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On Performance Bounds for Space-Time Coded Modulation on Fading Channels

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Abstract

In this paper, we evaluate truncated union bounds on the frame error rate (FER) performance of selected space-time codes for three different channel scenarios and compare them to results of computer simulation. Channels with AWGN, uncorrelated Rayleigh fading, and block Rayleigh fading are considered.

To calculate the bounds, we use a superstate trellis approach to characterize codeword differences according to performance measures derived from a general expression for the exact pairwise error probability (PEP) [1]. We observe that the computed bounds are very accurate for AWGN and uncorrelated fading, but quite loose for block fading. We then generalize the results of [2] to obtain a significantly tighter bound for the block fading case.

1. Introduction

Space-time coded modulation, introduced in [3], imposes a spatio-temporal structure onto the transmitted signal by allocating different symbols to different antennas. This structure is designed to guarantee a particular level of transmitter diversity, and to provide forward error correction capability when communicating over fading channels. To predict code performance, a union bound on the frame error rate as a function of signal-to-noise ratio (SNR) can be calculated by summing the exact pairwise error probabilities for all codeword pairs.

In [3] and [4], the analysis of an upper bound on the PEP identified the structural properties that govern code performance over fading channels. This has led to code construction methods that exploit these properties (for example, see [5]). A general method for computing the exact PEP for channels with different degrees of spatial and temporal correlation was presented in [1].

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To evaluate the union bound for specific codes, the pairwise codeword differences must be characterized according to the appropriate performance measure for a given channel model. The enumeration of the codeword difference measures and their multiplicities is analogous to the classical code "distance" spectrum. The definition of this measure is dependent upon the channel, and with the exception of AWGN, is not a true distance in the mathematical sense.

In the first part of this work, we extend to space-time codes the "distance" spectrum evaluation methods outlined in [6] in order to accommodate the performance measures for AWGN, uncorrelated Rayleigh fading, and block Rayleigh fading. In the case of block fading, we extend to space-time codes the modified bound of [2] which was shown to offer significantly improved bounds for convolutional codes.

In the second part, we compare computer simulations of specific space-time codes to the truncated union bound. In AWGN and uncorrelated fading, exact PEP-based bounds are asymptotically tight at high SNR, but quite loose in block fading. We show that by extending the block-fading bound in [2] to space-time codes, we achieve a more accurate prediction of code performance.

2. System Model

We consider a space-time coded system that employs L transmit antennas and M receive antennas such that the fading is spatially uncorrelated across all the antennas. The baseband constellation symbol transmitted by antenna i during time epoch n is denoted by $\sqrt{E_s}d_i(n)$ where $d_i(n)$ has normalized magnitude with average symbol energy E_s. Each codeword spans N time epochs, and during each time slot, L symbols are simultaneously transmitted from the L transmit antennas. We let N be the frame size. Each receive antenna observes a noisy superposition of the L transmitted symbols impaired by Rayleigh fading. The unit energy channel attenuation coefficients between transmit antenna i and receive antenna j at time epoch n are $\gamma_{ij}(n)$. Thus, at receive antenna j, the post match-filtered sampled receive signal is

$$r_j(n) = \sqrt{E_s}\sum_{i=1}^{L} d_i(n)\,\gamma_{ij}(n) + \eta_j(n) \qquad j = 1,\cdots,M;\;\; n = 1,\cdots,N, \tag{1}$$

where $\eta_j(n)$ is zero mean additive complex Gaussian noise with complex variance $N_o/2$. This general expression includes the cases of both uncorrelated and block fading, where the latter assumes that the attenuation coefficients are constant over the duration of one codeword of N symbol epochs, i.e., $\gamma_{ij}(n) = \gamma_{ij}\ \forall i,j;\ n = 1,\cdots,N$. For a single receive antenna, $j = 1$, a codeword a of code $\mathcal{C}$, denoted $\mathbf{D}_a = [\mathbf{D}_a(1),\cdots,\mathbf{D}_a(L)]$, is defined as an $N \times LN$ block matrix where each $\mathbf{D}_a(i)$ is an $N \times N$ diagonal codeword matrix of complex constellation symbols $d_i(n)$ (see [1]). We may write (1) in matrix form for one receive antenna ($M = 1$ and ignoring index j), namely,

