Graceful Graphs and Graceful Labelings: Two Mathematical Programming Formulations and Some Other New Results

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Given a graph G consisting of vertices and edges, a *vertex labeling* of G is an assignment f of labels to the vertices of G that produces for each edge xy a label depending on the vertex labels $f(x)$ and $f(y)$. A vertex labeling f is called a *graceful labeling* of a graph G with e edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, e\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$ the resulting edge labels are distinct. A graph G is called *graceful* if there exists a graceful labeling of G (see Fig. 1). In this paper we present two mathematical programming formulations of the graceful labeling problem (first as an integer programming problem, second as a constraint programming problem), along with some new results on the gracefulness of three classes of graphs: generalized Petersen graphs $P(n, k)$, double cones $C_n + \overline{K_2}$, and product graphs of the form $K_4 \times P_n$.

Key Words: graceful labeling, graceful graph, generalized Petersen graph, double cone, $K_4 \times P_n$, integer programming, constraint programming

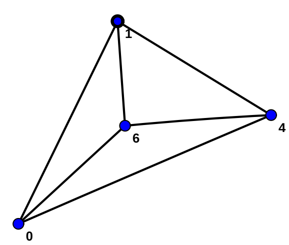

FIG. 1. A graceful labeling of a graceful graph: K_4

1. INTRODUCTION

The study of graceful graphs and graceful labeling methods was introduced by Rosa [12]. Rosa defined a β-*valuation* of a graph G with e edges as an injection from the vertices of G to the set $\{0, 1, \ldots, e\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$ the resulting edge labels are distinct. β-valuations are functions that produce graceful labelings. However, the term *graceful labeling* was not used until Golomb studied such labelings several years later [8].

The graceful labeling problem is to determine which graphs are graceful. Proving a graph G is or is not graceful involves either producing a graceful labeling of G or showing that no such labeling exists. Over the past 30 years, approximately 200 papers on graceful labeling methods have been published. An unpublished result of Erdös that was later proven by Graham and Sloane states that almost all graphs are not graceful [7], though it does appear that most graphs having some regularity of structure to them are graceful.

When studying graceful labelings, we need only consider simple graphs, or graphs without loops or parallel edges. A loop in a labeled graph would assume an edge label of 0, and in a graceful labeling of a graph G the resulting edge labels must be distinct and take values of $1, 2, \ldots, e$, where e is the number of edges in G. Parallel edges between a particular pair of vertices in a labeled graph would always assume the same edge label, a violation of the property that edge labels be distinct in a graceful labeling.

While the graceful labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, gracefully labeled graphs often serve as models in a wide range of applications. Such applications include coding theory and communication network addressing. Bloom and Golomb [2] give a detailed explanation of some of the applications of gracefully labeled graphs.

Most of the papers published to date on the subject of graceful labeling are theoretical, however, and principally focus on certain classes of graphs and labeling methods. Such papers often present arguments by either providing formulas for gracefully labeling graphs within a particular class, or proofs that graphs of a particular class are not graceful. The gracefulness of several classes of graphs has already been established. For example, all paths P_n are graceful [12]. Cycles C_n are graceful only when $n \equiv 0$ or 3 (mod 4) [12]. The complete graphs K_n are graceful only for $n \leq 4$ [8]. All wheels W_n (or equivalently, $C_{n-1} + K_1$) are graceful [4]. This last result illustrates that subgraphs of a graceful graph need not be themselves graceful, and that the addition of a single vertex to a graph may easily change its gracefulness or non-gracefulness. A complete and current summary of graceful and non-graceful results along with some unproven conjectures can be found in Gallian's dynamic survey of graceful labeling [6]. Included in Gallian's survey is the popular Ringel-Kotzig conjecture that *all* trees are graceful, a conjecture which remains an open problem in this area, and whose proof (or counterexample) has eluded researchers for many years.

2. SOME NECESSARY CONDITIONS FOR GRACEFUL GRAPHS

Rosa identified three basic reasons why a graph G fails to be graceful: (1) G has too many vertices and not enough edges, (2) G has too many edges, and (3) G has the wrong parity [6]. As an example of (3), Rosa developed a useful parity condition for a simple graph G with e edges. He proved that if every vertex of G has even degree and $e \equiv 1$ or 2 (mod 4), then G is not graceful [12]. Golomb restated this parity condition for graceful graphs as a necessary condition [8], illustrating that if G is a graceful, even (simple) graph with e edges, then necessarily $\lfloor (e+1)/2 \rfloor \equiv 0$ (mod 2).

