time at which the $i-$th transition of the petri-net is fired for the $k-$th time.

With the $\gamma-$transformation of an signal $\mathbf{x}(k)$ as an analogy to the z-transform,

$$\Gamma\{\mathbf{x}(k)\} = \bigoplus_{k\in\mathbb{Z}} \mathbf{x}(k)\gamma^k, \qquad (6)$$

signals are transformed into an image domain. The delay of a signal corresponds to the multiplication with the formal operator γ in the image domain:

$$\Gamma\{\mathbf{x}(k)\} = \mathbf{X}(\gamma) \Longrightarrow \Gamma\{\mathbf{x}(k-1)\} = \gamma\mathbf{X}(\gamma) \qquad (7)$$

In terms of this image domain calculations are simplified and the state space equations become

$$\begin{aligned}\mathbf{X}(\gamma) &= \mathbf{A}\gamma\mathbf{X}(\gamma) \oplus \mathbf{B}\mathbf{U}(\gamma)\\ \mathbf{Y}(\gamma) &= \mathbf{C}\mathbf{X}(\gamma) \oplus \mathbf{D}\mathbf{U}(\gamma).\end{aligned}$$

Applying theorem 2.1 the resulting input-output-behavior is given by:

$$\mathbf{Y}(\gamma) = \left(\mathbf{C}[\mathbf{A}\gamma]^*\mathbf{B} \oplus \mathbf{D}\right)\mathbf{U}(\gamma) \qquad (8)$$

Therefore the input-output behavior of the system is determined, calculating the right hand bracket. This computation requires knowledge of the matrix $[\mathbf{A}\gamma]^*$ and therefore an effective calculation of this term is essential.

4 Calculation of the Star-Operator

As proofed in [2], the calculation of the star-operation for a 2×2-matrix can be reduced to manipulation of the (scalar) matrix coefficients:

Theorem 4.1 *In $\mathcal{D}^{2\times 2}$ the following calculation of the star-operation takes place:*

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \mathbf{M}^* = \begin{pmatrix} a^* \oplus a^*b(ca^*b \oplus d)^*ca^* & a^*b(ca^*b \oplus d)^* \\ (ca^*b \oplus d)^*ca^* & (ca^*b \oplus d)^* \end{pmatrix}$$

Proof: The proof bases on the property of $\mathbf{M}^*$ being the least solution of an affine equation. The corresponding scalar equations are given by:

$$\begin{aligned} X_{1,1} &= aX_{1,1} \oplus bX_{2,1} \oplus e \\ X_{1,2} &= aX_{1,2} \oplus bX_{2,2} \\ X_{2,1} &= cX_{1,1} \oplus dX_{2,1} \\ X_{2,2} &= cX_{1,2} \oplus dX_{2,2} \oplus e \end{aligned}$$

Solving the first two equations with respect to the scalars $X_{1,1}, X_{1,2}$ the solutions

$$\begin{aligned} X_{1,1} &= a^* \left(bX_{2,1} \oplus e\right) \\ X_{1,2} &= a^* bX_{2,2} \end{aligned}$$

are obtained due to theorem 2.1. Substituting these results the other equations become:

$$\begin{aligned} X_{2,1} &= ca^* \left(bX_{2,1} \oplus e\right) \oplus dX_{2,1} \\ X_{2,2} &= ca^* bX_{2,2} \oplus dX_{2,2} \oplus e \end{aligned}$$

Thus the stated second line of matrix $\mathbf{M}$ is proofed. The first line results by substituting this values. It is obvious that the property of being the last solutions of the particular equations is guarded throughout the whole process. **q.e.d.**

At first sight this solution seems quite circumstantial. In fact, the calculation is significantly simplified. Considering the terms of matrix $\mathbf{M}^*$ it becomes obvious that the elements can be recursively computed using the similarities between the elements.

Theorem 4.2 *The star-operation for a $2 \times 2-$matrix can be calculated within two scalar additions, six scalar multiplications and 2 scalar star-operations.*

Proof: The proof is done by specification of an algorithm for the calculation of $\mathbf{M}^*$. The elements have to be computed in the following order:

1. a^*
2. ca^*, a^*b and ca^*b
3. $(ca^*b \oplus d)^*$
4. $a^*b(ca^*b \oplus d)^*$,$(ca^*b \oplus d)^*ca^*$ and $a^* \oplus a^*b(ca^*b \oplus d)^*ca^*$

q.e.d.

