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Hilbert Schemes, Separated Variables, and D-Branes

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We explain Sklyanin's separation of variables in geometrical terms and construct it for Hitchin and Mukai integrable systems. We construct Hilbert schemes of points on $T^*\Sigma$ for $\Sigma = \mathbf{C}, \mathbf{C}^*$ or elliptic curve in some detail and show that their complex deformations are integrable systems of Calogero-Sutherland type. We present the hyperkahler quotient constructions for Hilbert schemes of points on cotangent bundles to the higher genus curves, utilizing the results of Hurtubise, Kronheimer and Nakajima. Finally we discuss the connections to physics of D-branes and string duality.

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1. Introduction

One way of solving a complicated problem with many degrees of freedom is to reduce it to the problem with smaller number degrees of freedom. The solvable models allow to reduce the original system with N degrees of freedom to N systems with 1 degree of freedom which can be solved in quadratures. This approach is called a separation of variables (**SoV**). Recently, E. Sklyanin came up with a "magic recipe" for the **SoV** in the large class of integrable models with a Lax representation [1][2]. The basic strategy of his method is to look at the Lax eigen-vector (also called Baker-Akhiezer function) $\Psi(z,\lambda)$:

$$L(z)\Psi(z,\lambda) = \lambda(z)\Psi(z,\lambda) \tag{1.1}$$

with some choice of normalization (this is the artistic part of the method). The poles z_i of $\Psi(z,\lambda)$ together with the eigenvalues $\lambda_i = \lambda(z_i)$ are the separated variables. In all the examples studied so far the most naive way of normalization leads to the canonically conjugate coordinates λ_i, z_i.

The purpose of this paper is to explain the geometry behind the "magic recipe" in a broad class of examples, which include Hitchin systems [3], their deformations [4] and many-body systems considered as their degenerations [5][6]. We shall use the results of [7],[8],[9]. For a complex surface X let $X^{[h]}$ denote the Hilbert scheme of points on X of length h (if X is compact hyperkahler then so is $X^{[h]}$ [10]).

2. Hitchin systems

Hitchin systems can be thought of as a generalized many-body system. In fact, elliptic Calogero-Moser model as well as its various spin and some relativistic generalizations can be thought as of a particular degeneration of Hitchin system [5],[11],[6].

2.1. The integrable system

Recall the general Hitchin's setup [3]. One starts with the compact algebraic curve Σ of genus higher then one and a topologically trivial vector bundle V over it. Let $G = SL_N(\mathbf{C})$, $\underline{\mathbf{g}} = \mathrm{Lie}G$. The Hitchin system is the integrable system on the moduli space $\mathcal{N}$ of stable Higgs bundles. The point of $\mathcal{N}$ is the gauge equivalence class of a pair (an operator $\bar\partial_A = \bar\partial + \bar A$, a holomorphic section ϕ of $\mathrm{ad}(V)\otimes\omega^1_\Sigma$). The holomorphic structure on V is defined with the help of $\bar{\partial}_A$. The symplectic structure on $\mathcal{N}$ descends from the two-form $\int \mathrm{Tr}\delta\phi \wedge \delta\bar{A}$. The integrals of motion are the Hitchin's hamiltonians:

$$\mathrm{Tr}\phi^k \in \mathrm{H}^0(\Sigma, \omega^k) \approx \mathbf{C}^{(2k-1)(g-1)}, \quad k > 1$$

Their total number is equal to:

$$\sum_{k=2}^{N-1}(2k-1)(g-1) = (N^2-1)(g-1) = \frac{1}{2}\mathrm{dim}_{\mathbf{C}}\mathcal{N}$$

Thus, $\mathcal{N}$ can be represented as fibration over

$$\mathbf{B} = \bigoplus_{k=2}^{N-1} \mathrm{H}^0(\Sigma, \omega^k)$$

with the fibre over a generic point $b \in \mathbf{B}$ being an abelian variety E_b. This variety is identified by Hitchin with the quotient of the Jacobian variety J_b of the spectral curve C_b, defined as the divisor of zeroes of

$$R(z, \lambda) = \mathrm{Det}(\phi(z) - \lambda) \in \mathrm{H}^0(\Sigma, \omega^N)$$

in $T^*\Sigma$. The curve C_b has genus $N^2(g-1)+1$ which is by g higher then the dimension of E_b. In fact, $E_b = J_b/\mathrm{Jac}(\Sigma)$. The Jacobian of Σ is imbedded into the Jacobian of C since the spectral curve is a covers Σ and therefore the holomorphic 1-differentials on Σ are pulled back to C_b. An open dense subset of $\mathcal{N}$ is isomorphic to $T^*\mathcal{M}$, the cotangent bundle to the moduli space of holomorphic stable G-bundles on Σ.

The space $\mathcal{N}$ is a non-compact integrable system. One can compactify it by replacing $T^*\Sigma$ by a **K3** surface. This is a natural deformation of the original system in the sense that the infinitesimal neibourghood of Σ imbedded into **K3** is isomorphic to $T^*\Sigma$ (since $c_1(\mathbf{K3}) = 0$). Instead of studying the moduli space of the gauge fields on Σ together with the Higgs fields ϕ one studies the moduli of torsion free sheaves, supported on Σ [4].

It turns out that this model is important in the studies of the bound states of $D2$ branes in Type IIA string theory compactified on **K3**, which wrap a given holomorphic curve Σ. One can think of the bound state as of the vacuum in the gauge theory on Σ according to [12], [13]. It can also be represented classically by a smooth curve of the genus $h = N^2(g-1)+1$, which is an N-fold cover of Σ.

In the compactified case, the curve C (which is nothing but our fellow spectral curve C_b) is imbedded into $\mathbf{K3}$. It is also endowed with the line bundle $\mathcal{L}$ (from the point of view of Hitchin equations the bundle is simply the eigen-bundle of ϕ; from the D-brane point of view - the single D-brane carries a $U(1)$ gauge field on it and the vacuum configuration corresponds to a flat connection) which determines a point on its Jacobian J_b.

Take the generic section of this line bundle. It has h zeroes $p_1, \ldots, p_h$. Conversely, given a set S of h points in $\mathbf{K3}$ there is generically a unique curve C in a given homology class $\beta \in \mathrm{H}_2(\mathbf{K3}, \mathbf{Z})$ of genus h with a line bundle on it $\mathcal{L}$, such that the curve passes through these points and the divisor of $\mathcal{L}$ coincides with S.

This allows one to identify the open dense subset of the moduli space of pairs (a curve C_b; a line bundle $\mathcal{L}$ on it) with that in the symmetric power $\mathrm{Sym}^h(\mathbf{K3})$ of the $\mathbf{K3}$ surface itself. The symplectic form on the moduli space is therefore the direct sum of h copies of the symplectic forms on $\mathbf{K3}$.

Summarizing, the phase space of integrable system looks locally as $T^*\mathcal{M}$, where $\mathcal{M}$ is the moduli space of rank N vector bundles over Σ, where Σ is holomorphically imbedded in $\mathbf{K3}$. It can also be identified with the moduli space of pairs (a curve C_b; a line bundle $\mathcal{L}$ on it) where the homology class of C_b is N times that of Σ, and the topology $\mathcal{L}$ is fixed. This identification provides the action-angle coordinates on the phase space. Namely, the angle coordinates are (following [3]) the linear coordinates on the Jacobian of C_b, while the action coordinates are the periods of $\mathrm{d}^{-1}\omega$ along the A-cycles on C_b. The last identification of the phase space with the symmetric power of $\mathbf{K3}$ provides the $\mathbf{SoV}$ in the sense of Sklyanin. Notice the similarity of our description to his "magic recipe".