Rosa's parity condition (and equivalently, Golomb's necessary condition) significantly reduces the number of graphs that are graceful, and often serves to classify an infinite collection of graphs as being not graceful. It also leads us to the following theorem.

THEOREM 2.1. *Let G be a graceful, even (simple) graph having e edges, and let $\mathcal{E}$ and $\mathcal{O}$ denote the sums of the even edge labels and odd edge labels in a graceful labeling of G, respectively.*

Case 1. If $\frac{e(e+1)}{2} \equiv 0$ (mod 4) then

1. $\mathcal{E} \equiv 0$ *(mod 4)*
2. $\mathcal{O} \equiv 0$ *(mod 4)*
3. $\lfloor \frac{(e+1)}{2} \rfloor \equiv 0$ *(mod 2)*
4. $\lfloor \frac{(e+2)}{4} \rfloor \equiv 0$ *(mod 2)*

Case 2. If $\frac{e(e+1)}{2} \equiv 2$ (mod 4) then

1. $\mathcal{E} \equiv 2$ *(mod 4)*
2. $\mathcal{O} \equiv 0$ *(mod 4)*
3. $\lfloor \frac{(e+1)}{2} \rfloor \equiv 0$ *(mod 2)*
4. $\lfloor \frac{(e+2)}{4} \rfloor \equiv 1$ *(mod 2)*

Proof. Suppose G is a graceful, even (simple) graph (i.e., it is Eulerian) having e edges. Then there exists a graceful labeling L_G of G. Since G is graceful and Eulerian, the sum of the resulting edge labels of G, which equals $\sum_{i=1}^{e} i = \frac{e(e+1)}{2}$, must be even [8]. In other words, $\frac{e(e+1)}{2} \equiv 2k$ (mod 4), where $k = 0$ (Case 1) or $k = 1$ (Case 2). If $k = 0$ (Case 1), then $\frac{e(e+1)}{2} \equiv 0$ (mod 4) $\equiv \mathcal{E} + \mathcal{O}$. If

$k = 1$ (Case 2), then $\frac{e(e+1)}{2} \equiv 2 \pmod 4 \equiv \mathcal{E} + \mathcal{O}$. By analyzing partial sums of sequences of consecutive even integers and consecutive odd integers respectively, we observe that $\mathcal{E} \equiv 0$ (mod 4) or 2 (mod 4), and $\mathcal{O} \equiv 0$ (mod 4) if L_G has an even number of odd edge labels and $\equiv 1$ (mod 4) if L_G has an odd number of odd edge labels. Thus the only possibility for Case 1 ($k = 0$) is $\frac{e(e+1)}{2} \equiv 0$ (mod 4) $\equiv \mathcal{E} + \mathcal{O} \equiv 0$ (mod 4) + 0 (mod 4), and the only possibility for Case 2 ($k = 1$) is $\frac{e(e+1)}{2} \equiv 2$ (mod 4) $\equiv \mathcal{E} + \mathcal{O} \equiv 2$ (mod 4) + 0 (mod 4). From Golomb's necessary condition [8] we have that $\lfloor \frac{(e+1)}{2} \rfloor \equiv 0$ (mod 2). By inspection, we also see that L_G has $\lfloor \frac{(e+2)}{4} \rfloor$ edge labels that are $\equiv 2$ (mod 4). It follows that in Case 1, where $k = 0$ and $\mathcal{E} \equiv 0$ (mod 4), we have the number of edge labels that are $\equiv 2$ (mod 4) = $\lfloor \frac{(e+2)}{4} \rfloor \equiv 0$ (mod 2). It also follows that in Case 2, where $k = 1$ and $\mathcal{E} \equiv 2$ (mod 4), we have the number of edge labels that are $\equiv 2$ (mod 4) = $\lfloor \frac{(e+2)}{4} \rfloor \equiv 1$ (mod 2). We have therefore proven properties 1, 2, 3, and 4 for graphs in Case 1, and have also proven properties 1, 2, 3, and 4 for graphs in Case 2. This concludes the proof of the theorem. ∎

We conclude that if we have an even (simple) graph G that either falls under Case 1 and violates at least one of the properties 1, 2, 3, or 4 in Case 1, or falls under Case 2 and violates at least one of the properties 1, 2, 3, or 4 in Case 2, then G is not graceful. In the next three sections we define three classes of graphs—generalized Petersen graphs $P(n, k)$, double cones $C_n + \overline{K_2}$, and product graphs of the form $K_4 \times P_n$—and present new results on their gracefulness.

