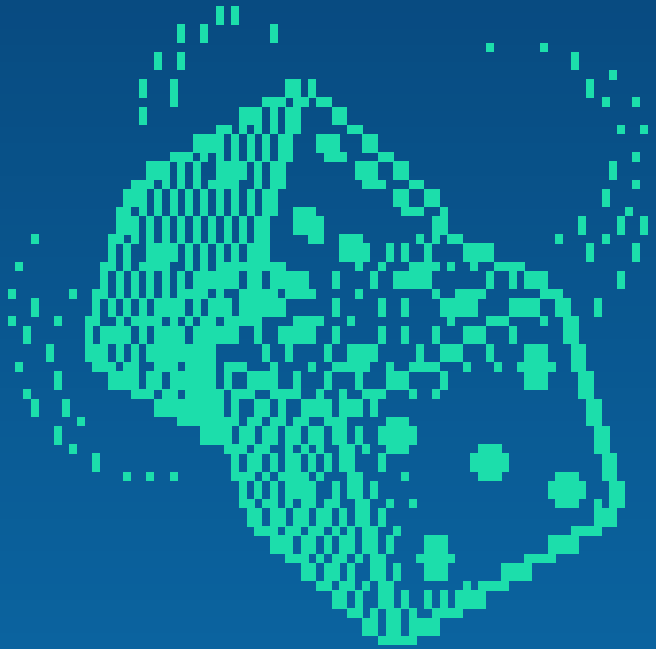
Entropy based Tests for Randomness and Applications to Cryptographic Generators



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  typical correlations: repetitive structures

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⇒ wanted:

universal tests that can detect
any systematic deviation from randomness

? cost of universality

Why Entropy

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- well understood concept for measuring randomness

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- well understood concept for measuring randomness
- applicable to finite sequences
- related to cryptographic security
- related to established tests for randomness in Monte Carlo

Interpretation

Interpretation: Per-Bit Entropy

$$\frac{1}{\log_2\ (\text{size of state space})} \cdot \text{Entropy}$$

of a cryptographic source

is equal to factor by which effective keysize is reduced

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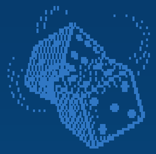

How to apply Entropy based tests

- different test principles (return times, frequencies)
- how to parameterize?
- what can we expect?
- correlations between tests?

Question:

Entropy based testing universal, i.e. no other tests are required?

Entropy Primer

Entropy of Ergodic Prozesses

The scaled logarithm of the probability of a random path of length n converges to a limit

$$-\frac{1}{n}\log\left(P[\text{ random Path }]\right) \xrightarrow{\text{a.s.}} H \quad \ldots \ \text{Entropy}$$

(Shannon-McMillan-Breimann)

Order κ Markov Chains

The process $X = (X_i)_{i \geq 0}, \quad X_i \in \mathcal{A} := \{1, 2, \ldots, m\}$ is a

homogenous Markov-Chain of order κ

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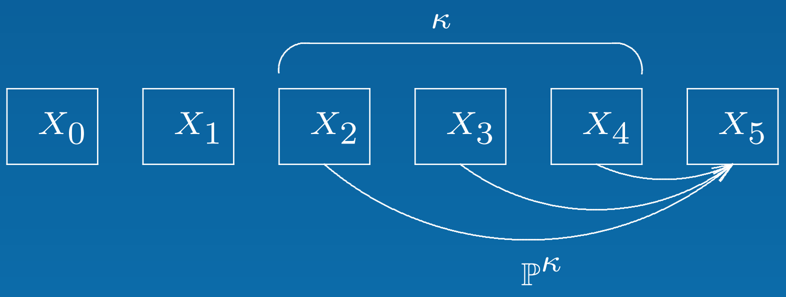
$$P[X_i = a_i | X_{i-1} = a_{i-1}, \ldots, X_0 = a_0] = p_{a_{i-\kappa}, \ldots, a_{i-1}, a_i}$$



Ergodic Order κ Markov Chains

If transition probabilities

$$\mathbb{P}^{\kappa} = \left(p_{a_{i-\kappa},\ldots,a_{i-1},a_i}\right)_{\mathbf{a}\in\mathcal{A}^{(\kappa+1)}}$$

define an irreducible and aperiodic chain,

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define an irreducible and aperiodic chain,

there exists a unique stable distribution

$$\left(\pi^{(\kappa)}_{a_1,\ldots,a_\kappa}\right)_{\mathbf{a}\in\mathcal{A}^{\kappa}} \quad \text{with} \quad \pi^{(\kappa)} = \pi^{(\kappa)} \cdot \mathbb{P}^{\kappa}$$

Derived Processes of Order κ Chains

$$\bar{X}_i^{(r)} = (X_{r\cdot i}, \ldots, X_{r\cdot i+r-1})$$ . . . . non-overlapping r-tuples

$$\underbrace{\boxed{X_0}\quad\boxed{X_1}\quad\boxed{X_2}}_{\bar{X}_0^{(3)}}\quad\underbrace{\boxed{X_3}\quad\boxed{X_4}\quad\boxed{X_5}}_{\bar{X}_1^{(3)}}\quad\underbrace{\boxed{X_6}\quad\boxed{X_7}\quad\boxed{X_8}}_{\bar{X}_2^{(3)}}$$

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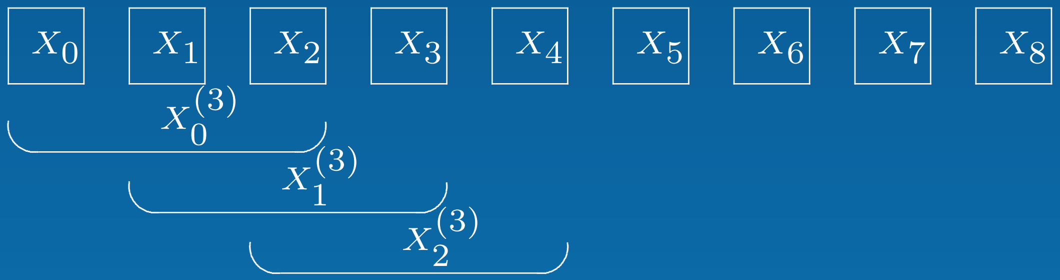
$X_i^{(r)} = (X_i, \ldots, X_{i+r-1})$ .... overlapping r-tuples



Derived Processes of Order κ Chains (ii)

If $r > \kappa$, the process of overlapping r tuples

$$(X_i^{(r)})_{i \geq 0} \quad \text{becomes ordinary (order 1) chain}$$

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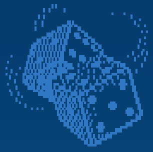
$(X_i^{(r)})_{i \geq 0}$ becomes ordinary (order 1) chain

with stable distribution

$$\pi^{(r)}_{a_1,\dots,a_r} = \pi^{(\kappa)}_{a_1,\dots,a_\kappa} \cdot \prod_{l=\kappa+1}^{r} p_{a_{l-\kappa},a_{l-\kappa+1},\dots,a_l}$$



Entropy of Order κ Chains

For $r > \kappa$ Birkhoff's theorem gives

$$H = H(\mathcal{A}, \mathbb{P}^{\kappa}) = -\sum_{\mathbf{a}\in\mathcal{A}^r} \pi_{\mathbf{a}}^{(r)} \log_2 \underbrace{\left(\frac{\pi_{\mathbf{a}}^{(r)}}{\pi_{\mathbf{a}'}^{(r-1)}}\right)}_{\text{prob. of last step}}$$

with $\mathbf{a}' = (a_1, \dots, a_{r-1})$.

Entropy of Order 1 Chains

For ordinary chains, this gives:

$$H(\mathcal{A}, \mathbb{P}) = -\sum_{i \in \mathcal{A}} \pi_i \sum_{j \in \mathcal{A}} p_{ij} \log_2 p_{ij}$$

Computing Entropy of Chains

Using stable distributions: Chain Rule

For $r > \kappa$

$$H(\mathcal{A}, \mathbb{P}^{\kappa}) = H(\pi^{(r)}) - H(\pi^{(r-1)}),$$

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where

$$H(\pi^{(r)}) = -\sum_{\mathbf{a} \in \mathcal{A}^r} \pi_{\mathbf{a}}^{(r)} \log_2 \pi_{\mathbf{a}}^{(r)}$$

is the Entropy of the distribution $\pi^{(r)}$

Using Return Times - Overlapping Case

Let

$$T^{(r)} = \min \left\{ i \geq r : X_i^{(r)} = X_0^{(r)} \right\}$$

be the first (overlapping) return time of order r.

