

# THERE IS NO CATEGORICAL METRIC CONTINUUM

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ABSTRACT. We show there is no categorical metric continuum. This means that for every metric continuum  $X$  there is another metric continuum  $Y$  such that  $X$  and  $Y$  have (countable) elementarily equivalent bases but  $X$  and  $Y$  are not homeomorphic. As an application we show that the chainability of the pseudoarc is not a first-order property of its lattice of closed sets.

## INTRODUCTION

Many properties of compact Hausdorff spaces can, naturally, be phrased in terms of their families of closed sets. For a fair number of these one can find even first-order formulas in the language of lattices that characterize them, see, e.g., [7].

In [1] and [4] it was shown that chainability is not a first-order property. In an earlier version of the former paper the question was raised whether there is any chainable continuum for which its chainability is expressible in first-order terms. The authors offered the pseudoarc as a candidate.

If the pseudoarc were ‘first-order chainable’ then it would at once become a categorical continuum. This is so because the pseudoarc is the only continuum that is both chainable and hereditarily indecomposable. This would imply that any continuum with a lattice-base for its closed sets that is elementarily equivalent to some lattice base for the closed sets of the pseudoarc would itself be the pseudoarc.

In this note we show that no categorical continuum exists and hence, indirectly, that the pseudoarc is not first-order chainable.

## 1. PRELIMINARIES

**1.1. Chainability.** A continuum is *chainable* if every finite open cover has a finite chain refinement, where a chain refinement is an indexed refinement  $\{V_i : i < n\}$  with the property that  $V_i \cap V_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ . The condition that  $\mathcal{V}$  is a chain refinement of  $\mathcal{U}$  can be expressed by a (rather long) first-order formula. The condition that  $\mathcal{U}$  has a chain refinement is, a priori, not first-order as one does not know beforehand how large the refinement is going to be. One gets a formula of the form  $(\exists \mathcal{V})(\bigvee_n \phi_n(\mathcal{U}, \mathcal{V}))$ , where  $\phi_n$  expresses that  $\mathcal{V}$  is an  $n$ -element chain refinement of  $\mathcal{U}$  — this is an  $L_{\omega, \omega_1}$ -formula. Chainability proper is then defined by infinitely many such formulas: one for each possible cardinality of  $\mathcal{U}$ .

The authors of [1] identified one way of defining first-order chainability: make sure the disjunction becomes finite. This would mean, in words: for every natural number  $m$  there is a natural number  $n$  such that every open cover of size  $m$  has an open chain refinement of size  $n$  or less.

The negation of this, namely that there is a natural number  $m$  such that for every  $n$  there is an open cover for which every chain refinement has at least  $n$  members, was called *elastically chainable* in [1].

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**1.2. Categoricity.** Categoricity is a model-theoretic notion; we refer to [5, Section 6.3] for a complete treatment of the countable case, which is the case that we shall need; we refer to [5] for other model-theoretic notions as well. A countable structure  $S$  (group, lattice, ordered set) is categorical if every other countable structure that satisfies the same first-order sentence as  $S$  is actually isomorphic to  $S$ . A prime example is the set  $\mathbb{Q}$  of rational numbers; it is, up to isomorphism, the only countable linearly ordered set that is densely ordered and without end points. Structures that satisfy the same first-order sentences are usually said to be *elementarily equivalent*.

We extend these notions to cover compact Hausdorff spaces: we call two such spaces elementarily equivalent if they have bases for the closed sets that are elementarily equivalent as lattices. A compact metric space is categorical if every compact metric space that is elementarily equivalent to it is homeomorphic to it.

**1.3. Creating surjections.** The following lemma is used to construct continuous surjections.

**Lemma 1.1** ([3, Theorem 1.2]). *Let  $X$  and  $Y$  be compact Hausdorff spaces and let  $\mathcal{C}$  be a base for the closed subsets of  $Y$  that is closed under finite unions and finite intersections. Then  $Y$  is a continuous image of  $X$  if and only if there is a map  $\phi : \mathcal{C} \rightarrow 2^X$  such that*

- (1)  $\phi(\emptyset) = \emptyset$  and if  $F \neq \emptyset$  then  $\phi(F) \neq \emptyset$ ;
- (2) if  $F \cup G = Y$  then  $\phi(F) \cup \phi(G) = X$ ; and
- (3) if  $F_1 \cap \dots \cap F_n = \emptyset$  then  $\phi(F_1) \cap \dots \cap \phi(F_n) = \emptyset$ . □

**1.4.  $K_0$ -functions.** Consider a metric space  $X$ , with metric  $d$ , and a closed subspace  $A$ . Define a map  $\kappa : 2^A \rightarrow 2^X$  by

$$\kappa(F) = \{x \in X : d(x, F) \leq d(x, A \setminus F)\}.$$

In [6, §21 XI] it is shown that for all closed sets  $F$  and  $G$  in  $A$  we have

- $\kappa(F) \cap A = F$ ;
- $\kappa(F \cup G) = \kappa(F) \cup \kappa(G)$ ; and
- $\kappa(A) = X$  and  $\kappa(\emptyset) = \emptyset$  — by the fact that  $d(x, \emptyset) = \infty$  for all  $x$ .