2.2. Separation of variables in Hitchin system

Let us present an explicit realization of the separation of variables in the Hitchin system. Let Σ be a compact smooth genus g algebraic curve and p be the projection $p : T^*\Sigma \to \Sigma$. Let V be complex vector bundle over Σ of rank N and degree k. Consider the moduli space $\mathcal{M}_{N,k}$ of the semi-stable holomorphic structures E on V. It can be identified with the quotient of the open subset of the space of $\bar{\partial}_A$ operators acting on the sections of V by the action of the gauge group $\bar{\partial}_A \to g^{-1}\bar{\partial}_A g$. The complex dimension of $\mathcal{M}_{N,k}$ is given by the Riemann-Roch formula

$$\dim_{\mathbf{C}}\mathcal{M}_{N,k} = N^2(g-1) + 1 := h \tag{2.1}$$

Explicitly, the points in $T^*\mathcal{M}_{N,k}$ are the equivalence classes of pairs (E, Φ) where E is the holomorphic bundle on Σ and Φ is the section of $\mathrm{End}(E) \otimes \omega_\Sigma$. The map π is given by the formula:

$$\pi(E, \Phi) = \{\mathrm{Tr}\Phi, \mathrm{Tr}\Phi^2, \ldots, \mathrm{Tr}\Phi^N\} \tag{2.2}$$

Let $\mathcal{H}$ denote the h-dimensional vector space $\mathbf{C}^h$:

$$\mathcal{H} = \oplus_{l=1}^N \mathrm{H}^0(\omega_\Sigma^{\otimes l})$$

N. Hitchin shows that the partial compactification of the cotangent bundle $T^*\mathcal{M}_{N,k}$ is the algebraiccaly integrable system, i.e. there exists a holomorphic map

$$\pi : T^*\mathcal{M}_{N,k} \to \mathcal{H}$$

whose fibers are abelian varieties which are Lagrangian with respect to the canonical symplectic structure on $T^*\mathcal{M}_{N,k}$. The generic fiber is compact. Geometric separation of variables in Hitchin system is the content of the following:

Theorem. There exists a birational map

$$\varphi : T^*\mathcal{M}_{N,k} \to (T^*\Sigma)^{[h]},$$

which is a symplectomorphism of the open dense subsets.

Remark. The open dense set in $X^{[h]}$ coincides with $(X^h - \Delta)/\mathcal{S}_h$ where Δ denotes the union of all diagonals and $\mathcal{S}_h$ is the symmetric group. The theorem implies that on this dense set one can introduce the coordinates $\{(z_i, \lambda_i)\}$, where $z_i \in \Sigma$, $\lambda_i \in T^*_{z_i}\Sigma$, such that the symplectic form in these coordinates have the separated form:

$$\Omega = \sum_{i=1}^h \delta\lambda_i \wedge \delta z_i \tag{2.3}$$

The coordinates (λ_i, z_i) are defined up to permutations.

Proof. First we construct φ and prove that this map is biholomorphic. In order to do that we need t o choose special k. As the moduli spaces $\mathcal{M}_{N,k}$ are isomorphic for different k (although not canonically!) this is not a problem. Fix the pair (E, Φ). Consider the spectral curve $C \subset T^*\Sigma$ defined as the zero set of the characteristic polynomial

$$P(\lambda, z) = \mathrm{Det}(\Phi(z) - \lambda) \in \Gamma(\omega_\Sigma^{\otimes N}) \tag{2.4}$$

Its genus equals h as can be seen from the adjunction formula or from Riemann-Hurwitz formula. The pullback p^*E restricted to C contains the line sub-bundle L of the eigen-lines of Φ. Our choice of k is such that the degree of L equals h. Generically L has unique up to a multiple non-vanishing section s. Its zeroes $l_1, \dots, l_h$ determine h points on C and therefore on $T^*\Sigma$. Clearly they are determined uniquely up to permutations. This allows us to define:

$$\varphi(E, \Phi) = (l_1, \dots, l_h) \tag{2.5}$$

Now let us prove that the image of φ coincides with $(T^*\Sigma)^{[h]}$. We do it at the level of the open dense sets. Fix the set of points $(l_1 = (\lambda_1, z_1), \dots, l_h = (\lambda_h, z_h))$ in $T^*\Sigma$. Let $\omega_\alpha^{(j)}$ denote a basis in the space of holomorphic j-differentials on Σ. For given j the index α runs from 1 to $(2j-1)(g-1)$ for $j \geq 2$, to g for $j = 1$ and to 1 for $j = 0$. Consider the space $\mathcal{V}$ of sections of $p^*\omega_\Sigma^{\otimes N}$. The vectors of $\mathcal{V}$ are:

$$P(\lambda, z) = \sum_{j=0}^{N} \sum_{\alpha} V_{j,\alpha} \omega_\alpha^{(j)}(z) \lambda^{N-j} \tag{2.6}$$

The dimension of $\mathcal{V}$ equals $h+1$. Consider the subspace of P's such that $P(\lambda_i, z_i) = 0$ for any $i = 1, \dots, h$. It is one-dimensional if the points $l_1, \dots, l_h$ are distinct. In that case choose any P in this one dimensional subspace and consider the curve C defined by the equation $P(\lambda, z) = 0$ through the points $l_1, \dots, l_h$. They form a divisor of the unique line bundle L on C of degree h which can be considered as a point of $(T^*\Sigma)^{[h]}$. Then we can consider a direct image p_*L which is a vector bundle over Σ and a Higgs form $\Phi : p_*L \to p_*L \otimes \omega_\Sigma$ which is nothing but the multiplication by the elements $\lambda \in C$. Hence we obtain desired one-to-one correspondence at least at the level of an open dense sets.

Now let us prove the equality of symplectic forms on $T^*\mathcal{M}_{N,k}$ and on $(T^*\Sigma)^{[h]}$. We show it at the generic point of $(T^*\Sigma)^{[h]}$ corresponding to the set of distinct points $(l_1, \dots, l_h)$. To this end recall the construction of the symplectic form on $T^*\mathcal{M}_{N,k}$ in terms of the data (C, L): Let A_a be a choice of the basis of A-cycles on C. The Jacobian $\mathrm{Jac}(C) = \mathrm{H}^{(0,1)}(C, \mathbf{C})/\mathrm{H}^1(C, \mathbf{Z})$ has the local linear coordinates φ_b associated to A_a. They are normalized in such a way that:

$$\int_{A_a} d\varphi_b = \delta_b^a \tag{2.7}$$

where $d\varphi_b$ is represented as a $(1,0)$-form on C. Introduce the coordinates I^a:

$$I^a = \int_{A_a} \lambda dz. \tag{2.8}$$

Then:

$$\omega = \sum_{a=1}^{h} \delta I^a \wedge \delta\varphi_a \tag{2.9}$$

To make contact with (2.3) we recall the Abel map: Let Ω_a be the basis in the space of holomorphic 1-differentials on C which obeys:

$$\int_{A_a} \Omega_b = \delta_b^a. \tag{2.10}$$

Then

$$\varphi_a = \sum_{i=1}^{h} \int_{l_*}^{l_i} \Omega_a$$

Notice that the normal bundle $N_{T^*\Sigma|C}$ to C is isomorphic to T^*C. The deformed curve $\tilde{C}$ can be identified with the holomorphic section $p(x)dx$ of T^*C. It can be expanded as follows:

$$p(x)dx = \sum_{a=1}^{h} p^a \Omega_a, \quad p^a \in \mathbf{C}$$

Using (2.10) we get:

$$p^a = \oint_{A_a} p(x)dx = I^a \tag{2.11}$$

Let (p_i, x_i), $i = 1, \ldots, h$ be a set of distinct points in T^*C. Let $C_{a,i} = \Omega_a(x_i) \in T^*_{x_i}C$.