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Wyner & Ziv:

$$\frac{\log_2\left(T^{(r)}\right)}{r} \xrightarrow{\mathrm{p}} H, \text{ as } r \to \infty$$

Pointwise Theorem: Ornstein, Weiss

Using Return Times - Non-Overlapping Case

Let

$$\bar{T}^{(r)} = \min\left\{ i \geq 1 : \bar{X}_i^{(r)} = \bar{X}_0^{(r)} \right\}$$

be the first (non-overlapping) return time of order r.

Expected Non-Overlapping Return Times

If $r \geq \kappa$,

$$E\left[\bar{T}^{(r)} | \bar{X}_0^{(r)} = \mathbf{a}\right] = \frac{1}{\pi_{\mathbf{a}}^{(r)}}, \quad \mathbf{a} \in \mathcal{A}^r$$

and, consequently,

$$E\left[\bar{T}^{(r)}\right] = m^r$$

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and, consequently,

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$E[\cdot]$ contains no information on chain!

Expected Logarithm of Return Times

Thus: use $E[\log_2(\cdot)]$

Maurer, similar to Willems:

$$\frac{E\left[\log_2\left(\bar{T}^{(r)}\right)\right]}{r} \rightarrow H$$

Expected Logarithm of Return Times (ii)

Speed of convergence: Let

$$E(r) = (r-1) \cdot H(\mathcal{A}, \mathbb{P}^{\kappa}) + H(\pi^{(r)}),$$

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$$\lim_{r \to \infty} \left(E\left[\log_2 \left(\bar{T}^{(r)} \right) \right] - E(r) \right) = \frac{\gamma}{\ln(2)}$$

Wegenkittl, IEEE 2002

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- Compute by either
  - stable distribution of overlapping tuples and chain rule
  - expected logarithm of return time of non-overlapping tuples

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$$x = (x_0, x_1, \ldots, x_{n-1})$$

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estimate Entropy H (or related property) of X

Reject H_0 if result unlikely under H_0.

Tests for Randomness based on Entropy (ii)

Universality:

H_0: $H = \log_2(m)$

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Universality:

H_0: $H = \log_2(m)$

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Estimator of H will be able to

detect any possible deviation from H_0

provided n is large enough.

Construction of Entropy Tests

Goal: estimate Entropy

Tests based on relative frequency

Let

$$\hat{\pi}_{\mathbf{a}}^{(r)} = \frac{1}{n} \# \left\{ 0 \leq i < n \ : \ \bar{X}_i^{(r)} = \mathbf{a} \right\}$$

then $\hat{\pi}_{\mathbf{a}}^{(r)}$ is unbiased consistent estimator for $\pi_{\mathbf{a}}^{(r)}$

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Apply Chain-Rule: For $r > \kappa$,

$$\hat{H}_f := H(\hat{\pi}^{(r)}) - H(\hat{\pi}^{(r-1)})$$

is asymptotically unbiased consistent estimator for H.

Tests based on relative frequency (ii)

Normalization:

dense : $2n(\log_2(m) - \hat{H}_f) \xrightarrow{\mathrm{d}} \chi^2_{m^r - m^{r-1}}$, as $n \to \infty$

sparse : $\frac{\hat{H}_f - \mu}{\sigma} \xrightarrow{\mathrm{d}} N(0, 1)$ as $n, r \to \infty, \frac{m^r}{n} \to \lambda$

r . . . dimension of test m . . . granularity

Tests based on relative frequency (iii)

Examples for H_f:

Approx. Entropy	Pincus et. al., NIST battery: discrete case, which is equivalent to $\hat{H}_f$
Modified ApEn	Ruhkin, asymptotically and practically equivalent to $\hat{H}_f$
Overlapping I-Divergence	Wegenkittl: connection to serial testing

Tests based on relative frequency (iv)

H_f belongs to the class of Power-Divergence Tests :

- statistical goodness-of-fit tests
- various likelyhood functions
- statistically equivalent in dense case
- very efficient in sparse case (L'Ecuyer, Simard, Wegenkittl)

Tests based on relative frequency (v)

Standard overlapping serial test (Pearson, Good): let

$$\psi^{(r)} := n \sum_{\mathbf{a} \in \mathcal{A}^r} \frac{\left(\hat{\pi}_{\mathbf{a}}^{(r)} - \pi_{\mathbf{a}}^{(r)}\right)^2}{\pi_{\mathbf{a}}^{(r)}}$$

and put

$$\hat{\chi} = \psi^{(r)} - \psi^{(r-1)} \xrightarrow{\mathrm{d}} \chi^2_{m^r - m^{r-1}}$$

Examples: Monkey Tests in Diehard, Serial test in NIST

Tests based on relative frequency (vi)

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Tests based on relative frequency (vii)

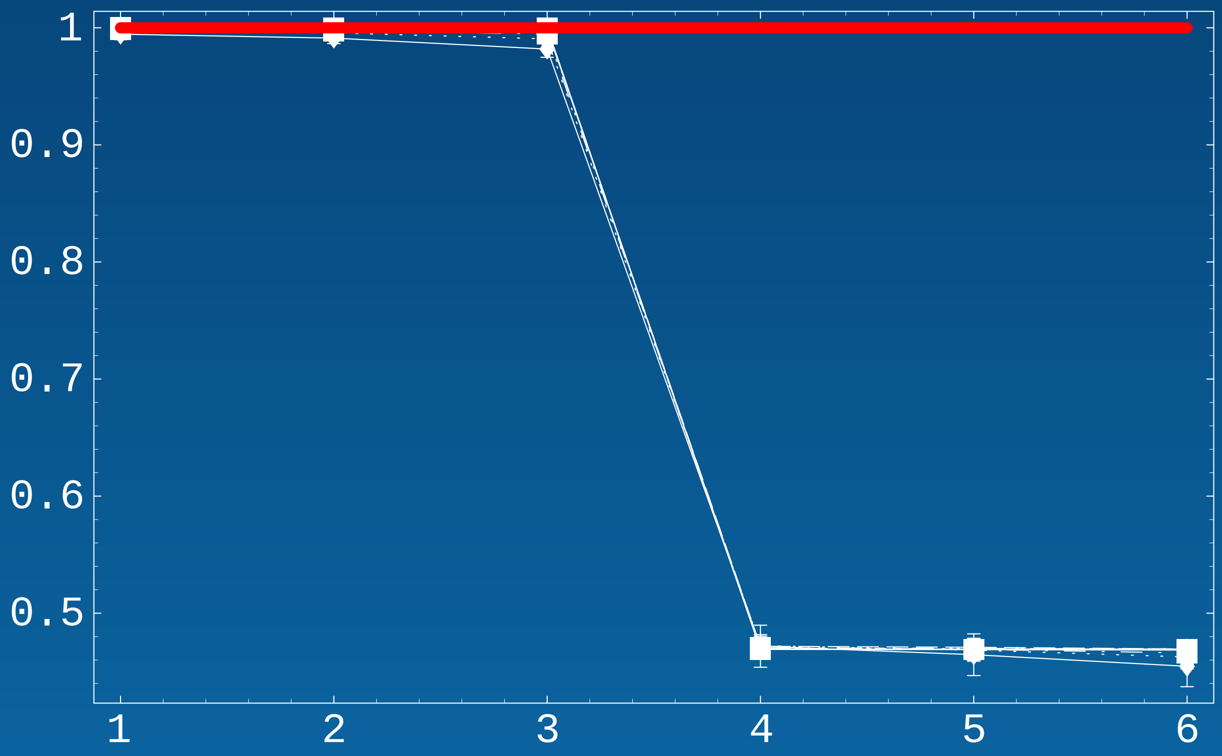

Fig. 1: Samples of $\hat{H}_f$ for an order-3 chain with $H = 0.469$

Tests based on return times

Maurer: estimate $E[\log_2(\bar{T}^{(r)})]$ by

$$\hat{H}_r := \frac{1}{r \cdot n} \sum_{i=Q+1}^{Q+n} \log_2 T(i),$$

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with (letting $\bar{X}_{-1}^{(r)} := \bar{X}_i^{(r)}$)

$$T(i) = \min\left\{1 \leq j \leq i+1 \;:\; \bar{X}_{i-j}^{(r)} = \bar{X}_i^{(r)}\right\}$$