$$\mathbf{r} = \mathbf{D}_a\boldsymbol{\gamma} + \boldsymbol{\eta}, \tag{2}$$

where $\mathbf{r} = [r(1),\cdots,r(N)]^{\mathrm{T}}$, $\boldsymbol{\eta} = [\eta(1),\cdots,\eta(N)]^{\mathrm{T}}$, and $\boldsymbol{\gamma} = \left[\boldsymbol{\gamma}_1^{\mathrm{T}},\cdots,\boldsymbol{\gamma}_L^{\mathrm{T}}\right]^{\mathrm{T}}$ with $\boldsymbol{\gamma}_i = [\gamma_i(1),\cdots,\gamma_i(N)]^{\mathrm{T}}$. (The superscript T indicates transposition.) We assume the receiver performs coherent detection and maximum likelihood decoding with perfect channel state information.

The encoding process is performed using a space-time convolutional encoder with state space Q and B state transitions (trellis branches) per state. Trellis branches are labeled with the L encoder output words, which are subsequently mapped to L constellation symbols.

2.1. Codeword Difference Covariance Matrix

The PEP, denoted $P(\mathbf{D}_a \to \mathbf{D}_b)$, is the probability of choosing codeword $\mathbf{D}_b$ instead of codeword $\mathbf{D}_a$ provided that $\mathbf{D}_a$ and $\mathbf{D}_b$ were the only two possible decoder outcomes. In [1], it was proved that the general expression for the exact PEP is a polynomial function of E_s/N_o and the eigenvalues of the codeword difference covariance matrix (CDCM),

$$\mathbf{C}_{\mathbf{D}} = \mathbf{D}_{ba}\mathbf{C}_{\boldsymbol{\gamma}}\mathbf{D}_{ba}^{\mathrm{H}}, \tag{3}$$

where $\mathbf{D}_{ba} = \mathbf{D}_b - \mathbf{D}_a$ is the codeword difference matrix for codewords a and b, and $\mathbf{C}_{\boldsymbol{\gamma}} = \mathrm{E}\left\{\boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{H}}\right\}$ is the channel covariance. (The superscript H indicates complex conjugate transposition, and E denotes the expectation operator.) By specializing $\mathbf{C}_{\boldsymbol{\gamma}}$ for a particular channel scenario, we modify the structure of $\mathbf{C}_{\mathbf{D}}$, resulting in different polynomial equations for the exact PEP [1].

For multiple spatially independent receive antennas, the CDCM is equal to

$$\mathbf{C}_{\mathbf{D},M} = \mathrm{diag}\left(\mathbf{C}_{\mathbf{D}},\cdots,\mathbf{C}_{\mathbf{D}}\right). \tag{4}$$

This has the effect of increasing the multiplicity of all the roots of the exact PEP eigenvalue polynomial equation by M.

3. Space-Time Code Error-Event Characterization

To characterize all error events (all pairwise codeword differences) according to the value of the performance measure, we choose to implement an extended version of the "General Algorithm" discussed in [6]. The complexity of this method grows with the square of the number of encoder trellis states. The algorithm considers all possible pairs of merged codeword paths by operating on a superstate trellis with $|Q|^2$superstates $\widehat{\sigma}_{mn}$ corresponding to all pairs of states in the original trellis. An error event in the original trellis, consisting of a pair of paths diverging in state σ_m and remerging in state σ_n, corresponds to a path in the new trellis that starts in superstate $\widehat{\sigma}_{mm}$ and ends in superstate $\widehat{\sigma}_{nn}$.

The performance measures are a function of the eigenvalues of $\mathbf{C}_{\mathbf{D}}$. We denote the non-zero elements of matrix $\mathbf{D}_{ba}$ by $d_i^{(ba)}(n)$, $i = 1,\cdots,L;\ n = 1,\cdots,N$, and the nth eigenvalue of $\mathbf{C}_{\mathbf{D}}$ as $\lambda_n(\mathbf{C}_{\mathbf{D}})$. In the remainder of this section, we describe the methods used to compute these eigenvalues for the three different channel scenarios. In this analysis, we assume one receive antenna. Using (4), the results can be generalized to multiple independent receive antennas.