Lemma.

$$\delta\varphi_a = \sum_{i=1}^{h} C_{a,i}\delta x_i \tag{2.12}$$

Proof of the Lemma.

$$\delta\varphi_a = \sum_{i=1}^{h} \delta \int_{l_*}^{(p_i,x_i)} \Omega_a = \sum_{i=1}^{h} \Omega_a(x_i)\delta x_i$$

Thus,

$$\sum_{i=1}^{h} \delta p_i \wedge \delta x_i = \sum_{a=1}^{h}\sum_{i=1}^{h} \delta p^a C_{a,i}\delta x_i = \sum_{a=1}^{h} \delta I^a \wedge \delta\varphi_a$$

The theorem is proven.

Remark. J. Hurtubise in [7] gave the pure algebro-geometric proof of the main part of the theorem. The basic motivation of our remarks is that some of our arguments looks more direct and more in the spirit of the approach of integrable systems.

3. Gaudin model

Consider the space

$$(\mathcal{O}_1 \times \ldots \times \mathcal{O}_k)//G \tag{3.1}$$

where $\mathcal{O}_l$ are the complex coadjoint orbits of $G = \mathrm{SL}_N(\mathbf{C})$ and the symplectic quotient is taken with respect to the diagonal action of G.

This moduli space parameterizes Higgs pairs on $\mathbf{P}^1$ with singularities at the marked points $z_i \in \mathbf{P}^1, \quad i = 1, \ldots, k$. This is a natural analogue of the Hitchin space for genus zero. Concretely, the connection to the bundles on $\mathbf{P}^1$ comes about as follows: consider the moduli space of Higgs pairs: $(\bar{\partial}_A, \phi)$ where ϕ is a *meromorphic* section of $\mathrm{ad}(V) \otimes \mathcal{O}(-2)$, with the restriction that $\mathrm{res}_{z=z_i}\phi \in \mathcal{O}_i$. The moduli space is isomorphic to (3.1). This space is integrable system, studied in [14][15]. Indeed, consider the solution to the equation

$$\bar{\partial}_A \phi = \sum_i \mu_i^c \delta^{(2)}(z - z_i) \tag{3.2}$$

in the gauge where $\bar{A} = 0$ (such gauge exists for stable bundles on $\mathbf{P}^1$ due to Grothendieck's theorem). We get:

$$\phi(z) = \sum_i \frac{\mu_i^c}{z - z_i} \tag{3.3}$$

provided that $\sum_i \mu_i^c = 0$ and is defined up to a global conjugation by an element of G hence the Hamiltonian reduction in (3.1). Now, consider the following polynomial:

$$\mathrm{Det}\,(\lambda - \phi(z)) = \sum_{i,l} A_{i,l} \lambda^i z^{-l} \tag{3.4}$$

It is an easy count to check that the number of functionally independent coefficients $A_{i,l}$ is precisely equal to

$$k\left(\frac{N(N-1)}{2}\right) + 1 - N^2$$

Now let us treat explicitly the case $N = 2$. In this case the coadjoint orbits $\mathcal{O}_i$ can be explicitly described as the surfaces in $\mathbf{C}^3$ given by the equations:

$$\mathcal{O}_i : Z_i^2 + X_i^+ X_i^- = \zeta_i^2 \tag{3.5}$$

with the symplectic forms:

$$\omega_i = \frac{dZ_i \wedge dX_i^+}{X_i^+} \tag{3.6}$$

and the complex moment maps:

$$\mu_i^c = \begin{pmatrix} Z_i & X_i^+ \\ X_i^- & -Z_i \end{pmatrix} \tag{3.7}$$

The phase space of our interest is $\mathcal{P} = \times_{i=1}^k \mathcal{O}_i//SL_2$. It is convenient to work with a somewhat larger space $\mathcal{P}_0 = \times_{i=1}^k \mathcal{O}_i/\mathbf{C}^*$, where $\mathbf{C}^* \in SL_2(\mathbf{C})$ acts as follows:

$$t : (Z_i, X_i^+, X_i^-) \mapsto (Z_i, tX_i^+, t^{-1}X_i^-)$$

The moment map of the torus $\mathbf{C}^*$ action is simply $\sum_i Z_i$. The complex dimension of $\mathcal{P}_0$ is equal to $2(k-1)$. The Hamiltonians are obtained by expanding the quadratic invariant:

$$\begin{aligned} T(z) &= \tfrac{1}{2}\mathrm{Tr}\phi(z)^2, \qquad \phi(z) = \sum_i \frac{\mu_i^c}{z - z_i} \\ T(z) &= \sum_i \frac{\zeta_i^2}{(z-z_i)^2} + \sum_i \frac{H_i}{z - z_i} \\ H_i &= \tfrac{1}{2}\sum_{j\neq i} \frac{X_i^+X_j^- + X_i^-X_j^+ + 2Z_iZ_j}{z_i - z_j} \end{aligned} \tag{3.8}$$

The separation of variables proceeds in this case as follows: write $\phi(z)$ as

$$\phi(z) = \begin{pmatrix} h(z) & f(z) \\ e(z) & -h(z) \end{pmatrix}.$$

Then Baker-Akhiezer function is given explicitly by:

$$\Psi(z) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi_+ = f, \quad \psi_- = \sqrt{h^2 + ef} - h \tag{3.9}$$

and its zeroes are the roots of the equation

$$f(p_l) = 0 \Leftrightarrow \sum_i \frac{X_i^+}{p_l - z_i} = 0, \quad l = 1, \ldots, k-1 \tag{3.10}$$

The eigen-value $\lambda(p)$ of the Lax operator ϕ at the point p is most easily computed using the fact that $T(z) = \lambda(z)^2$. Hence, $\lambda_l = \sum_i \frac{Z_i}{p_l - z_i}$,

$$\begin{aligned} X_i^+ &= u\frac{P(z_i)}{Q'(z_i)}, \quad u \in \mathbf{C} \\ Z_i &= \sum_l \frac{\lambda_l}{z_i - p_l}\frac{Q(p_l)P(z_i)}{Q'(z_i)P'(p_l)} \\ P(z) &= \prod_{l=1}^{k-1}(z - p_l) \\ Q(z) &= \prod_{i=1}^{k}(z - z_i) \end{aligned} \tag{3.11}$$

The value of u can be set to 1 by the $\mathbf{C}^*$ transformation. The (λ_l, p_l)'s are the gauge invariant coordinates on $\mathcal{P}_0$. They are defined up to a permutation. It is easy to check that the restriction of the symplectic form $\sum \omega_i$ onto the set $\sum_i Z_i = 0$ is the pullback of the form

$$\sum_{l=1}^{k-1} d\lambda_l \wedge dp_l. \tag{3.12}$$

In the last section we present the quantum analogue of this separation of variables.

4. Many-Body Systems: Rational and Trigonometric Cases

In this section we study the Hilbert scheme of points on $S = \mathbf{C}^2$, $S = \mathbf{C}^2/\Gamma$ for $\Gamma \approx \mathbf{Z}_N, \mathbf{Z}$. We show that $S^{[v]}$ has a complex deformation $S^{[v]}_\zeta$ and that each $S^{[v]}_\zeta$ is an integrable model including the complexification of Sutherland model [16].