Q . . . warm-up, initialization

Tests based on return times (ii)

Normalization: compute (tables available)

$E[\hat{H}_r]$ (rather easy)

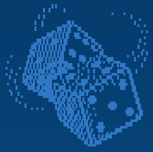
$V[\hat{H}_r]$ (rather complicated, only approx. available)

under H_0, then

$$\frac{\hat{H}_r - E[\cdot]}{\sqrt{V[\cdot]}} \xrightarrow{\mathrm{d}} N[0,1]$$

Tests based on return times (iii)

Universality: Maurer: $E\left[\log_2\left(\bar{T}^{(r)}\right)\right]$ detects any significant deviation from truly random bit source



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? also true for $r \in \mathbb{N}$ fixed

For any fixed $r \in \mathbb{N}$, we do not estimate H but only related property.
Is this property sensitive to H_1?

Tests based on return times (vii)

Some precomputed values of $E[\log_2(.)]$ for $m = 2$

r	$E[r \cdot \hat{H}_r]$	$E[\hat{H}_r] - \frac{\gamma}{r \cdot \ln(2)}$
1	0.73265	1.5654
2	1.53744	1.1851
3	2.40161	1.0781
4	3.31122	1.0360
5	4.25343	1.0172
6	5.21771	1.0084

Tests based on return times (viii)

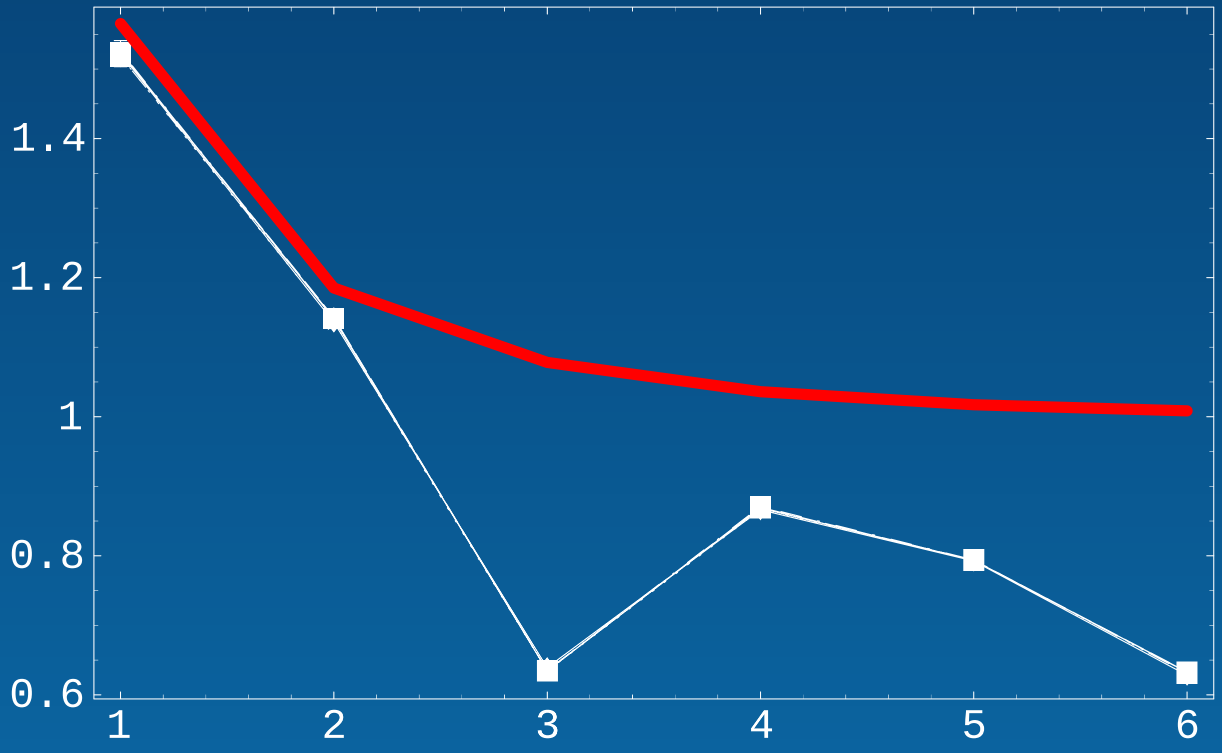

Fig. 2: Samples of $\hat{H}_r$ for an order-3 chain with $H = 0.469$

Summary

3 Types of Tests:

- Return Time based: Universal Test
- Relative Frequency based:

  a) Approximate Entropy, I-Divergence
  b) Serial Test, Monkey Test

Wegenkittl (IEEE 2002, PhD. Thesis): a) $\approx$ b)

Return Time vs. Frequency: Sample Study

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Idea: Simulate order κ chain with non-maximal Entropy

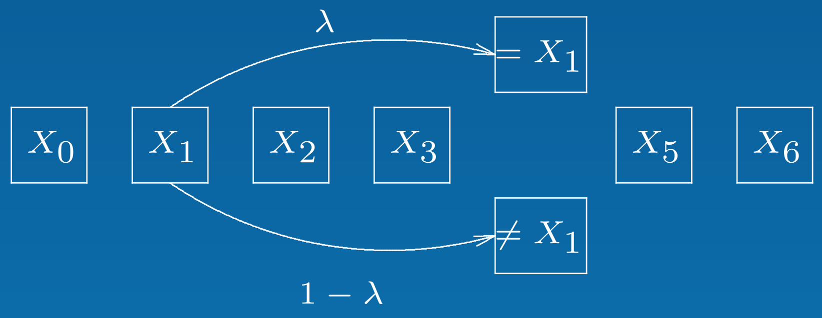
Vary “memory” κ of chain and “dimension” r of test

Sample Study: Defective Source

Let $m = 2$ (binary source) and

$$\tilde{\mathbb{P}} = (\tilde{p}_{ij})_{ij \in \mathcal{A}^2} = \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$$

and define transition probabilities $p_{a_{i-\kappa},\ldots,a_{i-1},a_i} = \tilde{p}_{a_{i-\kappa}a_i}$



Defective Source Analysis

Then, for the chain $(\mathcal{A}, \mathbb{P}^{\kappa}(\lambda))$

- equidistribution on $\mathcal{A}^{\kappa}$ is the stable distribution
- $H(\text{stable distribution}) = 1$, consequently
- $H(\mathcal{A}, \mathbb{P}^{\kappa}) = H(\lambda, 1-\lambda) < \ 1 \ \Leftrightarrow \lambda \neq \frac{1}{2}$

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Some values of $H(\mathcal{A}, \mathbb{P}^{\kappa})$:

λ	0.5	0.49	0.4	0.25	0.1
$H(\mathcal{A}, \mathbb{P}^{\kappa})$	1.0	0.999711	0.970951	0.811278	0.468996

Sensitivity of $\hat{H}_f$ and $\hat{\chi}$

- $\pi^{(r)}$ is equidistribution for all $r \leq \kappa$
- so that $\hat{H}_f$ and $\hat{\chi}$ will not be able to see any defect until $r > \kappa$
- but (Chain Rule) $\hat{H}_f$ (and $\hat{\chi}$) will see defect for all $r > \kappa$

Sensitivity of $\hat{H}_r$

- $\hat{H}_r$ should be sensible to $\lambda \neq 1/2$ for all $r \geq 1$.
- is there any dependency on r with respect to fixed κ?