3. THREE CLASSES OF GRAPHS

In this paper we study the gracefulness of three different classes of graphs: generalized Petersen graphs $P(n, k)$, double cones $C_n + \overline{K_2}$, and product graphs of the form $K_4 \times P_n$.

3.1. GENERALIZED PETERSEN GRAPHS

The generalized Petersen graph $P(n, k)$, where $n \geq 5$ and $1 \leq k < n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_i a_{i+1} | i = 0, 1, \ldots, n-1\} \cup \{a_i b_i | i = 0, 1, \ldots, n-1\} \cup \{b_i b_{i+k} | i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo n. $P(n, k)$ has $2n$ vertices and $3n$ edges, as long as $k \neq \frac{n}{2}$, where n is even. If $k = \frac{n}{2}$, then $P(n, k) = P(n, \frac{n}{2})$ has $2n$ vertices and $3n - \frac{n}{2} = \frac{5n}{2}$ edges. By definition, we observe that $P(n, k) = P(n, n-k)$. A graceful labeling of the Petersen graph $P(5, 2) = P(5, 3)$ has already been obtained, and appears in a book by Bosák, among other places [3].

Frucht and Gallian define a *prism* $D_n (n \geq 3)$ as the Cartesian product $P_2 \times C_n$ of a path with 2 vertices and a cycle of length n, and subsequently prove that all prisms are graceful [5]. The result of their paper provides the proof for the following proposition, illustrating the gracefulness of a subclass of generalized Petersen graphs, namely those of the form $P(n,1) = P(n,n-1)$.

PROPOSITION 3.1. *All generalized Petersen graphs* $P(n,1) = P(n,n-1)$ *are graceful.*

Proof. By the definitions of generalized Petersen graphs and prisms, we have $P(n,1) = D_n$. Since all prisms D_n are graceful [5], it follows that all generalized Petersen graphs of the form $P(n,1)$ (and equivalently, $P(n,n-1)$) are also graceful. ∎

We conclude this section with a proposition concerning the gracefulness of various generalized Petersen graphs, which leads to a question regarding the gracefulness of the entire class of generalized Petersen graphs $P(n,k)$.

PROPOSITION 3.2. *The generalized Petersen graphs* $P(n,k)$ *are graceful for* $n = 5, 6, 7, 8, 9,$ *and 10.*

Proof. To prove the gracefulness of the generalized Petersen graphs $P(n,k)$ for $n = 5, 6, 7, 8, 9,$ and 10, it suffices to provide graceful labelings for these graphs. If we order the $2n$ vertices of $P(n,k)$ as $\{x_1,\ldots,x_{2n}\} = \{a_0,a_1,\ldots,a_{n-1},b_0,b_1,\ldots,b_{n-1}\}$ with edge set $\{a_i a_{i+1} | i = 0,1,\ldots,n-1\} \cup \{a_i b_i | i = 0,1,\ldots,n-1\} \cup \{b_i b_{i+k} | i = 0,1,\ldots,n-1\}$ where all subscripts are taken modulo n, then the following vertex labelings are graceful labelings for these graphs:

$P(5,1)$: $\{x_1,\ldots,x_{10}\} = 0,1,6,2,15,14,7,4,12,3$
$P(5,2)$: $\{x_1,\ldots,x_{10}\} = 0,1,3,6,14,15,13,10,2,4$
$P(6,1)$: $\{x_1,\ldots,x_{12}\} = 0,1,3,9,4,18,17,5,13,6,15,2$
$P(6,2)$: $\{x_1,\ldots,x_{12}\} = 0,1,4,14,3,18,17,5,10,6,5,2$
$P(6,3)$: $\{x_1,\ldots,x_{12}\} = 0,1,3,11,4,15,14,13,12,8,9,2$
$P(7,1)$: $\{x_1,\ldots,x_{14}\} = 0,1,3,6,15,4,20,21,9,16,10,5,19,2$
$P(7,2)$: $\{x_1,\ldots,x_{14}\} = 0,1,3,12,5,10,21,20,19,6,4,18,14,2$
$P(7,3)$: $\{x_1,\ldots,x_{14}\} = 0,1,3,7,12,4,20,21,19,17,18,2,10,5$
$P(8,1)$: $\{x_1,\ldots,x_{16}\} = 0,1,3,6,11,20,4,23,24,7,18,14,21,8,22,2$
$P(8,2)$: $\{x_1,\ldots,x_{16}\} = 0,1,3,6,20,4,17,24,23,22,8,12,16,21,5,2$
$P(8,3)$: $\{x_1,\ldots,x_{16}\} = 0,1,3,10,5,21,11,23,24,22,20,4,18,2,19,9$
$P(8,4)$: $\{x_1,\ldots,x_{16}\} = 0,1,3,9,14,17,2,20,19,18,11,16,5,7,15,4$
$P(9,1)$: $\{x_1,\ldots,x_{18}\} = 0,1,3,6,11,7,25,4,27,26,12,22,13,5,20,8,24,2$
$P(9,2)$: $\{x_1,\ldots,x_{18}\} = 0,1,3,6,19,9,4,11,27,26,25,22,14,7,23,24,5,2$
$P(9,3)$: $\{x_1,\ldots,x_{18}\} = 0,1,3,6,13,7,23,4,27,26,25,24,21,5,20,9,14,2$
$P(9,4)$: $\{x_1,\ldots,x_{18}\} = 0,1,3,6,10,18,24,4,26,27,25,22,23,20,2,13,9,5$
$P(10,1)$: $\{x_1,\ldots,x_{20}\} = 0,1,3,6,10,15,5,27,4,29,30,16,24,12,21,8,25,9,28,2$
$P(10,2)$: $\{x_1,\ldots,x_{20}\} = 0,1,3,6,13,24,8,12,25,30,29,28,23,18,5,9,27,26,4,2$
$P(10,3)$: $\{x_1,\ldots,x_{20}\} = 0,1,3,6,10,24,5,26,4,30,29,28,27,12,21,15,25,11,20,2$

$P(10,4)$: $\{x_1,\ldots,x_{20}\} = 0,1,3,6,10,23,11,26,12,19,30,28,27,25,4,18,2,5,20,7$
$P(10,5)$: $\{x_1,\ldots,x_{20}\} = 0,1,3,8,12,21,9,20,2,25,24,23,17,14,19,5,6,7,22,4$
▮

Question 1. What other generalized Petersen graphs $P(n,k)$ are graceful? Are possibly all generalized Petersen graphs $P(n,k)$ graceful?

3.2. DOUBLE CONES

The class of double cones $C_n + \overline{K_2}$, where $n = 1,2,\ldots,\infty$, is among the classes of graphs that are the joins of other graphs. $C_n + \overline{K_2}$ has $n+2$ vertices and $3n$ edges. By definition, it is clear that the double cone $C_1 + \overline{K_2}$ is the graph that joins two vertices to a loop. Being a non-simple graph, it cannot be gracefully labeled, for reasons discussed above in Section 1. Similarly, the double cone $C_2 + \overline{K_2}$ is not graceful, as it too is a non-simple graph. However, some double cones have been found to be graceful, as illustrated by the following proposition.

PROPOSITION 3.3. *The double cones $C_n + \overline{K_2}$ are graceful for n = 3, 4, 5, 7, 8, 9, 11, and 12.*

Proof. If we order the $n+2$ vertices of $C_n + \overline{K_2}$ as $\{x_1,\ldots,x_{n+2}\} = \{a_0,a_1,b_0,b_1,\ldots,b_{n-1}$ with edge set $\{a_0b_i | i = 0,1,\ldots,n-1\} \cup \{a_1b_i | i = 0,1,\ldots,n-1\} \cup \{b_ib_{i+1} | i = 0,1,\ldots,n-1\} \cup \{b_0b_{n-1}\}$, then the following vertex labelings are graceful labelings for these graphs:

$C_3 + \overline{K_2}$: $\{x_1,\ldots,x_5\} = 0,1,2,6,9$
$C_4 + \overline{K_2}$: $\{x_1,\ldots,x_6\} = 0,3,6,10,12,11$
$C_5 + \overline{K_2}$: $\{x_1,\ldots,x_7\} = 0,2,4,10,15,14,11$
$C_7 + \overline{K_2}$: $\{x_1,\ldots,x_9\} = 0,1,2,5,19,13,21,10,17$
$C_8 + \overline{K_2}$: $\{x_1,\ldots,x_{10}\} = 0,1,2,9,22,12,15,20,14,18$
$C_9 + \overline{K_2}$: $\{x_1,\ldots,x_{11}\} = 0,2,3,9,26,16,27,22,4,23,15$
$C_{11} + \overline{K_2}$: $\{x_1,\ldots,x_{13}\} = 0,1,2,5,16,29,7,27,10,31,19,33,25$
$C_{12} + \overline{K_2}$: $\{x_1,\ldots,x_{14}\} = 0,1,2,5,16,34,10,27,7,32,13,36,22,30$ ▮

