NUMERICALLY STABLE DIRECT LEAST SQUARES FITTING OF ELLIPSES

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ABSTRACT

This paper presents a numerically stable non-iterative algorithm for fitting an ellipse to a set of data points. The approach is based on a least squares minimization and it guarantees an ellipse-specific solution even for scattered or noisy data. The optimal solution is computed directly, no iterations are required. This leads to a simple, stable and robust fitting method which can be easily implemented. The proposed algorithm has no computational ambiguity and it is able to fit more than 100,000 points in a second.

Keywords: ellipses, fitting, least squares, eigenvectors

INTRODUCTION

One of basic tasks in pattern recognition and computer vision is a fitting of geometric primitives to a set of points (see [Duda73] for a summary). The use of primitive models allows reduction and simplification of data and, consequently, faster and simpler processing. A very important primitive is an ellipse, which, being a perspective projection of a circle, is exploited in many applications of computer vision like 3-D vision and object recognition, medical imaging, industrial inspections, etc.

Regarding the importance of ellipses, many different methods have been proposed for their detection and fitting. The approaches exploit various ideas (for example Hough transform [Leave92, Yuen89, Yin92, Wu93], RANSAC [Rosin93, Werma95], Kalman filtering [Porri90, Rosin95], fuzzy clustering [Davé92, Gath95], or least squares approach [Haral93, Books79, Taubi91, Samps92, Gande94]), but in principle they can be divided into two main groups: voting/clustering and optimization methods. The methods belonging to the first group (Hough transform, RANSAC, and fuzzy clustering) are robust against outliers and they can detect multiple primitives at once. Unfortunately, these methods are slow, require large memory and their accuracy is low. Moreover, regarding the popular Hough transform, there are problems with a detection of objects due to blurred and spurious peaks in the accumulators [Grims90].

The second group of fitting methods are based on optimization of an objective function which characterizes a goodness of a particular ellipse with respect to the given set of data points. The main advantages of these methods are their speed and accuracy, on the other hand the methods can fit only one primitive at time (i.e. the data should be pre-segmented before the fitting). Also the sensitivity to outliers is higher than in the clustering methods.

An analysis of the optimization approaches was done in [Fitzg95a] and [Fitzg96a]. It was shown that the methods are typically based on a general conic fitting (such as [Books79, Taubi91, Samps92]) with additional constraints ensuring that the solution will be an ellipse rather than a general conic. Although the general conic fitting can be computed directly, the constraint on the ellipse-specific solution makes the complete methods iterative. There were many attempts to make the fitting process computationally effective. Finally, Fitzgibbon *et al.* proposed a direct least squares based ellipse-specific method in [Fitzg96b].

In this paper we analyze the Fitzgibbon's approach, characterize its drawbacks and propose an improved method for a direct fitting of ellipses. The paper is organized as follows: First, the original approach is described. We investigate the method and discuss situations when it fails or provides non-optimal results. Then, based on a block decomposition of matrices, an improved fitting method is proposed. Finally, a practical realization of our approach is presented together with its evaluation in several experiments. A comparison of the proposed method with currently used approaches concludes the whole paper.

FITZGIBBON'S APPROACH TO LEAST SQUARES FITTING OF ELLIPSES

This approach was proposed in [Fitzg96b]. The method works on segmented data (that means that all data points are assumed to belong to one ellipse) and it is stated to be the first non-iterative ellipse-specific fitting. In this section we provide an overview of the method. Further details and comparisons with another approaches can be found in the technical report [Fitzg96a].

An ellipse is a special case of a general conic which can be described by an implicit second order polynomial

$$F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{1}$$

with an ellipse-specific constraint

$$b^2 - 4ac < 0 \tag{2}$$

where a,b,c,d,e,f are coefficients of the ellipse and (x,y) are coordinates of points lying on it. The polynomial $F(x,y)$ is called the *algebraic distance* of the point (x,y) to the given conic. By introducing vectors

$$\begin{aligned} \mathbf{a} &= [a,b,c,d,e,f]^{\mathrm{T}} \\ \mathbf{x} &= [x^2,xy,y^2,x,y,1] \end{aligned} \tag{3}$$