3.1. AWGN Model

In AWGN, $\mathbf{C}_{\boldsymbol{\gamma}}$ is an all-ones matrix. This specializes the CDCM to $\mathbf{C}_{\mathbf{D}} = \widetilde{\mathbf{d}}_{ba}\widetilde{\mathbf{d}}_{ba}^{\mathrm{H}}$, where $\widetilde{\mathbf{d}}_{ba} = \left[\sum_{i=1}^{L} d_i^{(ba)}(1),\cdots,\sum_{i=1}^{L} d_i^{(ba)}(N)\right]^{\mathrm{T}}$. Since the rank of $\mathbf{C}_{\mathbf{D}}$ is one, it has a single non-zero eigenvalue, obtained by taking the trace of $\mathbf{C}_{\mathbf{D}}$, $\lambda(\mathbf{C}_{\mathbf{D}}) = \mathrm{tr}(\mathbf{C}_{\mathbf{D}}) = \widetilde{\mathbf{d}}_{ba}^{\mathrm{H}}\widetilde{\mathbf{d}}_{ba}$. To characterize error events in AWGN, we compute the eigenvalue using the expression $\lambda(\mathbf{C}_{\mathbf{D}}) = \sum_{n=1}^{N}\left|\sum_{i=1}^{L} d_i^{(ba)}(n)\right|^2$. Since $\mathrm{tr}(\mathbf{C}_{\mathbf{D}})$ is the Euclidean distance between codewords $\mathbf{D}_a$ and $\mathbf{D}_b$, it follows that the exact PEP in AWGN is $P(\mathbf{D}_a \to \mathbf{D}_b) = \mathrm{erfc}\left(\sqrt{\mathrm{tr}(\mathbf{C}_{\mathbf{D}})\frac{E_s}{4N_o}}\right)/2$, where $\mathrm{erfc}(\cdot)$ is the Gaussian error function.

3.2. Uncorrelated Fading Model

In uncorrelated fading, each channel is spatially and temporally independent. This reduces $\mathbf{C}_{\boldsymbol{\gamma}}$ to the identity matrix, and the CDCM simplifies to $\mathbf{C}_{\mathbf{D}} = \sum_{i=1}^{L}\mathbf{D}_{ba}(i)\,\mathbf{D}_{ba}^{\mathrm{H}}(i)$. The nth eigenvalue of $\mathbf{C}_{\mathbf{D}}$ is $\lambda_n(\mathbf{C}_{\mathbf{D}}) = \|\mathbf{d}_{ba}(n)\|^2$, where $\mathbf{d}_{ba}(n) =$

$\left[d_1^{(ba)}(n), \cdots, d_L^{(ba)}(n)\right]$ is the nth row of matrix $\mathbf{D}_{ba}$. (The notation, $\|\cdot\|^2$, is the standard vector inner product.) The measure for an error event is the product of these eigenvalues for the corresponding time epochs. The form of $\lambda_n(\mathbf{C_D})$ permits the calculation of an eigenvalue at each successive epoch of an error event.

3.3. Block Fading Model

In block fading, $\mathbf{C}_{\boldsymbol{\gamma}}$ is a block diagonal matrix of $N \times N$ all-ones matrices. Therefore, the CDCM is $\mathbf{C_D} = \sum_{n=1}^{N} \mathbf{C_D}(n)$, where $\mathbf{C_D}(n) = \mathbf{d}_{ba}^{\mathrm{H}}(n)\,\mathbf{d}_{ba}(n)$. This expression suggests an efficient epoch-by-epoch approach to computing $\mathbf{C_D}$. The measure of interest for each error event is the product of the non-zero eigenvalues of $\mathbf{C_D}$.

4. Probability of Frame Error in Block Fading

Terminated trellis codes of frame length N have a frame error rate that can be upper bounded using (e.g., [7])

$$P_f(e) \le 1 - (1 - P(e))^N \le NP(e), \qquad (5)$$

where $P(e)$ is the error-event probability. For space-time codes, $P(e)$ can be union bounded by

$$P(e) \le P_{\mathrm{ub}}(e) \stackrel{\mathrm{def}}{=} \sum_{\substack{\mathbf{D}_a,\, \mathbf{D}_b \in \mathcal{C} \\ \mathbf{D}_b \neq \mathbf{D}_a}} H(\mathbf{D}_a \to \mathbf{D}_b)\, P(\mathbf{D}_a \to \mathbf{D}_b), \qquad (6)$$

where $H(\mathbf{D}_a \to \mathbf{D}_b)$ is the multiplicity of the PEP $P(\mathbf{D}_a \to \mathbf{D}_b)$. We use the error-event characterization method of Section 3.3 to determine the collection of PEPs and multiplicities.