As we observed with the class of generalized Petersen graphs, there exists an infinite subclass of non-graceful double cones. The proof of the following proposition is a direct result of Rosa's parity condition [12].

PROPOSITION 3.4. *All double cones $C_n + \overline{K_2}$, where $n \equiv$ 2 (mod 4), are not graceful.*

Proof. By definition, we see that double cones $C_n + \overline{K_2}$, where $n \equiv 2$ (mod 4), are simple, even graphs with e edges, with $e \equiv 2$ (mod 4). Such graphs, by Rosa's parity condition [12], cannot be gracefully labeled. ▮

We conclude this section with a question regarding the gracefulness of the entire class of double cones $C_n + \overline{K_2}$.

Question 2. What other double cones $C_n + \overline{K_2}$ are graceful? Are possibly all double cones $C_n + \overline{K_2}$, where $n \geq 3$ and $n \equiv 0$, 1, or 3 (mod 4), graceful?

3.3. A CLASS OF PRODUCT GRAPHS

Product graphs of the form $K_4 \times P_n$, where $n = 1, 2, \ldots, \infty$, have $4n$ vertices and $10n - 4$ edges. A graceful labeling of $K_4 \times P_1$, which is just K_4, is provided by Golomb [8] along with his proof that complete graphs K_n are graceful only for $n \leq 4$.

$K_1 \times P_n$, $K_2 \times P_n$, and $K_3 \times P_n$ are all graceful. $K_1 \times P_n$ are simply the path graphs P_n on n vertices, proved graceful for all n by Rosa [12]. Also, $K_2 \times P_n$ = $P_2 \times P_n$, the $2 \times n$ grids. All grids $P_m \times P_n$ are graceful, as proven by Acharya and Gill [1]. Finally, $K_3 \times P_n$ = $C_3 \times P_n$, which Singh [13] defines as a subclass of prisms (different from the definition of prisms in [5]) and proves graceful for all n. The above results provide the motivation to study graphs of the form $K_4 \times P_n$. Unfortunately, very little else is known about the gracefulness of these graphs, with the exception of the following proposition.

PROPOSITION 3.5. *Product graphs of the form $K_4 \times P_n$ are graceful for $n =$ 1, 2, 3, 4, and 5.*

Proof. If we order the $4n$ vertices of $K_4 \times P_n$ as $\{x_1, \ldots, x_{4n}\}$ where the n copies of K_4 have vertices $\{x_1, \ldots, x_4\}$ (1st copy of K_4), $\{x_5, \ldots, x_8\}$ (2nd copy of K_4), $\ldots, \{x_{4n-3}, \ldots, x_{4n}\}$ (nth copy of K_4 and are connected to each other by edges $\{x_i x_{i+4} | i = 1, 2, \ldots, 4n-4\}$, then the following vertex labelings are graceful labelings for these graphs:

$K_4 \times P_1$: $\{x_1, \ldots, x_4\} = 0, 1, 4, 6$
$K_4 \times P_2$: $\{x_1, \ldots, x_8\} = 0, 1, 5, 16, 6, 15, 13, 3$
$K_4 \times P_3$: $\{x_1, \ldots, x_{12}\} = 0, 1, 4, 26, 24, 18, 13, 6, 10, 2, 23, 25$
$K_4 \times P_4$: $\{x_1, \ldots, x_{16}\} = 0, 1, 3, 36, 24, 32, 7, 2, 13, 6, 22, 34, 27, 33, 4, 14$
$K_4 \times P_5$: $\{x_1, \ldots, x_{20}\} = 0, 1, 3, 46, 19, 42, 7, 2, 6, 14, 38, 44, 45, 34, 24, 8,$
$12, 5, 19, 30$ ∎

We conclude this section with a question regarding the gracefulness of the entire class of product graphs of the form $K_4 \times P_n$.