Example revisited

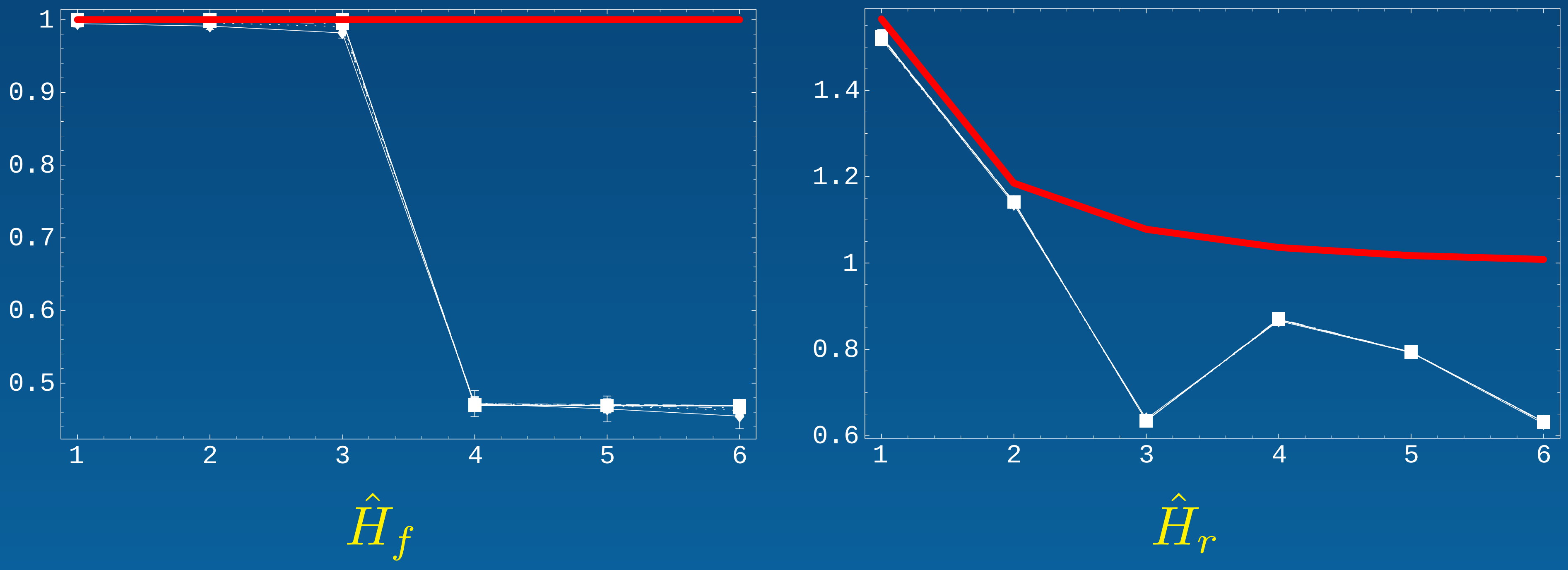


Fig. 3: Samples of $\hat{H}_f$ and $\hat{H}_r$ for $(\mathcal{A}, \mathbb{P}^3(1/2 - 0.4))$ with $H = 0.469$

Speaking of samplesizes

In our setup,

ApEn and Serial Test use $\approx n$ samples,

Universal Test uses $\approx n \cdot r$ samples plus warm-up

Speaking of samplesizes (ii)

$\log_2(n)$ needed to be able to detect Entropy defect (at 1% level of significance using $\hat{\chi}$):

r	1	2	4	8	12	16
$H = 0.999711$	15	15	16	19	23	27
$H = 0.468995$	4	4	5	8	12	16

Test Setup and Visualization

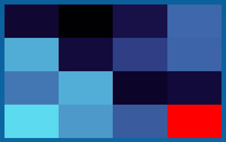
Compute 2-sided p-values according to given asymptotics

$\hat{H}_f$: dense case with χ^2 distribution

$\hat{\chi}$: dense case with χ^2 distribution

$\hat{H}_r$: normal distribution

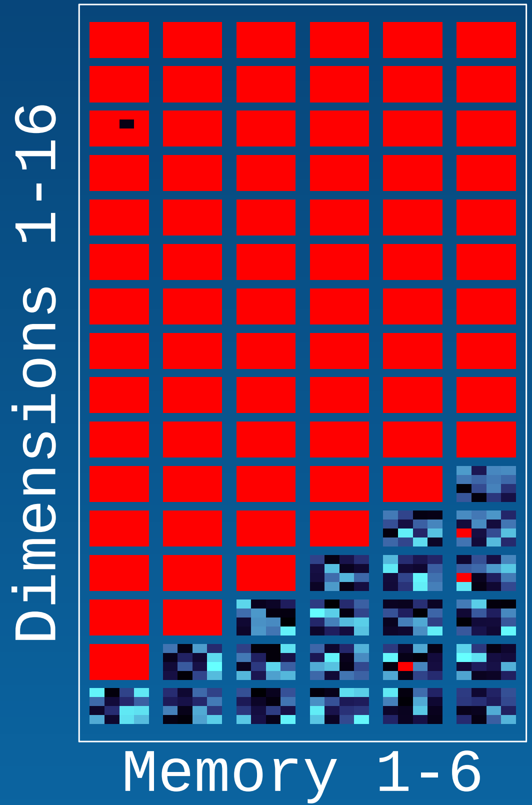
Repeat each test 16 times and arrange results into small rectangle:



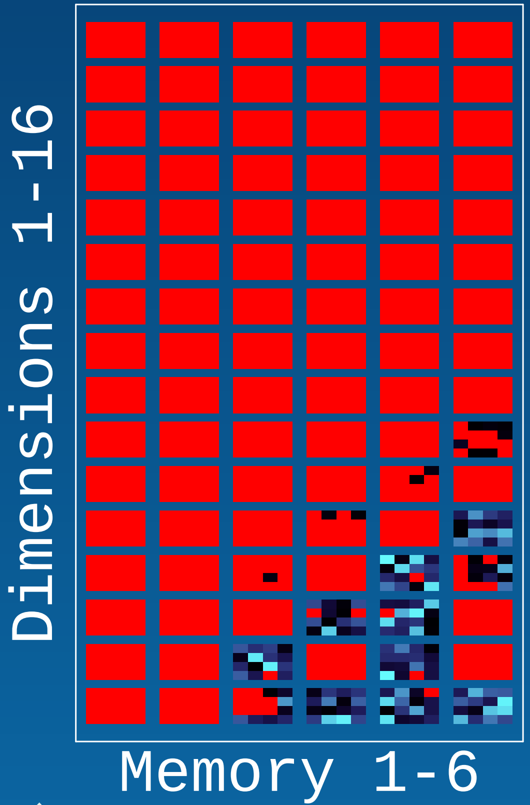
scale:

Red color indicates significance at 1% level.

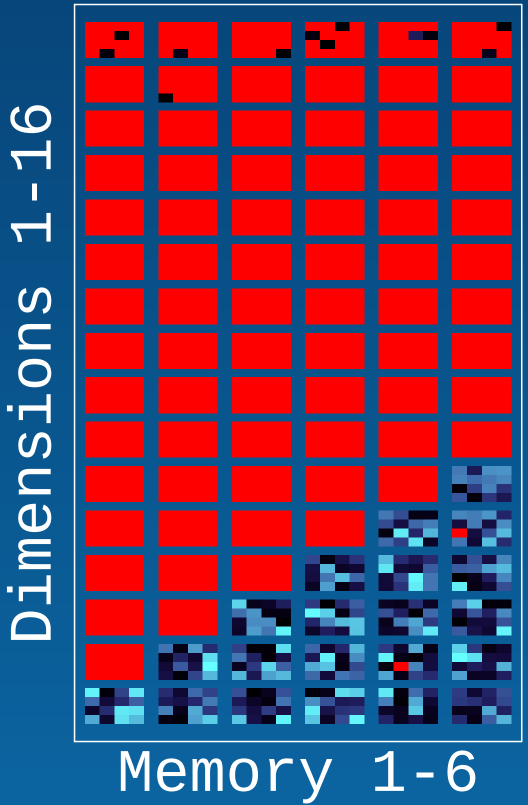
Easy case: $\lambda = 0.4,\ H = 0.970951$