Following [2] we call such a function a  *$K_0$ -function*.

## 2. THE MAIN LEMMA

**Lemma 2.1.** *Let  $X$  and  $Y$  be metric continua and let  $\mathcal{B}$  and  $\mathcal{C}$  be a lattice bases for their respective families of closed sets. Let  $u$  be any free ultrafilter on  $\omega$ . There is a map  $\phi$  from  $\mathcal{C}$  to the ultrapower  $\mathcal{B}_u$  that satisfies the conditions in Lemma 1.1.*

*Proof.* We consider  $Y$  embedded in the Hilbert cube  $Q$  and we let  $\kappa : 2^Y \rightarrow 2^Q$  be a  $K_0$ -function. Furthermore, fix a continuous surjection  $f : X \rightarrow [0, 1]$ .

Enumerate  $\mathcal{C}$  as  $\langle C_n : n \in \omega \rangle$  and put, for each  $n$ ,  $E = \{e \subseteq \omega : \bigcap_{i \in e} C_i = \emptyset\}$ . Observe that  $Y \cap \bigcap_{i \in e} \kappa(C_i) = \emptyset$  whenever  $e \in E$ .

Fix  $n < \omega$  and take a positive number  $\epsilon_n$  less than  $2^{-n}$  and all distances between  $Y$  and  $\bigcap_{i \in e} \kappa(C_i)$  for those  $e \in E$  that are subsets of  $n$ . Take a continuous map  $g_n : [0, 1] \rightarrow Q$  such that the image is a subset of  $B(Y, \epsilon_n)$  and such that it meets every ball  $B(y, \epsilon_n)$  with  $y \in Y$  (here we use that  $Y$  is a continuum: it has arbitrarily small arcwise connected neighbourhoods).

For  $i < n$  let  $D_i^n$  be the preimage  $h_n^{-1}[g_n^{-1}[\kappa(C_i)]]$ . Because  $\kappa$  is a  $K_0$ -function we know that  $D_i^n \cup D_j^n = X$  whenever  $C_i \cup C_j = Y$ . Also, by the choice of  $\epsilon_n$ , we know that  $\bigcap_{i \in e} D_i^n = \emptyset$  whenever  $e \in E$  and  $e \subseteq n$ . Now expand the sets  $D_i^n$  to get members  $B_i^n$  of  $\mathcal{B}$ , retaining the property that  $\bigcap_{i \in e} D_i^n = \emptyset$  whenever  $e \in E$  and  $e \subseteq n$ .

The definition of  $\phi$  is now straightforward: define  $\phi(C_i)$  to be the  $u$ -equivalence class of  $\langle B_i^n : n > i \rangle$ . Note that  $\phi$  has the required properties even when we take the reduced power modulo the co-finite filter.  $\square$

### 3. THE MAIN RESULT

The following proposition is the key to the main result.

**Proposition 3.1.** *Let  $X$  and  $Z$  be two metric continua. There is a third metric continuum  $Y$  such that*

- (1)  $Z$  is a continuous image of  $Y$ ; and
- (2)  $Y$  and  $X$  have elementarily equivalent bases for the closed sets.

*Proof.* Take countable bases  $\mathcal{B}$  and  $\mathcal{D}$  respectively for the closed sets of  $X$  and  $Z$ . Fix a free ultrafilter  $u$  on  $\omega$  and apply Lemma 2.1 to find a map  $\phi : \mathcal{D} \rightarrow \mathcal{B}_u$  as in Lemma 1.1. Next apply the Löwenheim-Skolem theorem to obtain a countable elementary substructure  $\mathcal{C}$  of  $\mathcal{B}_u$  that contains  $\phi[\mathcal{D}]$ . We let  $Y$  be the Wallman space of the lattice  $\mathcal{C}$ . Then  $Y$  is as required: the lattice  $\mathcal{C}$  is elementarily equivalent to  $\mathcal{B}_u$  and hence to  $\mathcal{B}$  itself. The map  $\phi$  enables us, via Lemma 1.1, to map  $Y$  onto  $Z$ .  $\square$

**3.1. The proof.** It is now straightforward to prove the main assertion of this note. In [8] Waraszkiewicz constructed a family of continua such that no single metric continuum maps onto all of them. Let  $X$  be any metric continuum and fix a continuum  $Z$  from that family that is not a continuous image of  $X$ . Apply Proposition 3.1 to find a metric continuum  $Y$  that does map onto  $Z$  and yet has a base for the closed sets that is elementarily equivalent to a base for the closed sets of  $X$ . Clearly  $X$  and  $Y$  are not homeomorphic.

**3.2. The pseudoarc.** In an earlier version of [1] it was asked whether the pseudoarc is inelastically chainable. If it were it would show that the pseudoarc is categorical.

The results of this paper imply that this corollary does not hold and hence that the pseudoarc *is* elastically chainable. This argument simply shows that a natural number  $m$  as in the definition exists, it does not provide a definite value.

The present version of [1] contains a proof that *every* chainable continuum is elastically chainable with the smallest possible value of  $m$ , to wit 3.

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