it can be rewritten to the vector form

$$F_{\mathbf{a}}(\mathbf{x}) = \mathbf{x}\cdot\mathbf{a} = 0 \tag{4}$$

The fitting of a general conic to a set of points $(x_i,y_i),\ i = 1\ldots N$ may be approached [Haral93] by minimizing the sum of squared algebraic distances of the points to the conic which is represented by coefficients $\mathbf{a}$:

$$\begin{aligned} \min_{\mathbf{a}} \sum_{i=1}^{N} F(x_i,y_i)^2 &= \min_{\mathbf{a}} \sum_{i=1}^{N} \left(F_{\mathbf{a}}(\mathbf{x_i})\right)^2 \\ &= \min_{\mathbf{a}} \sum_{i=1}^{N} (\mathbf{x_i}\cdot\mathbf{a})^2 \end{aligned} \tag{5}$$

The problem Eq. 5 can be solved directly by the standard least squares approach, but the result of such fitting is a general conic and it needs not to be an ellipse.

To ensure an ellipse-specificity of the solution, the appropriate constraint Eq. 2 has to be considered. In the Fitzgibbon's paper it was shown that such system is hard to solve in general. However, because $\alpha\cdot\mathbf{a}$ represents the same conics as $\mathbf{a}$ for any $\alpha \neq 0$, we have a freedom to arbitrarily scale the coefficients $\mathbf{a}$. Under a proper scaling, the inequality constraint Eq. 2 can be changed into an equality constraint

$$4ac - b^2 = 1 \tag{6}$$

and the ellipse-specific fitting problem can be reformulated as

$$\min_{\mathbf{a}} \|\mathbf{D}\mathbf{a}\|^2 \quad \text{subject to} \quad \mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a} = 1 \tag{7}$$

where the *design matrix* $\mathbf{D}$ of the size $N\times 6$,

$$\mathbf{D} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_i^2 & x_iy_i & y_i^2 & x_i & y_i & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & x_Ny_N & y_N^2 & x_N & y_N & 1 \end{pmatrix} \tag{8}$$

represents the least squares minimization Eq. 5 and the *constraint matrix* $\mathbf{C}$ of the size 6×6,

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

express the constraint Eq. 6. The minimization problem Eq. 7 is ready to be solved by a quadratically constrained least squares minimization as proposed in [Gande81]. First, by applying the Lagrange multipliers we get the following conditions for the optimal solution $\mathbf{a}$

$$\begin{aligned} \mathbf{S}\mathbf{a} &= \lambda\mathbf{C}\mathbf{a} \\ \mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a} &= 1 \end{aligned} \tag{10}$$

where $\mathbf{S}$ is the *scatter matrix* of the size 6×6,

$$\begin{aligned} \mathbf{S} &= \mathbf{D}^{\mathrm{T}}\mathbf{D} \\ &= \begin{pmatrix} S_{x^4} & S_{x^3y} & S_{x^2y^2} & S_{x^3} & S_{x^2y} & S_{x^2} \\ S_{x^3y} & S_{x^2y^2} & S_{xy^3} & S_{x^2y} & S_{xy^2} & S_{xy} \\ S_{x^2y^2} & S_{xy^3} & S_{y^4} & S_{xy^2} & S_{y^3} & S_{y^2} \\ S_{x^3} & S_{x^2y} & S_{xy^2} & S_{x^2} & S_{xy} & S_{x} \\ S_{x^2y} & S_{xy^2} & S_{y^3} & S_{xy} & S_{y^2} & S_{y} \\ S_{x^2} & S_{xy} & S_{y^2} & S_{x} & S_{y} & S_{1} \end{pmatrix} \end{aligned} \tag{11}$$

in which the operator S denotes the sum

$$S_{x^ay^b} = \sum_{i=1}^{N} x_i^a y_i^b \tag{12}$$

```
1 function a = fit_ellipse(x, y)
2 D = [x.*x x.*y y.*y x y ones(size(x))];                  % design matrix
3 S = D' * D;                                              % scatter matrix
4 C(6, 6) = 0; C(1, 3) = 2; C(2, 2) = -1; C(3, 1) = 2;    % constraint matrix
5 [gevec, geval] = eig(inv(S) * C);                        % solve eigensystem
6 [PosR, PosC] = find(geval > 0 & ~isinf(geval));  % find positive eigenvalue
7 a = gevec(:, PosC);                              % corresponding eigenvector
```