$\hat{H}_f$, ApEn, I-Divergence



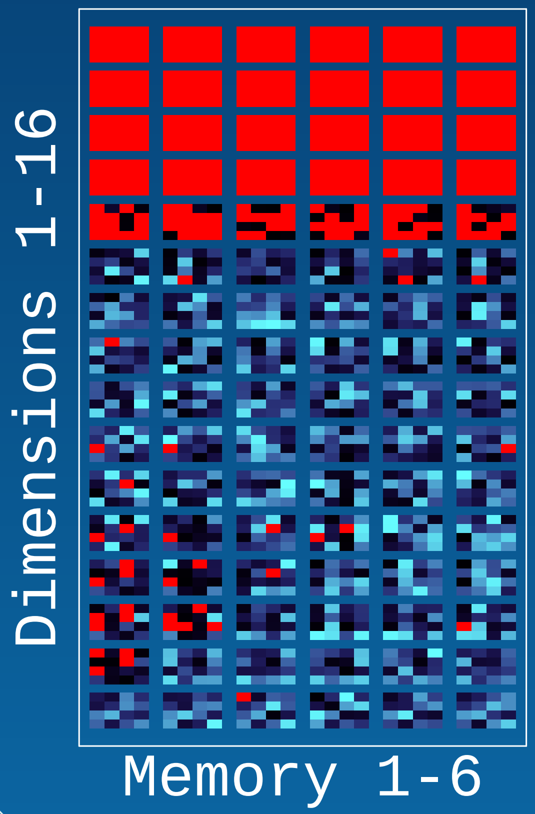
$\hat{H}_r$, Universal-Test



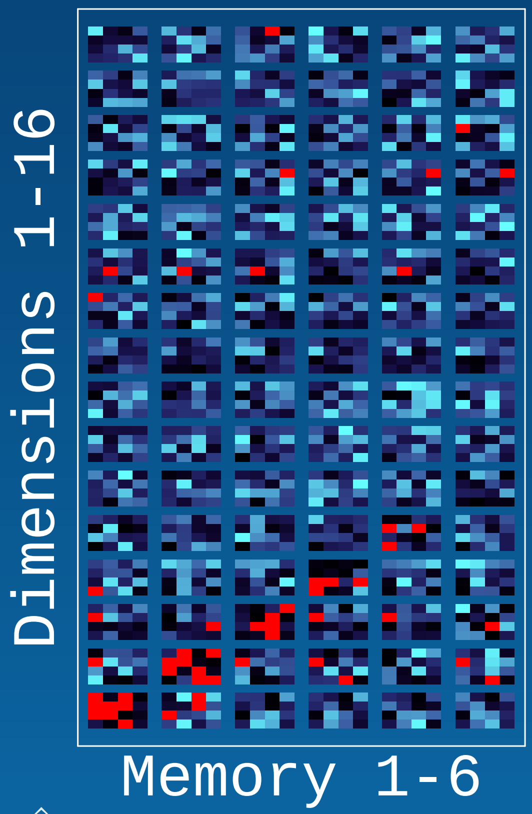
$\hat{\chi}$, Monkey-, Serial-Test

Parameters: $n = 2^{14}$, $1 \leq \kappa \leq 6$, $1 \leq r \leq 16$

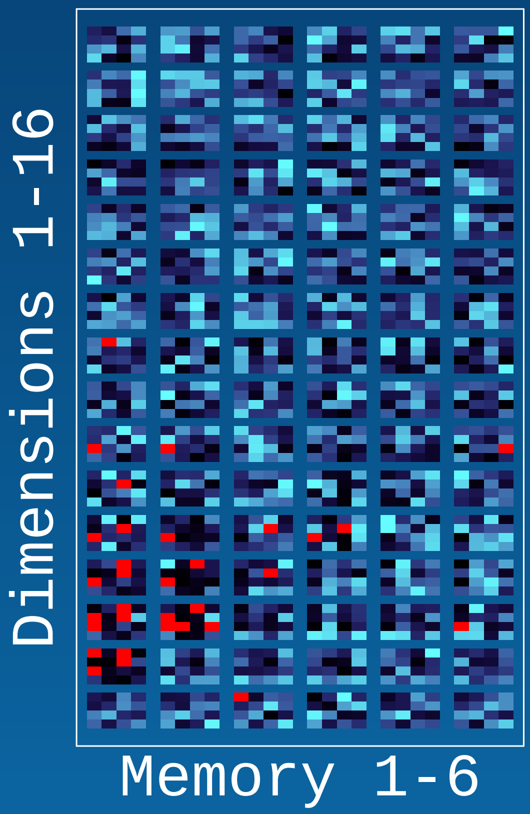
Hard case: $\lambda = 0.5$, $H = 1$, n too small



$\hat{H}_f$, ApEn, I-Divergence



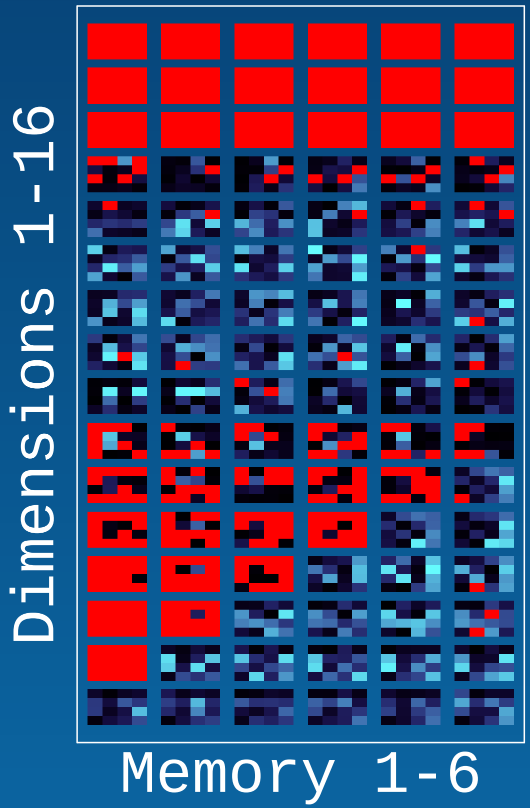
$\hat{H}_r$, Universal-Test



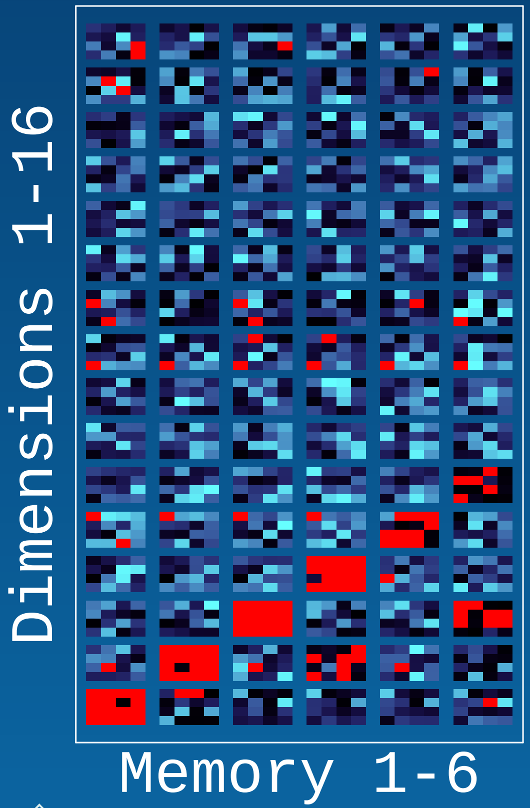
$\hat{\chi}$, Monkey-, Serial-Test

Parameters: $n = 2^{14}$, $1 \leq \kappa \leq 6$, $1 \leq r \leq 16$

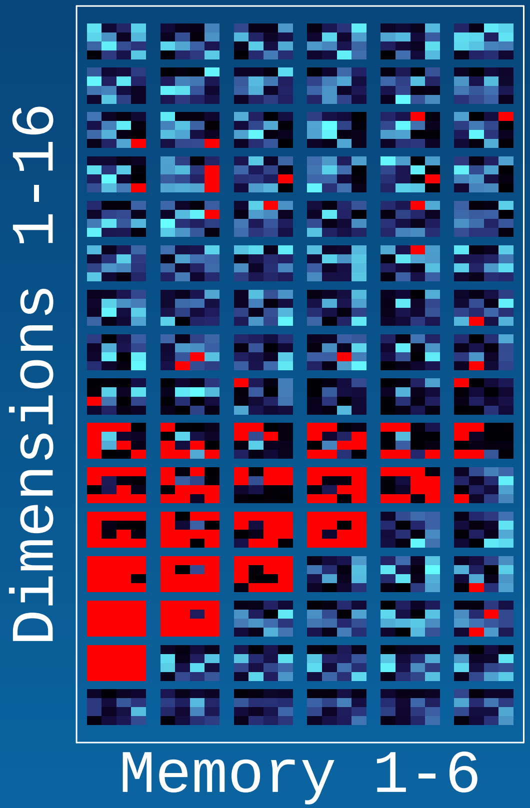
Hard case: $\lambda = 0.49$, $H = 0.999711$, n still small