4.1. Points on $\mathbf{C} \times \mathbf{C}$

Let us start with $\mathbf{C}^2$. As is well-known [17] the Hilbert scheme of points on $\mathbf{C}^2$ has ADHM-like description: it is the set of stable triples (B_1, B_2, I), $I \in V \approx \mathbf{C}^v, B_1, B_2 \in \mathrm{End}(V)$, $[B_1, B_2] = 0$ modulo the action of $\mathrm{GL}(V)$: $(B_1, B_2, I) \sim (gB_1g^{-1}, gB_2g^{-1}, gI)$ for $g \in \mathrm{GL}(V)$. Stability means that by acting on the vector I by arbitrary polynomials in B_1, B_2 one can generate the whole of V.

The meaning of the vector I and the operators B_1, B_2 is the following. Let z_1, z_2 be the coordinates on $\mathbf{C}^2$. Let Z be a zero-dimensional subscheme of $\mathbf{C}^2$ of length v. It means that the space $\mathrm{H}^0(\mathcal{O}_Z)$ of functions on Z which are the restrictions of holomorphic functions on $\mathbf{C}^2$ has dimension v. Let V be this space of functions. Then it has the canonical vector I which is the constant function $f = 1$ restricted to Z and the natural action of two commuting operators: multiplication by z_1 and by z_2, which are represented by the operators B_1 and B_2. Conversely, given a stable triple (B_1, B_2, I) the scheme Z, or, rather the corresponding ideal $\mathcal{I}_Z \subset \mathbf{C}[z_1, z_2]$ is reconstructed as follows: $f \in \mathcal{I}_Z$ iff $f(B_1, B_2)I = 0$.

Now let us discuss the notion of stability. In the Geometric Invariants Theory (GIT) the notion of a stable triple (B_1, B_2, I) would be the following: *there exists a holomorphic function* ψ *on the space of all triples* $(\tilde{B}_1, \tilde{B}_2, \tilde{I})$ *such that:*

1. *For any* $g \in \mathrm{GL}(V)$, $\quad \psi(g\tilde{B}_1g^{-1}, g\tilde{B}_2g^{-1}, g\tilde{I}) = \det(g)\psi(\tilde{B}_1, \tilde{B}_2, \tilde{I})$
2. $\psi(B_1, B_2, I) \neq 0.$

Let us show the equivalence of the two definitions of stability. Choose any v-tuple $\vec{f}$ of polynomials $f_1, \ldots, f_v \in \mathbf{C}[z_1, z_2]$. Choose any non-zero element $\omega \in (\Lambda^v \mathbf{C}^v)^*$. Define a function

$$\tau_{\vec{f}}(\tilde{B}_1, \tilde{B}_2, \tilde{I}) = \omega \left(f_1(\tilde{B}_1, \tilde{B}_2)\tilde{I} \wedge \ldots \wedge f_v(\tilde{B}_1, \tilde{B}_2)\tilde{I} \right) \tag{4.1}$$

Clearly it obeys the property **1**. If the triple (B_1, B_2, I) is stable in the sense of the first definition then there exist a v-tuple $\vec{f}$ for which the vectors $f_1(B_1, B_2)I, \ldots, f_v(B_1, B_2)I$ form a basis in $\mathbf{C}^v$ and therefore $\tau_{\vec{f}}(B_1, B_2, I) \neq 0$. Conversely, if $\tau_{\vec{f}}(B_1, B_2, I) = 0$ for any $\vec{f}$ then the span S of $\{\mathbf{C}[B_1, B_2]I\}$ is strictly less then V. On the other hand S is an invariant subspace.

Now let us discuss another aspect of the space $\left(\mathbf{C}^2\right)^{[v]}$. It is symplectic manifold. To see this let us start with the space of quadruples, (B_1, B_2, I, J) with B_1, B_2, I as above and $J \in V^*$. It is a symplectic manifold with the symplectic form

$$\Omega = \mathrm{Tr}\left[\delta B_1 \wedge \delta B_2 + \delta I \wedge \delta J\right] \tag{4.2}$$

which is invariant under the naive action of $G = \mathrm{GL}(V)$. The moment map for this action is

$$\mu = [B_1, B_2] + IJ \in \mathrm{Lie}G. \tag{4.3}$$

Let us perform the Hamiltonian reduction, that is:
take the zero level set of μ, choose a subset of stable points in the sense of GIT and take the quotient of this subset with respect to G. One can show [17] that the stability implies that $J = 0$ and therefore the moment equation reduces to the familiar $[B_1, B_2] = 0$.

Moreover, $\left(\mathbf{C}^2\right)^{[v]}$ is an integrable system. Indeed, the functions $\mathrm{Tr}B_1^l$ Poisson-commute and are functionally independent for $l = 1, \ldots, v$.

It turns out that $\left(\mathbf{C}^2\right)^{[v]}$ has an interesting complex deformation which preserves its symplecticity and integrability. Namely, instead of $\mu^{-1}(0)$ in the reduction one should take $\mu^{-1}(\zeta \cdot \mathrm{Id})$ for some $\zeta \in \mathbf{C}$. Now $J \neq 0$. The resulting quotient $S_\zeta^{[v]}$ no longer parametrizes subschemes in $\mathbf{C}^2$ but rather sheaves on a non-commutative $\mathbf{C}^2$, that is the "space" where functions are polynomials in z_1, z_2 with the commutation relation $[z_1, z_2] = \zeta$ (see [18] for more details). Nevertheless, the quotient itself is a perfectly well-defined symplectic manifold with an integrable system on it: the functions $H_l = \mathrm{Tr}B_1^l$ still Poisson-commute and are functionally independent for $l = 1, \ldots, v$. On the dense open subset of $S_\zeta^{[v]}$ where

B_2 can be diagonalized: $B_2 = \mathrm{diag}\,(q_1, \ldots, q_v)$ the Hamiltonians H_1, H_2 can be written as follows:

$$H_1 = \sum_i p_i, \quad H_2 = \sum_i p_i^2 + \sum_{i<j} \frac{\zeta^2}{(q_i - q_j)^2} \tag{4.4}$$

where $p_i = (B_1)_{ii}$. These Hamiltonians describe a collection of indistinguishable particles on a (complex) line with a pair-wise potential interaction $\frac{1}{x^2}$. This system is called rational Calogero model [19]. It is shown in [9] that the space $S_\zeta^{[v]}$ can be used for compactifying the Calogero flows in the complex case and moreover that the same compactification is natural in the KdV/KP realization of Calogero flows [20][21].

4.2. Rounding off to $\mathbf{C} \times \mathbf{C}^*$ *and to* $\mathbf{C}^* \times \mathbf{C}^*$

Now let z_1, z_2 be the coordinates on $\mathbf{C} \times \mathbf{C}^*$, i.e. $z_2 \neq 0$. Then the description of the previous section is still valid except that B_2 must be invertible now. So in this case the Hilbert scheme of points is obtained by a complex Hamiltonian reduction from the space $T^*(G \times V)$. We skip all the details as they are well-known by now (the complex analogue of the real reduction studied in [22][23] can be found for example in [5]). The moment map in our notations will be:

$$\mu = B_2^{-1} B_1 B_2 - B_1 + IJ \tag{4.5}$$

which corresponds to the symplectic form:

$$\Omega = \delta \mathrm{Tr}\left[B_1 B_2^{-1} \delta B_2 + I \delta J\right] \tag{4.6}$$

The reduction at the non-zero level $\mu = \zeta \cdot \mathrm{Id}$ leads to the the complex analogue of either Sutherland [16] or rational Ruijsenaars model [24][25]. In the former case $H_2 = \mathrm{Tr} B_1^2$ while in the latter $H_{rel} = \mathrm{Tr}\left(B_2 + B_2^{-1}\right)$. On the open dense subset where B_2 diagonalizable: $B_2 = \mathrm{diag}\,(\exp(2\pi i q_1), \ldots, \exp(2\pi i q_v))$ the Hamiltonian H_2 equals:

$$H_2 = \sum_i p_i^2 + \sum_{i<j} \frac{\zeta^2}{\sin^2(\pi q_i - q_j)} \tag{4.7}$$

The expression for H_{rel} can be found in [5].