Figure 1: MATLAB implementation of the direct ellipse-specific fitting algorithm proposed by Fitzgibbon *et al.* in [Fitzg96b]

Next, the system Eq. 10 is solved by using generalized eigenvectors. There exist up to six real solutions $(\lambda_j, \mathbf{a}_j)$, but because

$$||\mathbf{Da}||^2 = \mathbf{a}^T\mathbf{D}^T\mathbf{Da} = \mathbf{a}^T\mathbf{Sa} = \lambda\mathbf{a}^T\mathbf{Ca} = \lambda \tag{13}$$

we are looking for the eigenvector $\mathbf{a}_k$ corresponding to the minimal positive eigenvalue λ_k. Finally, after a proper scaling ensuring $\mathbf{a}_k^T\mathbf{Ca}_k = 1$, we get a solution of the minimization problem Eq. 7 which represents the best-fit ellipse for the given set of points.

Based on this derivation, Fitzgibbon proposed an effective and robust algorithm for an ellipse-specific fitting of data points. The algorithm was implemented in MATLAB [Mathw] and it is available as a part of the package [Fitzg95b]. The appropriate code is presented in Fig. 1.

IMPROVED LEAST SQUARES METHOD FOR FITTING ELLIPSES

Apart from its theoretical correctness, the original Fitzgibbon's approach described in the previous section has several drawbacks. The matrix $\mathbf{C}$ (Eq. 9) is singular and the matrix $\mathbf{S}$ (Eq. 11) is also nearly singular (it is singular if all data points lie exactly on an ellipse). Regarding that, the computation of the eigenvalues of Eq. 10 is numerically unstable and can produce wrong results (as infinite or complex numbers). It should be noted that using of an inverse as proposed in the original code (see line 5 in Fig. 1) does not solve this problem.

Another problematic part of the algorithm is a localization of the optimal solution of the fitting. In [Fitzg96a] authors proved that Eq. 10 has exactly one positive eigenvalue and they stated that the corresponding eigenvector is an optimal solution of Eq. 7. Unfortunately, this is not true. In an ideal case, when all data points lie exactly on an ellipse, the eigenvalue is zero. Moreover, regarding a numerical computation of eigenvalues the optimal eigenvalue can even be a small negative number. In such situations, the proposed localization (see line 6 in Fig. 1) can produce non-optimal or completely wrong solutions.

To overcome the drawbacks of the original approach, we should follow on a theoretical analysis of Eq. 10. Both matrices $\mathbf{S}$ (Eq. 11) and $\mathbf{C}$ (Eq. 9) have special structures which allow a simplification of the problem of finding the eigenvalues. First, we decompose the design matrix $\mathbf{D}$ (Eq. 8) into its quadratic and linear parts:

$$\mathbf{D} = (\mathbf{D}_1 \mid \mathbf{D}_2) \tag{14}$$

where

$$\mathbf{D}_1 = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_i^2 & x_iy_i & y_i^2 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_Ny_N & y_N^2 \end{pmatrix} \tag{15}$$

and

$$\mathbf{D}_2 = \begin{pmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_i & y_i & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{pmatrix} \tag{16}$$

Next, the scatter matrix $\mathbf{S}$ (Eq. 11) can be split as follows:

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^T & \mathbf{S}_3 \end{array}\right) \quad \text{where} \quad \begin{array}{l} \mathbf{S}_1 = \mathbf{D}_1^T\mathbf{D}_1 \\ \mathbf{S}_2 = \mathbf{D}_1^T\mathbf{D}_2 \\ \mathbf{S}_3 = \mathbf{D}_2^T\mathbf{D}_2 \end{array} \tag{17}$$

Similarly, the constraint matrix $\mathbf{C}$ (Eq. 9) can be expressed as

$$\mathbf{C} = \left(\begin{array}{c|c} \mathbf{C}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \text{ where } \mathbf{C}_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{18}$$

Finally, we split the vector of coefficients $\mathbf{a}$ (Eq. 3) into

$$\mathbf{a} = \left(\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{array}\right) \text{ where } \mathbf{a}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathbf{a}_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \tag{19}$$