$\hat{H}_f$, ApEn, I-Divergence



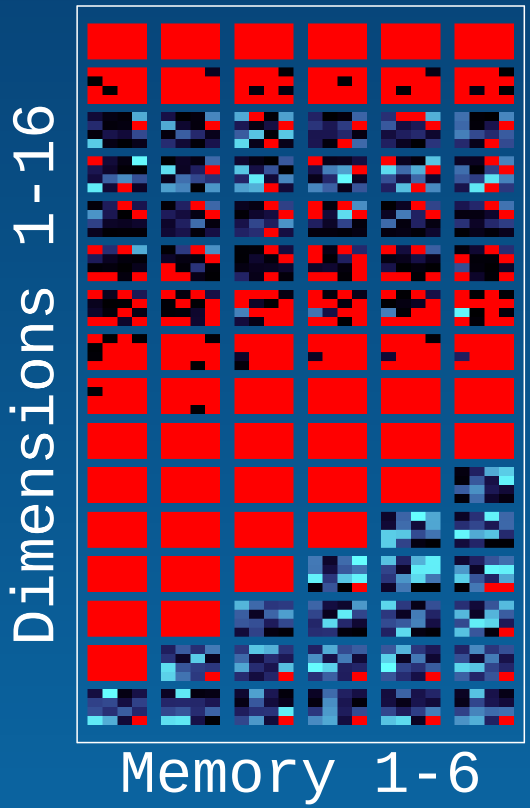
$\hat{H}_r$, Universal-Test



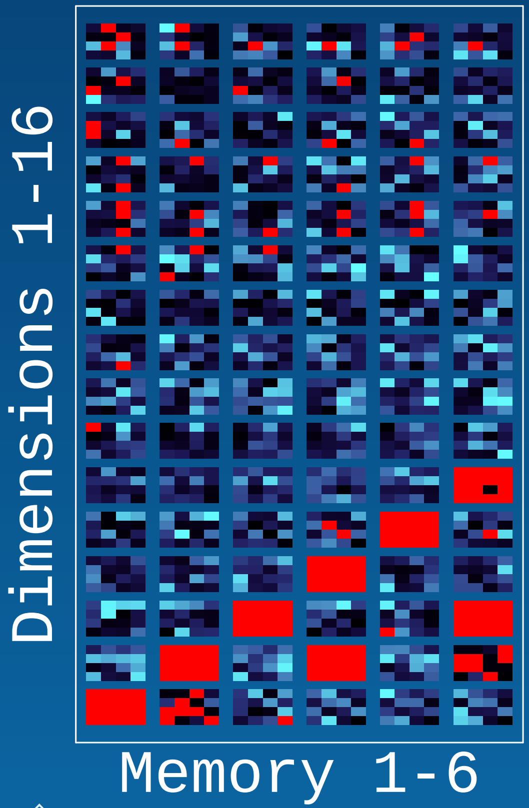
$\hat{\chi}$, Monkey-, Serial-Test

Parameters: $n = 2^{16}$, $1 \leq \kappa \leq 6$, $1 \leq r \leq 16$

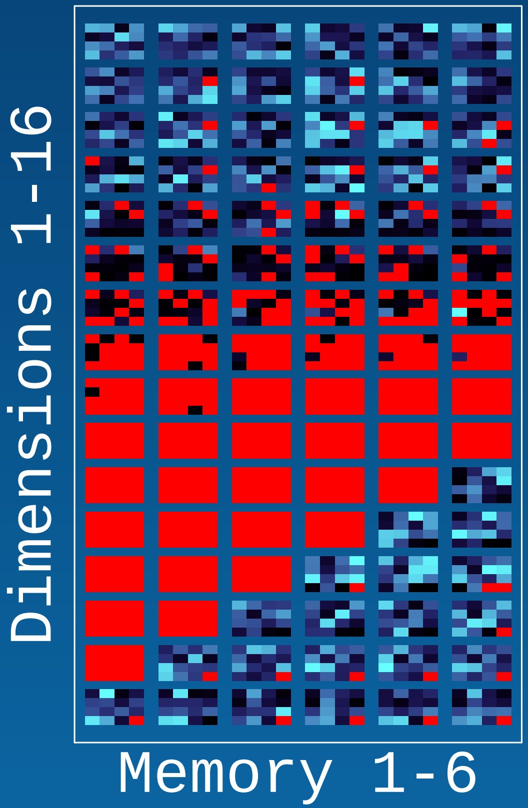
Hard case: $\lambda = 0.49$, $H = 0.999711$, n o.k.



$\hat{H}_f$, ApEn, I-Divergence



$\hat{H}_r$, Universal-Test



$\hat{\chi}$, Monkey-, Serial-Test

Parameters: $n = 2^{18}$, $1 \leq \kappa \leq 6$, $1 \leq r \leq 16$

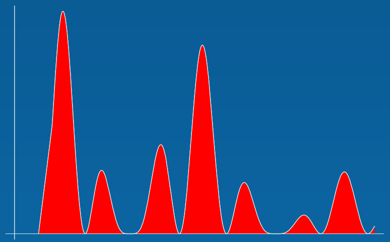
Summary: Degrees of Universality

special purpose multi purpose universal

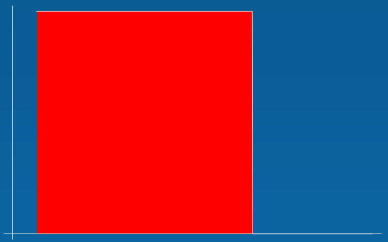
e.g. POKER TEST

UNIVERSAL TEST
vary DIMENSION

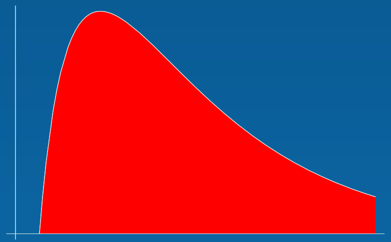
SERIAL TEST
vary DIMENSION, RESOLUTION



irregular



$r > \kappa$



$r \in \mathbb{R}$

Summary: Correlation between Tests

Ability to detect H_1:

- ApEn and Serial Tests: similar, good results if r large enough
- Universal Test: good results if r “correct” (critical!)

Summary: Correlation between Tests

Ability to detect H_1:

- ApEn and Serial Tests: similar, good results if r large enough
- Universal Test: good results if r “correct” (critical!)