Question 3. What other product graphs of the form $K_4 \times P_n$ are graceful? Are possibly all product graphs of the form $K_4 \times P_n$ graceful?

4. USING MATHEMATICAL PROGRAMMING TO COMPUTE GRACEFUL LABELINGS

Attempting to gracefully label a particular graph G by hand or theoretically prove that it is not graceful can be a tedious and/or difficult process. A much more practical, accurate and efficient route to take is one of computation. Below we present two mathematical programming formulations of the graceful labeling problem for a graph $G = (V, E)$ with n vertices and e edges. The first approach/formulation, described in Section 6.1, involves integer programming. The second approach/formulation, described in Section 6.2, involves constraint programming.

4.1. AN INTEGER PROGRAMMING FORMULATION AND ANALYSIS OF THE GRACEFUL LABELING PROBLEM USING CPLEX

Graceful Labeling Problem Formulation.

$$\text{minimize } f(x_1, x_2, \ldots, x_n) = 0$$

subject to

- (constraint 1) $|x_i - x_j| = a$ for some edge $ij \in E(G)$; $a = 1, 2, \ldots, e$
- (constraint 2) $|x_i - x_j| \geq 1$ for each edge $ij \in E(\overline{G})$
- (variable bound) $0 \leq x_i \leq e$; $i = 1, 2, \ldots, n$
- (integer requirement) x_i integer; $i = 1, 2, \ldots, n$

Analysis. In the above formulation of the graceful labeling problem (which will become an integer programming formulation once the constraints are linearized), we are only concerned with obtaining constraint feasibility, not a minimization or maximization of a specific objective function, and can therefore employ the use of a "dummy" (i.e., constant) objective function. The objective "minimize $f(x_1, x_2, \ldots, x_n) = 0$" is used here. If we can obtain a solution $\{x_1, \ldots, x_n\}$ that satisfies all of the above constraints for a particular graph G, then G is a graceful graph and $\{x_1, \ldots, x_n\}$ is a graceful labeling of the vertices $1, 2, \ldots, n$ of G. If no such feasible solution exists, then no such graceful labeling of G exists; that is, G is not a graceful graph.

Our formulation contains n integer variables $x_1, \ldots, x_n$, one for each of the n vertices of G. To linearize both of the constraints above we must introduce a number of additional binary variables to the formulation. In doing so we define e integer variables $z_1, z_2, \ldots, z_e$, one for each of the e edges of G, where $z_k = |x_i - x_j|$, for $k = 1, 2, \ldots, e$. With the inclusion of these e integer variables, we reduce the number of binary variables that must be added during constant linearization from $\frac{n^2-n+4e^2-2e}{2}$ to $\frac{n^2-n+2e^2}{2}$, a "savings" of $e^2 - e$ binary variables. As the size of our graph instance G increases, this "savings" can grow to be quite significant.

Constraint 1 above maintains that each element of the set $\{1, 2, \ldots, e\}$ must be an edge label in a graceful labeling of G. This constraint also illustrates the necessary requirement that in a graceful labeling of G, adjacent vertices must have distinct labels. Constraint 2 dictates that in a graceful labeling of G, non-adjacent vertices must also have distinct labels. Our variable bound is equivalent to the fact that in a graceful labeling of G, all vertex labels must assume values between 0 and e.

After constraint linearization, the above graceful labeling problem formulation is transformed into an integer programming formulation with a total of $n + e$ integer variables, $\frac{n^2-n+2e^2}{2}$ binary variables, $n^2 - n + 5e$ linear constraints, and $n + e$ variable bounds.

Implementation. The solving of specific graph instances G of the graceful labeling problem via integer programming is implemented as follows: a C++ program first converts the graph file "G.edg" containing the number of vertices in G, the number of edges in G, and a list of G's edges ij, where $i < j$, into a CPLEX input file "G.lp" based on the above formulation for CPLEX to read and solve as a integer programming problem. If a graceful labeling of G exists, then CPLEX will find a graceful labeling as a feasible solution to "G.lp". If G is not graceful, then CPLEX will find that "G.lp" is infeasible, from which we can conclude that no such graceful labeling of G exists.

Computational Results. Computational results have been obtained thus far for a number of graphs (see Table 1 at end of paper).

4.2. A CONSTRAINT PROGRAMMING FORMULATION AND ANALYSIS OF THE GRACEFUL LABELING PROBLEM USING OPL and OPLSTUDIO

Graceful Labeling Problem Formulation.