Based on these decompositions, the first condition

```
 1 function a = fit_ellipse(x, y)
 2 D1 = [x .ˆ 2, x .* y, y .ˆ 2];  % quadratic part of the design matrix
 3 D2 = [x, y, ones(size(x))];     % linear part of the design matrix
 4 S1 = D1' * D1;                  % quadratic part of the scatter matrix
 5 S2 = D1' * D2;                  % combined part of the scatter matrix
 6 S3 = D2' * D2;                  % linear part of the scatter matrix
 7 T = - inv(S3) * S2';            % for getting a2 from a1
 8 M = S1 + S2 * T;                % reduced scatter matrix
 9 M = [M(3, :) ./ 2; - M(2, :); M(1, :) ./ 2]; % premultiply by inv(C1)
10 [evec, eval] = eig(M);          % solve eigensystem
11 cond = 4 * evec(1, :) .* evec(3, :) - evec(2, :) .ˆ 2; % evaluate a'Ca
12 a1 = evec(:, find(cond > 0));   % eigenvector for min. pos. eigenvalue
13 a = [a1; T * a1];               % ellipse coefficients
```

Figure 2: MATLAB implementation of the improved ellipse-specific fitting algorithm proposed by the authors

of Eq. 10 can be rewritten as

$$\left(\begin{array}{c|c} \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^T & \mathbf{S}_3 \end{array}\right) \cdot \left(\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{array}\right) = \lambda \cdot \left(\begin{array}{c|c} \mathbf{C}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \cdot \left(\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{array}\right) \quad (20)$$

which is equivalent to the following two equations:

$$\mathbf{S}_1\mathbf{a}_1 + \mathbf{S}_2\mathbf{a}_2 = \lambda \mathbf{C}_1\mathbf{a}_1 \quad (21)$$

$$\mathbf{S}_2^T\mathbf{a}_1 + \mathbf{S}_3\mathbf{a}_2 = \mathbf{0} \quad (22)$$

The matrix $\mathbf{S}_3$,

$$\mathbf{S}_3 = \mathbf{D}_2^T\mathbf{D}_2 = \begin{pmatrix} S_{x^2} & S_{xy} & S_x \\ S_{xy} & S_{y^2} & S_y \\ S_x & S_y & S_1 \end{pmatrix} \quad (23)$$

is exactly a scatter matrix of the task of a fitting a line through a set of data points. It is known [Haral93] that this matrix is singular only if all the points lie on a line. In such situations there is no real solution of the task of a fitting an ellipse through these points. In all other cases the matrix $\mathbf{S}_3$ is regular. Regarding that, $\mathbf{a}_2$ can be expressed from Eq. 22 as

$$\mathbf{a}_2 = -\mathbf{S}_3^{-1}\mathbf{S}_2^T\mathbf{a}_1 \quad (24)$$

Including Eq. 24 into Eq. 21 yields

$$\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)\mathbf{a}_1 = \lambda \mathbf{C}_1\mathbf{a}_1 \quad (25)$$

Matrix $\mathbf{C}_1$ (Eq. 18) is regular, thus Eq. 25 can be rewritten as

$$\mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right)\mathbf{a}_1 = \lambda \mathbf{a}_1 \quad (26)$$

The second condition of Eq. 10 can also be reformulated by using the decomposition principle. Due to the special shape of matrix $\mathbf{C}$ (Eq. 18) we simply get

$$\mathbf{a}_1^T\mathbf{C}_1\mathbf{a}_1 = 1 \quad (27)$$

Regarding all the decomposition steps (Eq. 14–27), the conditions Eq. 10 can be finally expressed as the following set of equations:

$$\begin{aligned} \mathbf{M}\mathbf{a}_1 &= \lambda \mathbf{a}_1 \\ \mathbf{a}_1^T\mathbf{C}_1\mathbf{a}_1 &= 1 \\ \mathbf{a}_2 &= -\mathbf{S}_3^{-1}\mathbf{S}_2^T\mathbf{a}_1 \\ \mathbf{a} &= \left(\begin{array}{c} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{array}\right) \end{aligned} \quad (28)$$

where $\mathbf{M}$ is the *reduced scatter matrix* of the size 3×3,

$$\mathbf{M} = \mathbf{C}_1^{-1}\left(\mathbf{S}_1 - \mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^T\right) \quad (29)$$

Now we can return back to the task of a fitting an ellipse through a set of points. As we saw before, the task can be expressed as the constrained minimization problem (Eq. 7) whose optimal solution corresponds to the eigenvector $\mathbf{a}$ of Eq. 10 which yields a minimal non-negative value λ. Eq. 10 is equivalent with Eq. 28, thus it is enough to find the appropriate eigenvector $\mathbf{a}_1$ of the matrix $\mathbf{M}$.