Finally, if both B_1 and B_2 are invertible then we get the Hilbert scheme of points on $\mathbf{C}^* \times \mathbf{C}^*$. Its complex deformation is a bit more tricky, though. It turns out that it can be obtained via *Poisson reduction* of $G \times G \times V \times V$. The integrable system one gets in this case is the trigonometric case of Ruijsenaars model [26].

Slightly generalizing the results of [27][28][29] one may easily present the finite-dimensional symplectic quotient construction of the Hilbert scheme of points on $T^*\mathbf{P}^1$. Here it is: Take $\mathcal{V} = \mathbf{C}^v$, $A = T^*\mathrm{Hom}(\mathcal{V},\mathcal{V})$, $\tilde{A} = A \oplus A$ and $X = \tilde{A} \oplus T^*(\mathrm{Hom}(\mathcal{V},\mathbf{C}))$. The space X is acted on by the group $\mathcal{G} = GL(V) \times GL(V)$. The maximal compact subgroup $\mathcal{U}$ of $\mathcal{G}$ preserves the hyperkahler structure of X. The hyperkahler quotient of X with respect to $\mathcal{U}$ is the Hilbert scheme of points on $T^*\mathbf{P}^1$ of length v.

This space is an integrable system. We shall prove it in a slightly more general setting. Namely, let S be the deformation of the orbifold $\mathbf{C}^2/\mathbf{Z}_k$, where the generator $\omega = e^{\frac{2\pi i}{k}}$ of $\mathbf{Z}_k$ acts as follows: $(z_0, z_1) \mapsto (\omega z_0, \omega^{-1} z_1)$.

Theorem. $S^{[v]}$ is a holomorphic integrable system.

Proof.

The space $S^{[v]}$ can be described as a hyperkahler quotient. Let us take $k+1$ copy of the space $\mathbf{C}^v$, and denote the i'th vector space as V_i, $i = 0, \ldots, k$. Let us consider the space

$$X = \bigoplus_{i=0}^{k} \mathrm{Hom}(V_i, V_{i+1}) \oplus \mathrm{Hom}(V_{i+1}, V_i) \bigoplus \mathrm{Hom}(W, V_0) \oplus \mathrm{Hom}(V_0, W),$$

where $k+1 \equiv 0$. The space X has a natural hyperkahler structure, in particular it has a holomorphic symplectic form:

$$\omega = \mathrm{Tr}\delta I \wedge \delta J + \sum_{i=0}^{k} \mathrm{Tr}\delta B_{i,i+1} \wedge \delta B_{i+1,i} \tag{4.8}$$

where $B_{i,j} \in \mathrm{Hom}(V_j, V_i)$, $I \in \mathrm{Hom}(W, V_0)$, $J \in \mathrm{Hom}(V_0, W)$. The space X has a natural symmetry group $G = \prod_{i=0}^{k} U(V_i)$ which acts on X as:

$$B_{i,j} \mapsto g_i B_{i,j} g_j^{-1}, \quad I \mapsto g_0 I, \quad J \mapsto J g_0^{-1}$$

The action of the group G preserves the hyperkahler structure of X. The complex moment map has the form:

$$\mu_i = B_{i,i+1}B_{i+1,i} - B_{i,i-1}B_{i-1,i} + \delta_{i,0} IJ \tag{4.9}$$

The space $S^{[v]}$ is defined as a (projective) quotient of $\mu^{-1}(0)$ by the action of the complexified group G, which we denote as G_c. There is a deformation $S^{[v]}_{\zeta}$ which depends on k complex parameters $\zeta_0, \ldots, \zeta_k$, $\sum_i \zeta_i = 0$ defined as

$$S^{[v]}_{\zeta} = \cap_i \mu_i^{-1}(\zeta_i \mathrm{Id})/G_c \tag{4.10}$$

Now we present the complete set of Poisson-commuting functions on $S^{[v]}$: define the "monodromy":

$$\mathbf{B}_0 = B_{0,1}B_{1,2}\dots B_{k,0} \tag{4.11}$$

which transforms under the action of G in the adjoint representation. The invariants

$$f_l = \mathrm{Tr}\mathbf{B}_0^l, \quad l = 1,\dots,v \tag{4.12}$$

clearly Poisson-commute on X, are gauge invariant and therefore descend to the commuting functions on $S^{[v]}$ (and to $S^{[v]}_{\zeta}$ as well). The functional independence is easily checked on the dense open set where $S^{[v]}$ can be identified with the symmetric product of S's.

5. Elliptic models

5.1. Hilbert scheme of points on T^*E

Let E be the ellipic curve, $E^{\vee} \approx \mathrm{Jac}(E) = \mathrm{Pic}_0(E)$. For $t \in E^{\vee}$ let L_t denote the corresponding line bundle. In particular, let $0 \in E^{\vee}$ be the trivial line bundle: $L_0 = \mathcal{O}$. Let $S = T^*E$, let (z_1, z_2) be the coordinates on the universal cover of S, $z_2 \in T^*$, $\pi : S \to E$ be the projection.

Let $Z \in S^{[v]}$. For each $t \in E^{\vee}$ let V_t be the space $\mathrm{H}^0\left(Z, \pi^* L_t|_Z\right)$. The spaces V_t form a holomorphic rank v bundle $\mathcal{E}$ over $E^{\vee}$. Let $\phi : V \to V \otimes \Omega^1_{E^{\vee}}$ be the operator which multiplies a section of $\pi^* L_t$ by a covector (thus we get a map from $V \times \Omega^1_E$ to V which is the same as $V \to V \otimes \Omega^1_{E^{\vee}}$). The fiber at $t = 0$ contains a distinguished vector I which is the image of $1 \in \mathrm{H}^0(Z, \pi^*\mathcal{O}|_Z)$. The triple $(\mathcal{E}, \phi, I)$ is stable in the following sense: *There exists no holomorphic subbundle $\mathcal{F}$ of $\mathcal{E}$, such that $\phi(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1_{E^{\vee}}$ and $I \in \mathcal{F}_0$.*
This is an appropriate counterpart of the notion of a stable Higgs pair [30] in the case of genus one.

Let $Z \in S^{[v]}$. Consider the cover $p : \mathbf{C} \times \mathbf{C} \to T^*E$. Let us lift Z to the covering space. The space V of functions on p^*Z is $\mathbf{Z} \oplus \mathbf{Z}$-module. By Fourier transform we may view this space as a space of functions on a two-torus with values in a v-dimensional vector space. Let $t, \bar{t}$ be the coordinates on the torus. The elements of the v-dimensional space $V_{t,\bar{t}}$ are the functions on p^*Z which transform as follows:

$$f(z_1 + m + n\tau, z_2) = \exp \frac{2\pi i}{\tau - \bar{\tau}} \left(m\left(t - \bar{t}\right) + n\left(\tau\bar{t} - \bar{\tau}t\right)\right) f(z_1, z_2). \tag{5.1}$$

Clearly, the function $f \equiv 1$ belongs to $V_{0,0}$. It is represented by the vector $I \in V_{0,0}$.

To reconstruct a scheme Z given a stable triple $(\phi, \bar{\partial} + \bar{A}, I)$ one can go to a gauge where both ϕ and $\bar{A}$ are constant (t-independent). Then the matrices $(\phi, \bar{A})$ commute and together with I form a stable triple suitable for defining a subscheme of $\mathbf{C}^2$ of length v. In fact, the gauge where ϕ and $\bar{A}$ are constant allows extra gauge transformations which make the support of the subscheme of $\mathbf{C}^2$ invariant under the action of $\mathbf{Z} \oplus \mathbf{Z}$ by translations. The simplest yet instructive case is that of the diagonalizable $\phi, \bar{A}$.