In block fading, evaluation of the union bound using the exact PEP yields a very loose estimate of the FER. However, a tight bound can be obtained by extending the results of [2]. To this end, we use the conditional pairwise error probability for block fading derived in [3],

$$P(\mathbf{D}_a \to \mathbf{D}_b | \boldsymbol{\gamma}) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_s}{4N_o}\boldsymbol{\gamma}^{\mathrm{H}}\mathbf{D}_{ba}^{\mathrm{H}}\mathbf{D}_{ba}\boldsymbol{\gamma}}\right). \qquad (7)$$

Let $\mathbf{C}_{\mathbf{D}}^{\mathrm{H}} = \mathbf{D}_{ba}^{\mathrm{H}}\mathbf{D}_{ba} = [c_{mn}]$, where $c_{mn} \in \mathbb{C}$ and $m, n \in \{1, \cdots, LM\}$, and write the complex fade terms in their polar form, $\gamma_k = a_k \mathrm{e}^{-j\theta_k}$; $\theta_k \in (-\pi, \pi)$ for $k = \{1, \cdots, LM\}$. Defining $\theta_{mn} = \theta_m - \theta_n$, and making use of the Hermitian properties of $\mathbf{C_D}$, we obtain

$$\boldsymbol{\gamma}^{\mathrm{H}}\mathbf{D}_{ba}^{\mathrm{H}}\mathbf{D}_{ba}\boldsymbol{\gamma} = \sum_{m=1}^{L} a_m^2 c_{mm} + 2\sum_{m=1}^{L}\sum_{n=1,\, n>m}^{L} a_m a_n \mathrm{Re}\left\{\mathrm{e}^{-j\theta_{mn}} c_{mn}\right\}. \qquad (8)$$

In (8), the a_m are Rayleigh distributed random variables with unit variance, and the θ_{mn} have the density $f_{\theta_{mn}}(\theta) = \frac{1}{2\pi}\left(1 - \frac{\theta\,\mathrm{sign}(\theta)}{2\pi}\right)$ for $\theta \in (-2\pi, 2\pi)$, and $f_{\theta_{mn}}(\theta) = 0$, otherwise. (The function $\mathrm{sign}(x) = 1, \forall x \ge 0$ and $\mathrm{sign}(x) = -1, \forall x < 0$.)

Code	AWGN		Uncor. Fading		Block Fading	
	Trace	Mult.	$\prod_k \lambda_k$	Mult.	$\prod_k \lambda_k$	Mult.
A	4	4	4	2	4	2
	8	20	16	5	12	4
B	4	1	16	1	4	0.5
	8	14	36	2	12	4.5
C	2	0.1875	16	1	8	1.5
	4	0.984	24	1	12	2.125

Table 1: Two lowest error-event measures and their multiplicities for *Codes A, B, C.*

Let $P_{\mathrm{ub}}(e|\,\mathbf{a}, \boldsymbol{\theta})$ be the conditional form of the upper bound $P_{\mathrm{ub}}(e)$ obtained by substituting (7) for the PEP in (6). Combining (5) through (8) and averaging over the fading magnitudes $\mathbf{a}$ and phases $\boldsymbol{\theta}$, we have

$$P_f(e) \le 1 - \int_{\mathbf{a}}\int_{\boldsymbol{\theta}} \left[1 - \min\left(1, P_{\mathrm{ub}}(e|\,\mathbf{a}, \boldsymbol{\theta})\right)\right]^N \cdot f_{\mathbf{a}}(\mathbf{a})\, f_{\boldsymbol{\theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta}\, d\mathbf{a}, \qquad (9)$$

where the minimization limits the conditional error-event probabilities before averaging over the fading process. This integration is $M\left(L^2 + L\right)/2$-fold and is performed numerically using the algorithm discussed in [8].