PRACTICAL REALIZATION OF THE ELLIPSE SPECIFIC FITTING ALGORITHM

The improved fitting algorithm proposed in the previous section was implemented in the same straightforward way as the original Fitzgibbon's method. The appropriate MATLAB code is presented in Fig. 2. In the code, two tricks which improve numerical stability of the computation are used. First, due to the special structure of the matrix $\mathbf{C}_1$ (Eq. 9) the premultiplication by $\mathbf{C}_1^{-1}$ as requested in Eq. 29 is done directly as can be seen at line 9.

The second trick gets rid of the problem of the localization of the minimal non-negative eigenvalue of the reduced scatter matrix $\mathbf{M}$. The matrix has three real eigenvalues, typically two negative and one positive. Unfortunately, we cannot simply find the positive eigenvalue. When $\mathbf{M}$ is close to a singular matrix (i.e.

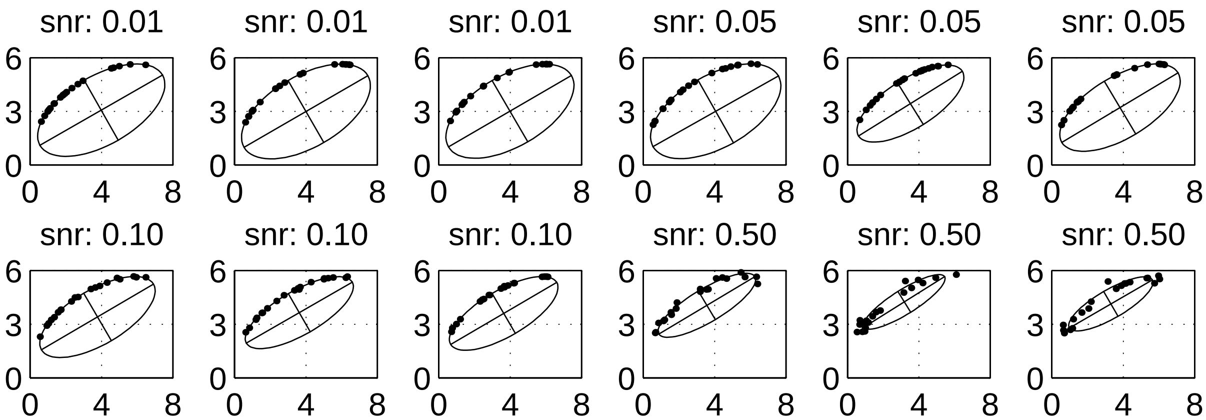


Figure 3: Results of the fitting algorithm on different data sets of 20 points which represent the same elliptical arc of the ellipse with the center $(4,3)$, semiaxes $(4,2)$ and tilt 30 degrees, but with an increasing amount of Gaussian noise added. The appropriate signal-to-noise ratio (snr) is given above each figure.