Find an integer label $vlabel(i)$ for each vertex i of $V(G)$, $i = 1, \ldots, n$

such that

Vertex label restrictions:

- $0 \leq vlabel(i) \leq e; \quad i = 1, 2, \ldots, n$
- Furthermore, all vertex labels must have different values.

Edge label restrictions:

- $elabel(ij) = |vlabel(i) - vlabel(j)|$ for each edge $ij \in E(G)$
- $1 \leq elabel(ij) \leq e$ for each edge $ij \in E(G)$
- Furthermore, all edge labels must have different values.

Analysis. The two mathematical programming formulations (both the integer programming formulation presented in Section 4.1 and the constraint programming formulation presented above) appear to be quite similar; indeed they are both formulations of the same problem. However, there are some key differences between the two.

Unlike the integer programming formulation, the above constraint programming formulation does not contain, nor does it require, an objective function. If we can obtain a solution $\{vlabel(1), \ldots, vlabel(n)\}$ that satisfies all of the above restrictions for a particular graph G, then G is a graceful graph and $\{vlabel(1), \ldots, vlabel(n)\}$ is a graceful labeling of the vertices $1, 2, \ldots, n$ of G. If no such solution exists, then no such graceful labeling of G exists; that is, G is not a graceful graph.

Another difference between the integer programming formulation and the above constraint programming formulation is that when constructing a constraint programming formulation we need not worry about linearizing constraints. For example, the OPLStudio constraint programming solver can handle nonlinear constraints involving absolute values and "not-equal-to" signs.

The above graceful labeling problem formulation can be written as an constraint programming formulation in OPL with a total of $n + e$ variables, $2e$ constraints, and $n + e$ variable bounds.

Implementation. The solving of specific graph instances G of the graceful labeling problem via constraint programming is implemented as follows: a C++ program first converts the graph file "G.edg" containing the number of vertices in G, the number of edges in G, and a list of G's edges ij, where $i < j$, into a OPL data file "G.dat". This data file is then loaded into the project "graceful.prj" and accompanies a general OPL model file "graceful.mod" which is based on the above formulation for OPLStudio to read and solve as a constraint programming problem. If a graceful labeling of G exists, then OPLStudio will find a graceful labeling as a solution to "graceful.prj" containing files "graceful.mod" and "G.dat". If G is not graceful, then OPL will terminate with the message "No solution found", from which we can conclude than so graceful labeling of G exists.

Computational Results. Computational results have been obtained thus far for a number of graphs. For some graphs, we are able to compute *all* the graceful labelings for that particular graph in a reasonable amount of time (see Table 2 at end of paper).

5. CONCLUSIONS / FUTURE WORK

This paper presents two mathematical programming formulations of the graceful labeling problem (first as an integer programming problem, second as a constraint programming problem), along with some new results on the gracefulness of three classes of graphs: generalized Petersen graphs $P(n, k)$, double cones $C_n + \overline{K_2}$, and product graphs of the form $K_4 \times P_n$. We look to obtain even greater results in the future regarding graphs in these three classes, as well as perhaps expand upon Theorem 2.1 to include even more types of graphs, and answer Questions 1, 2, and 3 presented above in Section 3. Implementation of both the integer and constraint programming formulations of the graceful labeling problem have already provided us with accurate and efficient computational results for some graceful and non-graceful graphs. Hopefully similar results can be achieved for larger graphs as well. Graceful labelings have been studied for over three decades, and the topic continues to be a fascinating one in the world of graph theory and discrete mathematics. An abundance of published papers and results exist, yet various unsolved problems and unproven conjectures continue to allow for the undertaking of even more research, with the hopes that new results will be obtained.

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TABLE 1.

Computing graceful labelings using an integer programming approach: all computations were performed on a Sun Enterprise 220R workstation with two 450-MHz UltraSPARC II cpu's, 2 Gb memory and 2 Gb swap space (harvey.caam.rice.edu)
$\star$ CPLEX v 5.0.0, $\star\star$ CPLEX v 6.0.1, $\star\star\star$ CPLEX v 6.5.2