5. Results

In this section, we evaluate the upper bounds on the FER for three 4-state, 4-PSK codes: the delay diversity code of [3, Fig. 4], an improved minimum determinant code found in [9, (14)], and an optimal block fading code discussed in [5, Table I]. These are identified as *Code A*, *Code B*, and *Code C,* respectively. We list in Table 1 the two lowest error-event measures and their multiplicities for each of the example codes in all three channel scenarios for error-event lengths $l \le 100$.

For AWGN, we computed truncated union bounds on FER, using error events of length $l \le 6$, for one receive antenna, as well as for two receive antennas. As shown in Figure 1, there is excellent agreement with computer simulation results. As expected, the code performance is governed by the magnitude and multiplicity of the error events with the minimum trace.

For the uncorrelated fading channel, we computed truncated union bounds on FER, using error events of length $l \le 3$, for one receive antenna, as well as for two receive antennas. As shown in Figure 1, there is excellent agreement with computer simulation results. In addition, one can see that performance in uncorrelated fading is governed by the minimum eigenvalue product, shown in Table 1.

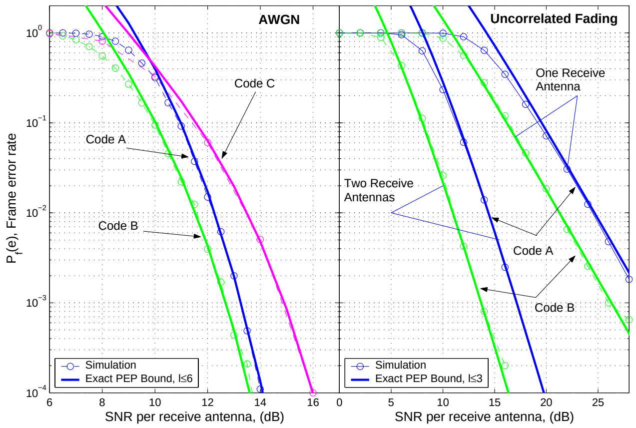

Figure 1: FER performance in AWGN and uncorrelated fading.

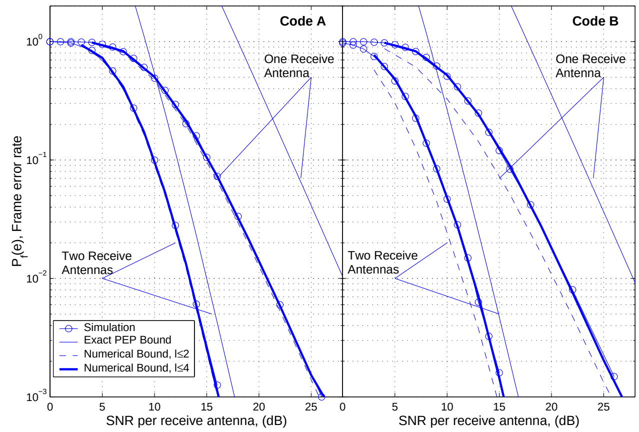

Figure 2: FER performance for *Code A* and *Code B* in block fading.

For block fading, we computed the exact-PEP bound for *Code A* and *Code B* using error events of length $l \leq 6$. As shown in Figure 2, these bounds are extremely loose, even at high SNR. We believe that this reflects the fact that in block fading there is no dominant error event, as was observed in [2]. Nevertheless, the bounds provide insight into relative performance. Figure 2 also shows modified bounds for *Code A* and *Code B*, computed using error events of length $l \leq 2$ and $l \leq 4$. For *Code A*, both bounds are very accurate over the full range of SNR, for both one and two receive antennas. For *Code B*, however, we observe that longer error events contribute significantly to the accuracy of the improved bound.

6. Conclusions

In this paper, we investigated the accuracy of truncated union bounds on the frame error rate (FER) for selected space-time codes in several channel conditions. For AWGN and uncorrelated Rayleigh fading channels, we show that bounds based upon the exact pairwise error probability (PEP) [1] are very tight at high SNR.

For the block fading channel, however, the PEP-based bound was found to be generally quite loose. Extending the results of [2], we derived a modified bound for the block fading case, and demonstrated its accuracy at both high and low SNR.

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