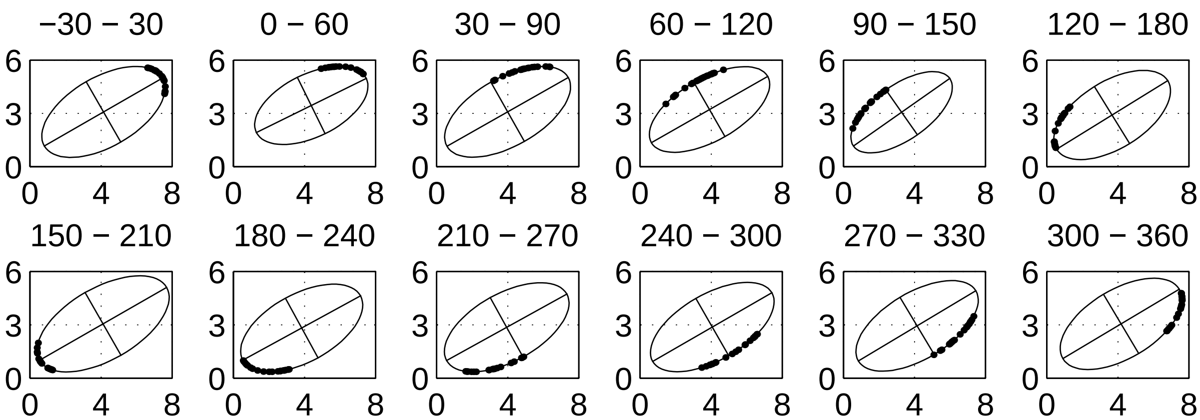


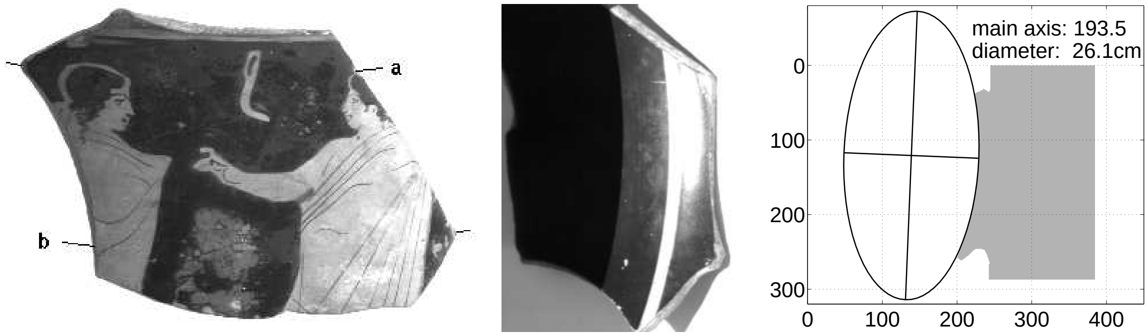
Figure 4: Results of the fitting algorithm for data sets of 20 points which represent different elliptical arcs of the same ellipse with the center $(4,3)$, semiaxes $(4,2)$ and tilt 30 degrees. The data sets are blurred by Gaussian noise with snr 0.025. The appropriate arc section interval (in degrees) is given above each figure.

when the data points lie exactly on an ellipse), the optimal value can be zero or even a small negative number. Instead of the localization of such eigenvalue, we evaluate the condition Eq. 27 for all eigenvectors of the matrix **M** (line 11 in Fig. 2). It can be proven that there exists only one eigenvector which gives a positive value — the one which corresponds to the optimal solution of our fitting problem. That eigenvector is localized at line 12. Finally, at line 13 the rest of the ellipse coefficients are computed and the complete solution is provided.

EXPERIMENTAL RESULTS

The proposed algorithm was evaluated in many experiments. Being an improvement of the original Fitzgibbon's method, our approach preserves its favorable properties such as guaranteed ellipse specific solution, robustness against noise, and invariance of the solution to an affine transformation of the data points. In addition, the proposed method brings numerical stability and removes ambiguity in the localization of the optimal solution.

The properties of the fitting algorithm were verified on synthetic data sets in the same way as was done in [Fitzg96a]. Some results of these experiments are presented in Fig. 3 and Fig. 4. The first experiment (Fig. 3) illustrates the stability of the fitting algorithm against noise. All the data sets were generated by adding an increasing amount of Gaussian noise to synthetic points which represent the same ellipse. These noisy points were fitted by the proposed algorithm with results shown in the figures. You can see that even for higher noise level the fitted conic is an ellipse with parameters conforming to the original elliptical arc. You can also note the tendency of the algorithm to shrink the solution with an increasing amount of noise.



(a) (b) (c)

Figure 5: Diameter estimation of archaeological pottery from fragments: (a) fragment from which the diameter of the original pot should be estimated in positions (a) and (b); (b) the fragment illuminated by the light plane in the position (a); (c) result of the estimation obtained by an ellipse fitting of the detected intersection. See the text for the description of the estimation method.

This bias is caused by the use of the algebraic distance of points (Eq. 1) instead of the geometric one in the minimization function Eq. 5 and cannot be simply corrected [Kanat94].

The second experiment (Fig. 4) depicts the stability of our fitting algorithm with respect to different data sets which represent the same ellipse. Note that the fitted ellipses have the same characteristics even if only a limited number of noisy points (20 in our case) which represent only a small portion of the ellipse (60 degrees) are available.