Handling:

- Serial Tests: well understood, fast convergence
- ApEn is easy to interpret

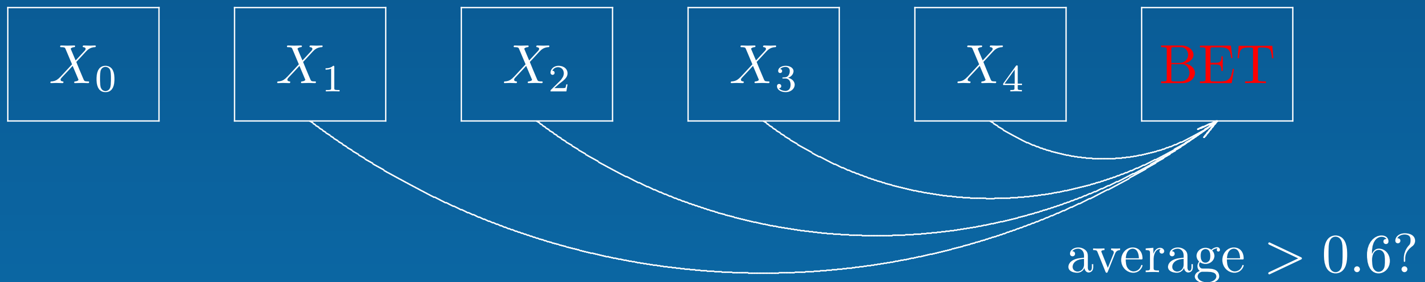
Serial Tests in Applications

Dimension Reduction: Gambling Tests

Need $r \in 1 \ldots 256$ (generating keys for cryptosystems)

Technique of Gambling Tests (Wegenkittl 2000):

reduce test dimension, preserve typical RNG correlations



X_0 X_1 X_2 X_3 X_4 BET

average > 0.6?

Application: RNG241 of PDH

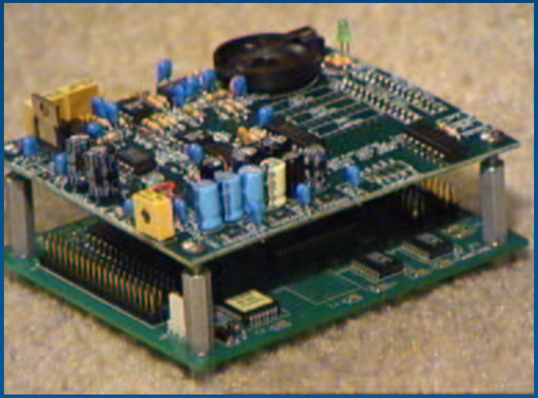
Study: Hardware Generators in Serial Tests

Example: RNG241

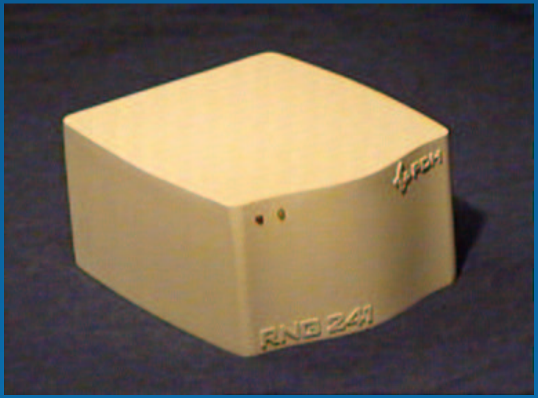
PDH International, Inc.
Precision Digital Hardware
1250 E. Hallandale Beach Blvd. PH2
Hallandale FL 33009, USA
www.pdhint.com
E-mail: contact@pdhint.com

Application: RNG241 of PDH (ii)

radioactive α-decay RNG hardware
1 Kbps naturally perfect entropy
unaffected by external noise



plug & play serial-port box
integrated software solutions



Application: RNG241 of PDH (iii)

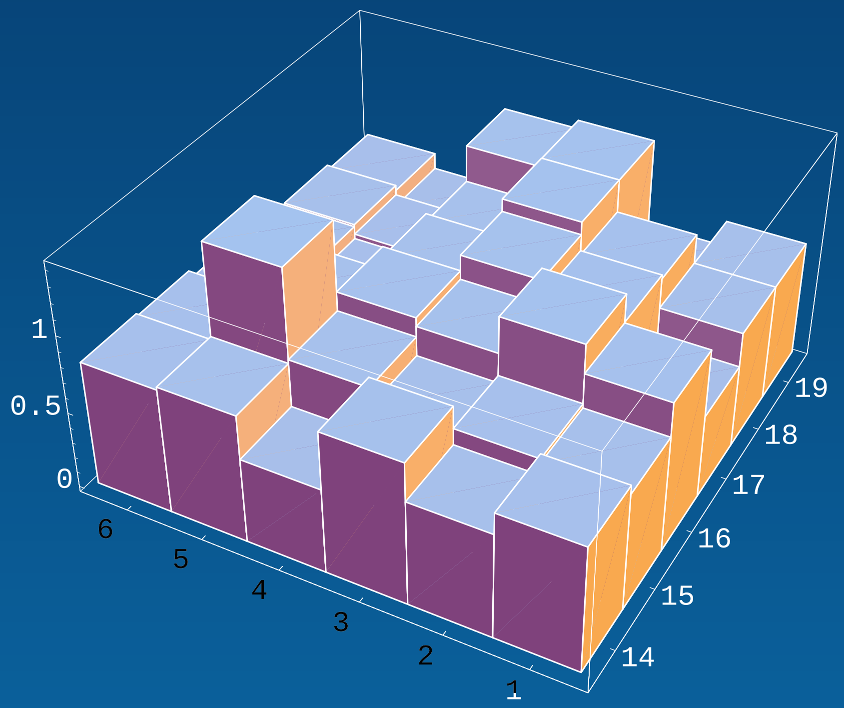


Fig. 4: Load Test: Each bar denotes the value of a single Kolmogorov-Smirnov statistic on 32 repetitions of the Load Test. Axes: dimension $s \in \{1, \dots, 6\}$ and the dual logarithm of the level-one sample size ranging from 14 to 19.

Application: RNG241 of PDH (iv)

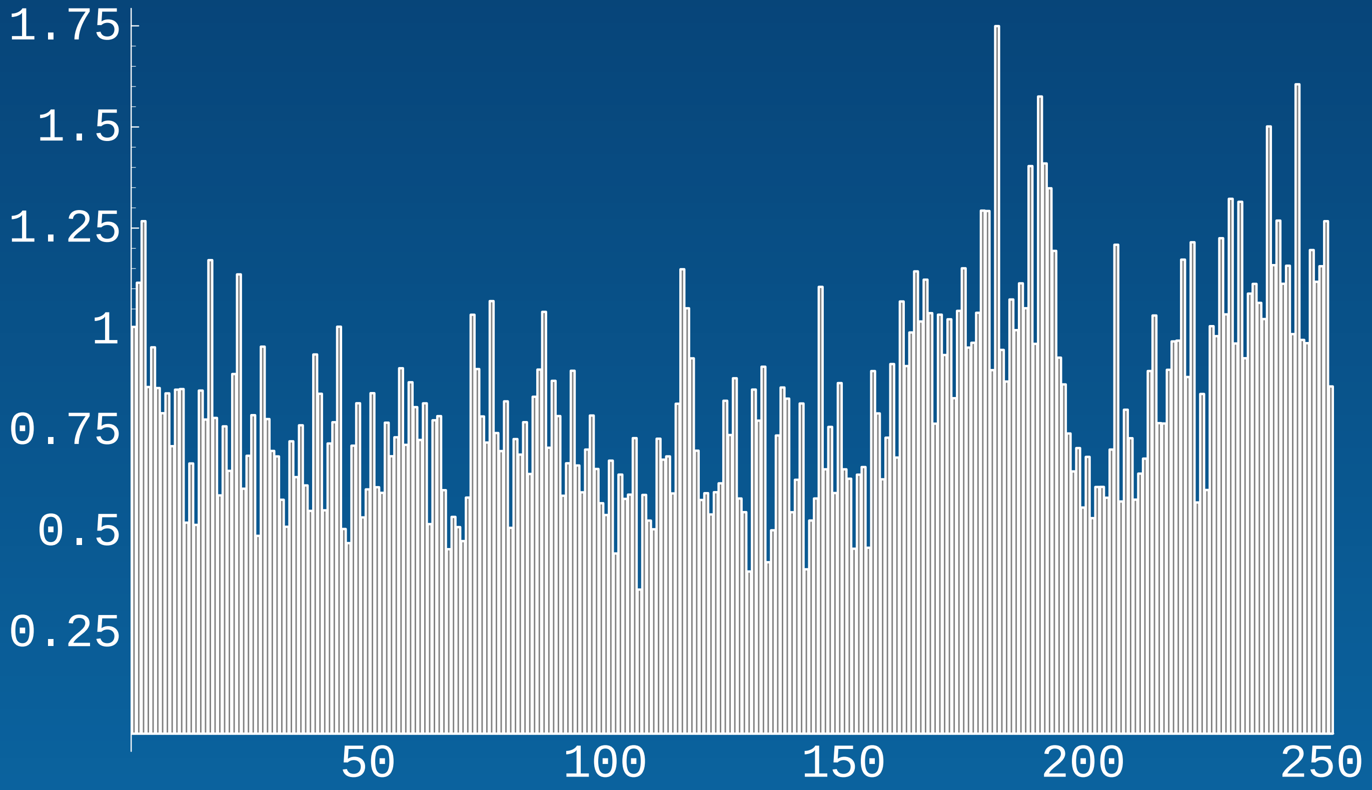


Fig. 5: A plot of the values of the Kolmogorov-Smirnov statistic for the Gambling Test with memory sizes $\lambda \in \{5, \ldots, 256\}$