Graph	n	e	Integer Variables	Binary Variables	Constraints	Graceful?	Time (s)
C_4	4	4	8	22	32	Yes	0.00 $\star\star$
P_4	4	3	7	15	27	Yes	0.01 $\star\star$
W_5	5	8	13	74	60	Yes	0.16 $\star\star$
$C_3 + \overline{K_2}$	5	9	14	91	65	Yes	0.17 $\star\star$
$C_4 + \overline{K_2}$	6	12	18	159	90	Yes	1.07 $\star\star\star$
K_5	5	10	15	110	70	No	3.44 $\star$
$K_4 \times P_2$	8	16	24	284	136	Yes	8.05 $\star\star$
$C_5 + \overline{K_2}$	7	15	22	246	117	Yes	37.01 $\star\star\star$
$P(5,1)$	10	15	25	270	165	Yes	72.30 $\star\star\star$
$P(5,2)$	10	15	25	270	165	Yes	154.97 $\star$

TABLE 2.

Computing graceful labelings using a constraint programming approach: all computations were performed on a Sun Enterprise 220R workstation with two 450-MHz UltraSPARC II cpu's, 2 Gb memory and 2 Gb swap space (harvey.caam.rice.edu)
⋆ OPLStudio v 2.1

Graph	n	e	Variables/Constraints	Graceful?	Time (s)	# of sols	Total time (s)
C_4	4	4	8/8	Yes	0.00 ⋆	16	0.00 ⋆
P_4	4	3	7/6	Yes	0.00 ⋆	4	0.00 ⋆
W_5	5	8	13/16	Yes	0.00 ⋆	64	0.17 ⋆
K_5	5	10	15/20	No	0.17 ⋆	0	0.17 ⋆
$P(5,1)$	10	15	25/30	Yes	0.04 ⋆		
$P(5,2)$	10	15	25/30	Yes	0.00 ⋆		
$P(6,1)$	12	18	30/36	Yes	0.06 ⋆		
$P(6,2)$	12	18	30/36	Yes	0.54 ⋆		
$P(6,3)$	12	15	27/30	Yes	0.05 ⋆		
$P(7,1)$	14	21	35/42	Yes	0.43 ⋆		
$P(7,2)$	14	21	35/42	Yes	3.69 ⋆		
$P(7,3)$	14	21	35/42	Yes	0.70 ⋆		
$P(8,1)$	16	24	40/48	Yes	3.95 ⋆		
$P(8,2)$	16	24	40/48	Yes	19.23 ⋆		
$P(8,3)$	16	24	40/48	Yes	77.04 ⋆		
$P(8,4)$	16	20	36/40	Yes	12.50 ⋆		
$P(9,1)$	18	27	45/54	Yes	71.07 ⋆		
$P(9,2)$	18	27	45/54	Yes	636.93 ⋆		
$P(9,3)$	18	27	45/54	Yes	144.78 ⋆		
$P(9,4)$	18	27	45/54	Yes	58.28 ⋆		
$P(10,1)$	20	30	50/60	Yes	16.26 ⋆		
$P(10,2)$	20	30	50/60	Yes	7311.20 ⋆		
$P(10,3)$	20	30	50/60	Yes	1648.89 ⋆		
$P(10,4)$	20	30	50/60	Yes	1109.34 ⋆		
$P(10,5)$	20	25	45/50	Yes	10481.40 ⋆		
$C_3 + \overline{K_2}$	5	9	14/18	Yes	0.00 ⋆	96	0.25 ⋆
$C_4 + \overline{K_2}$	6	12	18/24	Yes	0.04 ⋆	96	1.39 ⋆
$C_5 + \overline{K_2}$	7	15	22/30	Yes	0.13 ⋆	280	10.42 ⋆
$C_7 + \overline{K_2}$	9	21	30/42	Yes	0.01 ⋆		
$C_8 + \overline{K_2}$	10	24	34/48	Yes	0.52 ⋆		
$C_9 + \overline{K_2}$	11	27	38/54	Yes	498.22 ⋆		
$C_{11} + \overline{K_2}$	13	33	46/66	Yes	4.68 ⋆		
$C_{12} + \overline{K_2}$	14	36	50/72	Yes	39.61 ⋆		
$K_4 \times P_1$	4	6	10/12	Yes	0.00 ⋆	48	0.06 ⋆
$K_4 \times P_2$	8	16	24/32	Yes	0.02 ⋆	1440	65.17 ⋆
$K_4 \times P_3$	12	26	38/52	Yes	1.53 ⋆		
$K_4 \times P_4$	16	36	52/72	Yes	3.69 ⋆		
$K_4 \times P_5$	20	46	66/92	Yes	2592.50 ⋆		

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