One can do more. The Hilbert scheme of points on T^*E can be endowed with a hyperkahler structure. To see this we start with the space $\mathcal{X}$ of the pairs (A, Φ) - where A is $U(N)$ gauge field on the torus E and Φ is the adjoint-valued one-form. The space $\mathcal{X}$ is hyperkahler and the hyperkahler structure is preserved by the action of the gauge group $\mathcal{G}$. The latter may also act via evaluation at some points $t_1, \ldots, t_s$ on some finite-dimensional hyperkahler manifolds $\mathcal{O}_1, \ldots, \mathcal{O}_s$. Let $\vec{\mu}_k$ be the hyperkahler moment map for $\mathcal{O}_k$. Let us consider the hyperkahler reduction of the space $\mathcal{X} \times \mathcal{O}_1 \times \ldots \mathcal{O}_s$ with respect to $\mathcal{G}$. We first impose the hyperkahler moment map equations:

$$\begin{aligned} \bar{\partial}\Phi_t + [A_{\bar{t}}, \Phi_t] &= \sum_{k=1}^{s} \mu_k^c \delta^{(2)}(t - t_k) \\ F_{t\bar{t}} + [\Phi_t, \Phi_{\bar{t}}] &= \sum_{k=1}^{s} \mu_k^r \delta^{(2)}(t - t_k) \end{aligned} \tag{5.2}$$

These equations are the genus one analogues of Hitchin's self-duality equations [30]. They were studied from the point of view of complex reduction in [5][31][32]. They also appeared in the study of intersecting D-branes on tori [33](see also [34] for the general discussion of D-branes on tori), in the attempts to find appropriate field parameterization of the Yang-Mills theory [35]. The case of our interest here is $s = 1$, $\mathcal{O}_1 = T^*\mathbf{CP}^{N-1}$. The complex reduction in this case has been studied in [31] where it was shown that the solutions to the equations (5.2) (in fact of their deformation) form a holomorphic integrable system, which turns out to be elliptic Calogero-Moser system. The latter describes the system of non-relativistic particles on elliptic curve E, which pairwise interact via potential $\zeta_c^2 \wp(z_i - z_j)$ where ζ is a the period of a holomorphic symplectic form on $\mathcal{O}_1$ (see below).

The space $\mathcal{O}_1$ in turn can be described as a hyperkahler quotient of $\mathbf{C}^{2N}$ with respect to the action of $U(1)$ which has charges $(+1, -1)$ on $\mathbf{C}^N \oplus \mathbf{C}^N$. Let us denote the elements of $\mathbf{C}^N \oplus \mathbf{C}^N$ as $I \oplus J$, where J and I are the row and the column vectors respectively. Then the equations (5.2) can be written explicitly as:

$$\begin{aligned} \bar{\partial}\Phi_z + [A_{\bar{z}}, \Phi_z] &= (IJ - \zeta_c \mathrm{Id})\, \delta^{(2)}(z) \\ F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] &= \left(II^\dagger - J^\dagger J - \zeta_r \mathrm{Id}\right) \delta^{(2)}(z) \end{aligned} \tag{5.3}$$

The solutions to (5.3) are the gauge fields which have monodromy around the puncture $z = 0$, which is conjugate to $\exp\left(II^\dagger - J^\dagger J - \zeta_r \mathrm{Id}\right)$ and the covariantly constant Higgs fields which have a first order pole at $z = 0$ with the residue given by $(IJ - \zeta_c \mathrm{Id})$.

6. Beauville - Mukai's systems

Let S be the surface of $K3$ type, which contains a holomorphically embedded curve Σ of genus g. Fix the numbers N and p. Let $\mathcal{M}_{N,p,g}(S, \Sigma)$ be the moduli space of the pairs $(C, \mathcal{L})$, where $[C] = N[\Sigma]$, C is a holomorphically embedded curve and $\mathcal{L}$ is the line bundle on C of degree p. Let h be the genus of generic C. It is equal to

$$h = 1 + N^2(g - 1) \tag{6.1}$$

Let $\overline{\mathcal{M}}$ be the compactification of $\mathcal{M}_{N,h,g}(S, \Sigma)$ (the case of general p is left beyond the scope of our investigation) which is defined as the moduli space of pairs $(C, \mathcal{L})$ where C is a curve as above and $\mathcal{L}$ is a torsion free rank one coherent sheaf on C (if the curve C is smooth then this is the same as a line bundle), with $\int_C c_1(\mathcal{L}) = h$.

Theorem. $\overline{\mathcal{M}}$ is a symplectic manifold, birationally equivalent to $S^{[h]}$ with its standard symplectic structure.

Proof. The statement is actually well-known and is a compilation of two Beauville remarks [36], [37]. Similar considerations are contained in [38],[39]. Fix a smooth irreducible representative C. A. Beauville proves that under the conditions of our theorem there exists a morphism $\varphi : S \to \mathbf{P}^h$, such that its restriction to C coincides with the canonical embedding of C into $\mathbf{P}^{h-1}$ by the sections of the canonical sheaf ω_C of C: A hyperplane $H \subset \mathbf{P}^h$ intersects $\varphi(S)$ by a curve C_H. A restriction of the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^h$ to C_H gives a line bundle $\mathcal{L}_H$ of degree h.

Given a very ample line bundle $\mathcal{L}$ over C_H of the same degree h one may consider the corresponding linear system $|\mathcal{L}| = \left(\mathbf{P}^h\right)^*$ as a base of the fibration $\mathcal{J} \to |\mathcal{L}|$ with the fiber $\mathrm{Jac}(C_H)$ [36] over a point C_H. Moreover, as it was shown in [40] one can extend this fibration to a "compactified fibration" $\bar{\mathcal{J}}$ with the fibre $\overline{\mathrm{Jac}(C_H)}$ such that $\dim\bar{\mathcal{J}} = 2h$. This space is an algebraically completely integrable system with respect to a holomorphic symplectic structure introduced in [41]. Through h lineary independent points $p_1, \ldots, p_h$ in $S \subset \mathbf{P}^h$ the unique hyperplane H passes. Hence (generically) there exists a unique genus h curve C_H passing through the points and the line bundle $\mathcal{L}$ whose divisor coincides with $p_1 + \ldots + p_h$. This $\mathcal{L}$ has a unique up to a constant non-zero section with h zeroes at the points $p_1, \ldots, p_h$. In other words the pairs $(C, \mathcal{L}) \in \overline{\mathcal{M}}$ are parameterized by $\bar{\mathcal{J}}$. We have established a correspondence between the point of $\bar{\mathcal{M}}$ and the point of smooth open subset in symmetric product $\mathrm{Sym}^h(S)$. Proposition **1.3** of [37] shows that this correspondence actually extends to a birational isomorphism of our theorem. Thus the symplectic structure of Mukai's on $\bar{\mathcal{J}}$ maps to a direct sum of symplectic structures on the copies of the surface S.

Remark. The identification of Beauville-Mukai integrable systems phase space with the Hilbert scheme of points on the initial **K3** surface S provides an analogue of Sklyanin's **SoV** in terms of projective coordinates in $\mathbf{P}^h$. It again resembles his "magic recipe": the rôle of zeroes and poles of eigen-bundle sections is played by the zeroes of sections of linear bundles $\mathcal{L}$ arising under replacing of $T^*\Sigma$ by **K3** surface S.

The following example of Beauville-Mukai system which is proposed in [36] is an illustration of this *ad hoc* separation.