The proposed fitting algorithm was also tested on real data sets. As an example let us present its application in a method for the diameter estimation of archaeological pottery from fragments. Having only a small part of an original pot, we want to estimate the diameter of the pot in some given positions. Based on an assumption that the original pot is rotationally symmetric, the task can be accomplished by an active vision method as proposed in [Halíř96]. In that method, the fragment is manually oriented in the measurement area and illuminated by a light plane. The intersection of the light plane with a surface of the fragment is observed by a camera. If the fragment is properly oriented, the intersection forms a a circular arc which can be seen as an elliptical arc in the image. Regarding the fact that there exists a linear relation between the diameter of the circle and the length of the main axis of the ellipse, the diameter of the original pot is estimated by fitting an ellipse to the detected intersection and measuring its main axis.

The whole estimation process is depicted in Fig. 5. Figure (a) presents a fragment on which the estimation should be performed in the positions (a) and (b). In figure (b) we can see the fragment illuminated by the light plane in the position (a). Note that the intersection is observed as an elliptical arc. Finally, in figure (c) the intersection is fitted by an ellipse and the diameter of the original pot is estimated from the length of its main axis.

From a theoretical point of view, our algorithm is linear in both time and space complexities. The implementation presented in Fig. 2 requires about 5.7 floating operations per one data point[1] and it can fit 250,000 points in time less than two seconds[2]. The original Fitzgibbon's code (presented in Fig. 1) requires about 7.5 flops per point, thus in addition to the improved numerical stability our code is also reasonably faster.

CONCLUSION

In this paper we propose a numerically stable non-iterative algorithm for a fitting an ellipse to a set of data points. The method is based on a reformulation of the fitting task as an linear optimization problem with a quadratic constraint. Such problem can be solved directly by a standard least squares minimization. This leads to simplicity, stability and robustness of the fitting.

In our approach we started with the ideas proposed by Fitzgibbon *et al.* in [Fitzg96b]. The original method guarantees an ellipse-specific solution, but due to its bad practical realization their algorithm can produce unoptimal or even completely wrong results. Regarding that, we made further theoretical analysis of the problem and found an alternative formulation of the original task based on the block decomposition of matrices. We also proposed a more robust method for

[1] measured by the command `flops` in MATLAB

[2] MATLAB v5.0 on one-processor SPARC Ultra-1 running at 167MHz with 64MB of RAM

the localization of the optimal solution. The new algorithm has no computational ambiguity and it can be implemented in a numerically stable manner.

When compared with another methods for fitting ellipses, our approach has the following advantages: simplicity, stability and robustness. The solution of the fitting is guaranteed to be an ellipse even for a limited number of noisy data points. This feature can play an important role in all applications where a strictly elliptical solution is required. Many other approaches can produce a general conic such as hyperbola or parabola instead of an ellipse, thus an additional check and rejection of non elliptical solutions is required in them. Sometimes these method even cannot produce any elliptical fit. Using our approach, no such problems arise.

The proposed fitting method is direct, with no iterative steps and problems with local minima and numerical stability of the computation. Regarding that, the whole fitting is very fast. On the other hand, due to the use of algebraic distances of points instead of the geometric ones, the solutions are biased towards smaller ellipses. The algebraic distance "prefers" the points lying inside an ellipse, thus the algorithm tends to produce ellipses smaller as they should be. Unfortunately, this bias depends on the parameters of the fitted ellipse and cannot be simply corrected.

Due to its systematic bias, the proposed fitting algorithm cannot be used directly in applications where excellent accuracy of the fitting is required. But even in that applications our method can be useful as a fast and robust estimator of a good initial solution of the fitting problem. The optimal solution is then found by applying some more sophisticated method based on geometrical distances of points. These methods are typically iterative and their behavior depends strongly on the initial estimate. Regarding that, our method can help even in these problems.

REFERENCES

[Books79] Bookstein, F. L.: Fitting conic sections to scattered data. *Computer Graphics and Image Processing*, 9:56–71, 1979.

[Davé92] Davé, R. N. and Bhaswan, K.: Adaptive fuzzy c-shells clustering and detection of ellipses. *IEEE Trans. Neural Networks*, 3:643–662, 1992.

[Duda73] Duda, R. and Hart, P.: *Pattern Classification and Scene Analysis*. Wiley, 1973.