Using the latter theorem, the computation of matrices of higher dimensions is ascribed to sub-matrices of the half dimension. This is somehow similar to the derivation of the FFT out of the DFT. For this purpose the similarity between two matrix dioids is needed:

Theorem 4.3 *Dioids $\mathcal{D}^{2n \times 2n}$ and $(\mathcal{D}^{n \times n})^{2 \times 2}$ are isomorph.*

Proof: Consider $\mathrm{M} \in \mathcal{D}^{2n\times 2n}$. Then matrix M is given by:

$$\mathbf{M} = \begin{pmatrix} M_{1,1} & \cdots & M_{1,n} & M_{1,n+1} & \cdots & M_{1,2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & \cdots & M_{n,n} & M_{n,n+1} & \cdots & M_{n,2n} \\ M_{n+1,1} & \cdots & M_{n+1,n} & M_{n+1,n+1} & \cdots & M_{n+1,2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{2n,1} & \cdots & M_{2n,n} & M_{2n,n+1} & \cdots & M_{2n,2n} \end{pmatrix}$$

Now, define sub-matrices $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ and $\mathbf{S}_4$ as suggested above, i.e. the division of M in four main parts. It is quite obvious that these sub-matrices as a part of $(\mathcal{D}^{n\times n})^{2\times 2}$ guard the properties of addition and multiplication. **q.e.d.**

Due to this theorem the calculation of matrices of higher dimensions is reduced to calculations with matrices of smaller dimensions.

Theorem 4.4 *For* $\mathrm{M} \in \mathcal{D}^{n\times n}, n = 2^k$, *the star operation can be calculated with less effort than a matrix multiplication of the same dimension.*

Proof: The proof is done by induction with respect to k. In order to proof the stated efficiency, let $N_1(k), N_2(k), N_3(k)$ be the numbers of star-operations, multiplications and additions in $\mathcal{D}^{2^k\times 2^k}$ necessary for the calculation of the star-operation.

For $k = 0$ the statement is a direct consequence from theorem 4.2. Due to the fact, that the costs in $\mathcal{D}$ may assumed to be equal (this is justified considering the (max,plus)-algebra), one obtains the initial values $(N_1(0), N_2(0), N_3(0)) = (1, 1, 1)$.

Now assume the statement to be valid for a certain k. We have to proof the validity for the next index, $k \to k+1$. The calculation of the star-operation is done by dividing the matrix into four sub-matrices, applying the algorithm to each sub-matrix and rejoining the results.

The algorithm stated in the proof of theorem 4.2 and equations (1), (2) result in the recursive equation

$$\mathbf{N}(k+1) = \begin{pmatrix} N_1(k+1) \\ N_2(k+1) \\ N_3(k+1) \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 0 & 4 \end{pmatrix}}_{=\mathbf{A}} \cdot \begin{pmatrix} N_1(k) \\ N_2(k) \\ N_3(k) \end{pmatrix}$$

for the effort needed by the operations. Considering $\mathbf{N}(k) = \mathbf{A}^k\mathbf{N}(0)$ and the eigenvalue decomposition $\mathbf{A} = \mathbf{T}\lambda\mathbf{T}^{-1}$ the explicit solution for the effort is given by

$$\begin{pmatrix} N_1(k) \\ N_2(k) \\ N_3(k) \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 8^k \\ 4^k \\ 2^k \end{pmatrix}.$$

The quotient of the effort of star operations and multiplications $N_1(k)/N_2(k)$, which is steadily less than one, is depicted in figure 1. It is observed that the quotient of effort converges to one. Therefore for sufficient large indices k a star-operation is done with the same effort as a multiplication. **q.e.d.**

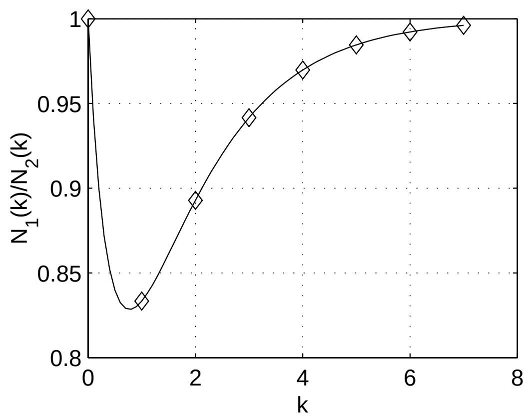

Figure 1: Relative effort for the calculation of the star operation

5 Summary

This article presents the basics of an efficient calculation for the star-operation in arbitrary dioids. After the motivation of the need of such considerations the effort is determined by an given algorithm for the simplification of the computations. As central theorem of this article it can be proofed that the relative effort of a star-operator is less than the multiplication of matrices of the same dimension. As a consequence the use of the star-operator is no time-critical part of the determination of a DES's behavior.

References

[1] F. Baccelli, G. Cohen, G. Olsder, and J. Quadrat. *Synchronization and Linearity*. Wiley, 1992.

[2] G. Cohen, P. Moller, J. Quadrat, and M. Viot. Algebraic tools for the performance evaluation of discrete event systems. *IEEE Proceedings: Special issue on Discrete Event Systems*, 77(1), Jan. 1989.

[3] H. Jäkel and P. Bort. A note on linear equations in common dioids. *Hänsel-Hohenhausen, Reports on Industrial Information Technology*, 3, 2000.

[4] H. Jäkel and C. Thierer. Discrete-event-system-theory based on (max,plus)-algebra - an introduction. *Hänsel-Hohenhausen, Reports on Industrial Information Technology*, 4, 2001.

[5] E. Menguy. *Contribution à la commande des systèmes linéaires dans les dioïdes*. Thèse, Université d'Angers, 1997.