Application: RNG241 of PDH (v)

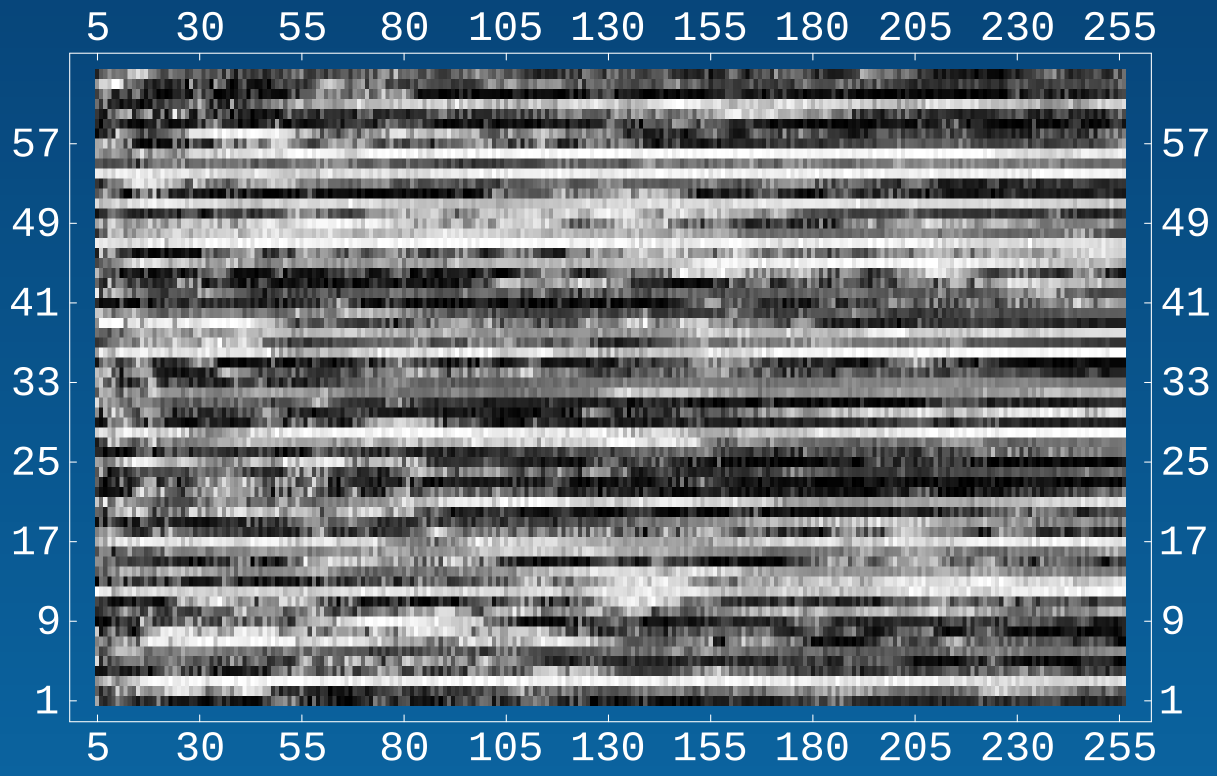


Fig. 6: Single Gambling Tests. Memory size $\lambda \in \{5, \ldots, 256\}$, 64 independent samples of upper-tail probabilities of the Gambling Test.

The End

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Slides on web server.

Links and References

References

[1] S. Wegenkittl. Monkeys, gambling, and return times: Assessing pseudorandomness. In P.A. Farrington, H.B. Nembhard, D.T. Sturrock, and G.W. Evans, editors, *Proceedings of the 1999 Winter Simulation Conference*, pages 625–631. IEEE Press, 1999.

[2] S. Wegenkittl. Entropy estimators and serial tests for ergodic chains. *IEEE Transactions on Information Theory*, **47**(6):2480–2489, Sep 2001.

[3] S. Wegenkittl. Gambling tests for pseudorandom number generators. *Mathematics and Computers in Simulation*, 55(1–3):281–288, 2001.

[4] S. Wegenkittl. A generalized ϕ-divergence for asymptotically multivariate normal models, 2002. To appear in *Journal of Multivariate Analysis, Vol. 84, No. 1, 2003*, available via IDEALFirst.

[5] S. Wegenkittl and M. Matsumoto. Getting rid of correlations among pseudorandom numbers: Discarding versus tempering. *ACM Trans. Modeling and Computer Simulation*, **9**(3):282–294, 1999.

[6] S. Wegenkittl, “Generalized ϕ-Divergence and Frequency Analysis in Markov Chains,”

Ph.D. thesis, Universität Salzburg, Österreich, 1998, HTML version: http://random.mat.sbg.ac.at

[7] P. L'Ecuyer, R. Simard, and S. Wegenkittl, "Sparse serial tests of uniformity for random number generators," Accepted for publication in SISC, 2002.

[8] H. Leeb and S. Wegenkittl, "Inversive and linear congruential pseudorandom number generators in empirical tests.," *ACM Trans. Modeling and Computer Simulation*, vol. **7**, no. 2, pp. 272–286, 1997.

[9] S. Wegenkittl, "The pLab picturebook: Load tests and ultimate load tests, part I," Report no. 1, pLab – reports, University of Salzburg, 1997.

[10] P. Hellekalek, "Good random number generators are (not so) easy to find," *Mathematics and Computers in Simulation*, vol. **46**, pp. 485–505, 1998.

[11] U.M. Maurer, "A universal statistical test for random bit generators," *J. Cryptology*, vol. **5**, pp. 89–105, 1992.

[12] I. Csiszár, "Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten," *Magyar Tud. Akad. Mat. Kutató Int. Közl*, vol. **8**, pp. 85–108, 1963.

[13] K. Pearson, "On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling.," *Philos. Magazine Series*, vol. **50** , no. 5, pp. 157–172, 1900.

[14] S. Kullback and R. Leibler, “On information and sufficiency,” *Ann. Math. Statist.*, vol. 22, pp. 79–86, 1951.

[15] I. J. Good, “The serial test for sampling numbers and other tests for randomness,” *Proc. Cambridge Philosophical Society*, vol. **49**, pp. 276–284, 1953.

[16] G. Marsaglia, “A current view of random number generators,” in *Computer Science and Statistics: The Interface*, L. Billard, Ed. 1985, pp. 3–10, Elsevier Science Publishers B.V.

[17] G.H. Choe and D.H. Kim, “The probability distribution of the first return time,” Submitted for publication, 1999.

[18] A. Wyner and J. Ziv, “Some asymptotic properties of the entropy of stationary ergodic data source with applications to data compression,” *IEEE Trans. Information Theory*, vol. **35**, pp. 1250–1258, 1989.

[19] D. Ornstein and B. Weiss, “Entropy and data compression schemes,” *IEEE Trans. Information Theory*, vol. **39**, pp. 78–93, 1993.

[20] J. S. Coron and D. Naccache, “An Accurate Evaluation of Maurer’s Universal Test,” in *Proceedings of Selected Areas in Cryptography 98*. 1998, Lecture Notes in Computer Science 1556, pp. 57–71, Springer.

[21] G.H. Choe and D.H. Kim, “Average convergence rate of the first return time,” Submitted for publication, 1999.

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