Example. Let us consider a quartic $S \subset \mathbf{P}^3$ given by the equation $F(X_0, X_1, X_2, X_3) = 0$ in homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$. This is an example of **K3** surface. The condition $\deg F = 4$ implies that S has a holomorphic symplectic form, whose associated Poisson bracket can be described quite explicitly. Suppose $X_0 = t \neq 0$. Let $f(x_1, x_2, x_3) = t^{-4}F(t, tx_1, tx_2, tx_3)$. Let g, h be the locally defined holomorphic functions on $S \backslash S \cap \{X_0 = 0\}$. Extend them to the functions in (x_1, x_2, x_3) defined in the neibourghood of $f^{-1}(0)$. Then:

$$\{g, h\} \equiv \frac{dg \wedge dh \wedge df}{dx_1 \wedge dx_2 \wedge dx_3} \tag{6.2}$$

evaluated at $f = 0$ is well-defined independently of the choices made and also the Poisson bivector defined in this way extends to the whole of S. Now we want to study the $h = 3$'d symmetric power of S, more precisely the Hilbert scheme $S^{[3]}$. Introduce the variables X_{ai}, $i = 1, \ldots, 3$, $\alpha = 0, 1, 2, 3$. We denote by $\vec{x}_i = (x_{1i}, x_{2i}, x_{3i})$, where $x_{ai} = X_{ai}/X_{0i}$, $a = 1, 2, 3$. The space $S^{[3]}$ has a natural sympectic form ω_h which is the induced from the symplectic form on S via the Hilbert-Chow morphism $S^{[3]} \to \mathrm{Sym}^3(S)$. Let $\pi = \omega_h^{-1}$

be the corresponding Poisson bivector. We now define three functions on $S^{[3]}$. Assume first that $\vec{x}_i \neq \vec{x}_j, i \neq j$. Then the construction of the previous section can be formulated very concretely as follows. There is a unique hyperplane in $\mathbf{P}^3$ which passes through the points $\vec{x}_1, \vec{x}_2, \vec{x}_3 \in S \subset \mathbf{P}^3$. The space of hyperplanes in $\mathbf{P}^3$ is the dual projective space $\mathcal{H} = \left(\mathbf{P}^3\right)^{\star}$:

$$c = (c^0 : c^1 : c^2 : c^3) \in \mathcal{H} \mapsto \sum_{a=0}^{3} c^a X_a = 0, \quad X = (X_0 : X_1 : X_2 : X_3) \in \mathbf{P}^3 \tag{6.3}$$

The Hilbert scheme of points contains regions where the points coincide. For a collision of a pair of points $\vec{x}_1, \vec{x}_2$ one glues a line $\mathbf{P}^1$ which determines the direction along which the points collided. In this case there still exists a unique hyperplane in $\mathbf{P}^3$ which passes through this line and the third point $\vec{x}_3$. Now, if the third point approaches the cluster formed by the first two then the plane $\mathbf{P}^2$ along which the configuration of the line and the third point collided is contained in the Hilbert scheme of points. Thus we have shown that by associating a plane to the triple of points and they limits one gets a well-defined birational map:

$$\varphi : S^{[3]} \to \mathcal{H} \tag{6.4}$$

The map φ is given by the explicit formulae:

$$\varphi^{\alpha} = \mathrm{Det}\, \Psi_{\alpha} \tag{6.5}$$

where Ψ_{α} is the 3×3 matrix obtained from $\|X_{ai}\|$ by removing the α'th column. In the domain where $\psi_0 := \mathrm{Det}\|x_{ai}\| \equiv \Psi_0 / \left(X_{10} X_{20} X_{30}\right) \neq 0$ we may introduce the functions:

$$H_a = \frac{\psi_a}{\psi_0} \tag{6.6}$$

where $\psi_a = \mathrm{Det} M_a$, $(M_a)_{bi} = X_{bi}$, for $b \neq a$ and $(M_a)_{ai} = 1$ for any i:

$$\begin{aligned} H_1 &= \frac{1}{\psi_0} \mathrm{Det} \begin{pmatrix} 1 & x_{21} & x_{31} \\ 1 & x_{22} & x_{32} \\ 1 & x_{23} & x_{33} \end{pmatrix}, \\ H_2 &= \frac{1}{\psi_0} \mathrm{Det} \begin{pmatrix} x_{11} & 1 & x_{31} \\ x_{12} & 1 & x_{32} \\ x_{13} & 1 & x_{33} \end{pmatrix}, \\ H_3 &= \frac{1}{\psi_0} \mathrm{Det} \begin{pmatrix} x_{11} & x_{21} & 1 \\ x_{12} & x_{22} & 1 \\ x_{13} & x_{23} & 1 \end{pmatrix}, \end{aligned} \tag{6.7}$$

These explicit Hamiltonians are defined in $\varphi^{-1}(\mathbf{A}^3)$, where $\mathbf{A}^3$ is the affine part of $\mathcal{H}$, corresponding to $\psi_0 \neq 0$. It follows from the identity:

$$\{\psi_a, \psi_b\} = \psi_a\{\psi_0, \psi_b\} - \psi_b\{\psi_0, \psi_a\} \tag{6.8}$$

that the Hamiltonians H_a, $a = 1, 2, 3$, Poisson commute. It would be interesting to investigate this system as an example of a "deformation" of a Hitchin system in the sense of [4] and to study its quantum analogues which as are hopefully related to some Feigin - Odesski - Sklyanin algebras.

7. Relations to the physics of D-branes and gauge theories

The abovementioned constructions of the separation of variables in integrable systems on moduli spaces of holomorphic bundles with some additional structures can be described as a symplectomorphism between the moduli spaces of the bundles (more precisely, torsion free sheaves) having different topology, e.g. Chern classes.

To be specific let us concentrate on the moduli space $\mathcal{M}_{\vec{v}}$ of stable torsion free coherent sheaves $\mathcal{E}$ on S. Let $\hat{A}_S = 1 - [\mathrm{pt}] \in H^*(S, \mathbf{Z})$ be the A-roof genus of S. The vector $\vec{v} = Ch(\mathcal{E})\sqrt{\hat{A}_S} = (r; \vec{w}; d - r) \in \mathrm{H}^*(S, \mathbf{Z}), \vec{w} \in \Gamma^{3,19}$ corresponds to the sheaves with the Chern numbers:

$$\begin{aligned} ch_0(\mathcal{E}) &= r \in \mathrm{H}^0(S; \mathbf{Z}) \\ ch_1(\mathcal{E}) &= \vec{w} \in \mathrm{H}^2(S; \mathbf{Z}) \\ ch_2(\mathcal{E}) &= d \in \mathrm{H}^4(S; \mathbf{Z}) \end{aligned} \tag{7.1}$$

Type IIA string theory compactified on S has BPS states, corresponding to the Dp-branes, with p even, wrapping various supersymmetric cycles in S, labelled by $\vec{v} \in \mathrm{H}^*(S, \mathbf{Z})$. The actual states correspond to the cohomology classes of the moduli spaces $\mathcal{M}_{\vec{v}}$ of the configurations of branes. The latter can be identified with the moduli spaces $\mathcal{M}_{\vec{v}}$ of appropriate sheaves.

The string theory, compactified on S has moduli space of vacua, which can be identified with

$$\mathbb{M}_A = O\left(\Gamma^{4,20}\right) \backslash O(4, 20; \mathbf{R}) / O(4; \mathbf{R}) \times O(20; \mathbf{R})$$

where the arithmetic group $O(\Gamma^{4,20})$ is the group of discrete authomorphismes. It maps the states corresponding to different $\vec{v}$ to each other. The only invariant of its action is $\vec{v}^2$. We have studied three realizations of an integrable system.