[Fitzg95a] Fitzgibbon, A. W. and Fischer, R. B.: A buyer's guide to conic fitting. In *Proc. of the British Machine Vision Conference*, pages 265–271, Birmingham, 1995.

[Fitzg95b] Fitzgibbon, A. W.: Set of MATLAB files for ellipse fitting. Dept. of Artificial Intelligence, The University of Edingburgh, *ftp://ftp.dai.ed.ac.uk/pub/vision/src/demofit.tar.gz*, September 1995.

[Fitzg96a] Fitzgibbon, A. W., Pilu, M and Fischer, R. B.: Direct least squares fitting of ellipses. Technical Report DAIRP-794, Department of Artificial Intelligence, The University of Edinburgh, January 1996.

[Fitzg96b] Fitzgibbon, A. W., Pilu, M and Fischer, R. B.: Direct least squares fitting of ellipses. In *Proc. of the 13th International Conference on Pattern Recognition*, pages 253–257, Vienna, September 1996.

[Gande81] Gander, W.: Least squares with a quadratic constraint. *Numerische Mathematik*, 36:291–307, 1981.

[Gande94] Gander, W., Golub, G. H. and Strebel R.: Least-square fitting of circles and ellipses. *BIT*, 43:558–578, 1994.

[Gath95] Gath, I and Hoory, D.: Fuzzy clustering of elliptic ring-shaped clusters. *Pattern Recognition Letters*, 16:727–741, 1995.

[Grims90] Grimson, W. E. L. and Huttenlocher, D. P.: On the sensitivity of the Hough transform for object recognition. *IEEE Trans. PAMI*, 12:2555–2574, 1990.

[Halíř96] Halíř, R. and Menard, Ch.: Diameter estimation for archaeological pottery using active vision. In Axel Pinz, editor, *Proc. of the 20th Workshop of the Austrian Association for Pattern Recognition (ÖAGM'96)*, pages 251–261, Schloss Seggau, Leibnitz, May 1996.

[Haral93] Haralick, R. M. and Shapiro, L. G.: *Computer and Robot Vision*, volume 1. Addison-Wesley, 1993.

[Kanat94] Kanatani, K.: Statistical bias of conic fitting and renormalization. *IEEE Trans. PAMI*, 16(3):320–326, 1994.

[Leave92] Leavers, V. F.: *Shape Detection in Computer Vision Using the Hough Transform*. Springer-Verlag, 1992.

[Porri90] Porrill, J.: Fitting ellipses and predicting confidence envelopes using a bias corrected Kalman filter. *Image Vision and Computing*, 8(1):1140–1153, February 1990.

[Rosin93] Rosin, P. L.: Ellipse fitting by accumulating five-point fits. *Pattern Recognition Letters*, 14:661–699, August 1993.

[Rosin95] Rosin, P. L. and West, G. A. W.: Non-parametric segmentation of curves into various representations. *IEEE Trans. PAMI*, 17:1140–1153, 1995.

[Samps92] Sampson, P. D.: Fitting conic sections to very scattered data: An iterative refinement of the bookstein algorithm. *Computer Graphics and Image Processing*, 18:97–108, 1992.

[Taubi91] Taubin, G.: Estimation of planar curves, surfaces and non-planar space curves defined by implicit equations with applications to edge and range image segmentation. *IEEE Trans. PAMI*, 13(11):1115–1138, November 1991.

[Mathw] The Mathworks Inc.: MATLAB: system for numerical computation and visualization. *http://www.mathworks.com*.

[Werma95] Werman, M. and Geyzel, G.: Fitting a second degree curve in the presence of error. *IEEE Trans. PAMI*, 17(2):207–211, 1995.

[Wu93] Wu, W. Y. and Wang, M. J. J.: Elliptical object detection by using its geometric properties. *Pattern Recognition*, 26:1499–1509, 1993.

[Yuen89] Yuen, H. K., Illingworth, J. and Kittler J.: Detecting partially occluded ellipses using the Hough transform. *Image Vision and Computing*, 7(1):31–37, 1989.

[Yin92] Yin, R. K. K., Tam, P. K. S. and Leung, N. K.: Modification of Hough transform for circles and ellipses detection using 2-D array. *Pattern Recognition*, 25:1007–1022, 1992.