The first one uses the non-abelian gauge fields on the curve Σ imbedded into symplectic surface S. Namely, the phase space of the system is the moduli space of stable pairs: $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is rank r vector bundle over Σ of degree l, while ϕ is the holomorphic section of $\omega_{\Sigma}^{1} \otimes \mathrm{End}(\mathcal{E})$.

The second realization is the moduli space of pairs $(C, \mathcal{L})$, where C is the curve (divisor) in S which realizes the homology class $r[\Sigma]$ and $\mathcal{L}$ is the line bundle on C.

The third realization is the Hilbert scheme of points on S of length h, where $h = \frac{1}{2}\mathrm{dim}\mathcal{M}$.

The equivalence of the first and the second realizations corresponds to the physical statement that the bound states of N $D2$-branes wrapped around Σ are represented by a single $D2$-brane which wraps a holomorphic curve C which is an N-sheeted covering of the base curve Σ. The equivalence of the second and the third descriptions is tempting to attribute to T-duality.

8. Discussion

We have attempted to formulate the separation of variables purely in geometric terms. It seems that the proper physical setup is the use of chain of dualities to get the system of $D0$ branes on some hyperkahler manifold.

A few subjects need further clarification. Evidently an interesting question to investigate is the quantization of the integrable system. According to the work of E. Frenkel and B. Feigin, the quantum separation of variables can be viewed as the geometrical Langlands transform. The latter maps the spectrum of the Hitchin $\mathcal{D}$-module (Hamiltonians) to the data connected to the local system on the dual object [42]. What we have studied is the classical analogue of the Langlands correspondence which maps the phase space of the Hitchin and Mukai systems to the Hilbert scheme of points.

The quantum separation of variables goes as follows. The wave function in the separated variables becomes the product of identical factors up to a determinant factor $\prod_{i,j}(\xi_i - \xi_j)^{\kappa}$:

$$\Psi(\xi_1, \ldots, \xi_n) = \prod_i Q(\xi_i)$$

where $Q(\xi)$ obeys the so-called Baxter's equation

$$\mathrm{Det}\,(L(\xi) - \eta)\, Q(\xi) = 0$$

where $L(\xi)$ is the Lax operator of the dynamical system, evaluated at the zero of the Baker-Akhiezer function. The variable η is viewed as an operator acting on $Q(\xi)$. The commutation relation between ξ and η implies the representation of η, which can be either differential, difference or integral operator.

In some sense $Q(\xi_i)$ can be considered as the wave function of a single $D0$ brane while the union of integrable systems with all N's is the second quantization.

As an example of the quantum separation of variables let us consider the Gaudin system. Consider $\mathcal{D}$-modules which describe a system of differential equations on the moduli space $\mathcal{M}_G(\Sigma)$ of principal G-bundles over a complex curve Σ with marked points which is given by a commuting set of differential operators whose symbols coincide with Hitchin hamiltonians. Hence they can be considered as a quantization of Hitchin systems [42]. If the curve Σ has genus zero then the corresponding $\mathcal{D}$-modules give rise to the Gaudin model of integrable quantum spin chain [43].

Consider the punctured sphere $\Sigma = \mathbf{CP}^1$ and attach to each of the distinct marked points $z_i, i = 1, ...n$ the $\underline{\mathbf{g}} = sl_2$ Verma module V_{λ_i} of the highest weight $\lambda_i \in \mathbf{C}$. Represent the Lie algebra sl_2 by the basic differential operators

$$e_i = -t_i \partial^2_{t_i} - \lambda_i \partial_{t_i}, \quad h_i = -2t_i \partial_{t_i} - \lambda_i, \quad f_i = t_i, \tag{8.1}$$

and denote by C_{ij} the Casimir operator $C_{ij} = e_i f_j + f_i e_j + \frac{1}{2} h_i h_j$. The Lax operator $\widehat{\phi(z)}$ is given by

$$\widehat{\phi(z)} = \sum_{i=1}^{n} \frac{\begin{pmatrix} h_i & f_i \\ e_i & -h_i \end{pmatrix}}{z - z_i} \tag{8.2}$$

Introduce the operators H_i :

$$H_i = \sum_{i \neq j} \frac{C_{ij}}{z_i - z_j}, \tag{8.3}$$

and $\nabla_i = (\kappa + 2)\partial_{z_i} - H_i$, where $\kappa \in \mathbf{C}$ is a "level". Consider the infinite-dimensional Lie algebras $\hat{\underline{\mathbf{g}}}$ and $\hat{\underline{\mathbf{g}}}_+$: $\hat{\underline{\mathbf{g}}} = \underline{\mathbf{g}} \otimes \mathbf{C}((z)) \oplus \mathbf{C}K$, where $\mathbf{C}((z))$ are meromorphic functions of a formal parameter z around a marked point and $\hat{\underline{\mathbf{g}}}_+ = \underline{\mathbf{g}} \otimes \mathbf{C}[[z]]$ is the Lie subalgebra of $\hat{\underline{\mathbf{g}}}$ which consists of the power seria in z. The quotient $U\hat{\underline{\mathbf{g}}}/(K - \kappa)$ is denoted by $U_\kappa \hat{\underline{\mathbf{g}}}$. By definition the conformal blocks are linear functionals:

$$f : \otimes_{i=1}^{n} (U_\kappa(\hat{\underline{\mathbf{g}}}) \otimes_{U(\hat{\underline{\mathbf{g}}}_+)} V_{\lambda_i}) \rightarrow \mathbf{C}, \tag{8.4}$$

which are invariant under the action of the Lie algebra $\hat{\underline{\mathbf{g}}}^{\mathrm{out}}(z_1, ..., z_n)$ of the Laurent seria at the marked points of regular $\underline{\mathbf{g}}$-valued functions on the marked sphere which vanish at infinity. It is well-known fact [43] that any such functional defines a number-valued function $f(z_1, \ldots, z_n; t_1, \ldots, t_n)$ obeying the Knizhnik-Zamolodchikov system

$$\nabla_i(f) = 0. \tag{8.5}$$

Let us perform the transition to the separated variables p_i: $(z, t_1, \ldots, t_n) \to (z, p_1, \ldots, p_{n-1}, u)$, where the variables (p, u) and t are defined through the following relation:

$$\sum_{i=1}^{n} \frac{t_i}{z - z_i} \, dz = u \frac{\prod_{l=1}^{n-1}(z - p_l)}{\prod_{i=1}^{n}(z - z_i)} \, dz. \tag{8.6}$$

It is clear that p_l which are the zeroes of the Φ_{12} element of the Lax matrix are zeroes of the Baker-Akhiezer function. Therefore the spectral curve

$$\eta^2 = \mathrm{Det}\Phi(z)$$

yields Baxter-Sklyanin equation for the Gaudin magnet

$$\left[\nabla_{p_i}^2 + \mathrm{Det}\Phi(p_i)\right] Q(p_i) = 0,$$

where $\nabla = -\partial_p - \sum_i \frac{\lambda_i}{p - z_i}$, which defines a projective connection. If one considers $\underline{\mathbf{g}} = \mathrm{sl}_N$ magnet instead then Baxter equation becomes the N-th order differential equation.

In generic situation there are a few subtle points. First, the proper ordering should be chosen in the quantum operators. Secondly, when discussing Hitchin system on higher genus curve one has to deal with globally defined objects with proper modular properties. There are no global solutions to the differential equations in this case. The differential operators are defined globally, though.

The explicit computations in the case of genus 1 [6][5] give a system which interpolates between the Gaudin model and Calogero-Moser elliptic many-body system. The geometrical approach to **SoV** for this system is formulated in [44] and elaborated on in [45].

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