Serial Concatenated Trellis Coded Modulation with Inner Rate-1 Accumulate Code

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***Abstract*— In this paper we propose a Serial Concatenated Trellis Coded Modulation system using an inner accumulate code and a Gray-labeled signal constellation. The simple inner code and the Gray labeling allow us to extend to higher-order constellations the coding theorems of [1], [2], stating that when the signal-to-noise ratio (SNR) exceeds a certain, system-specific threshold, the word error probability goes to zero as the blocklength goes to infinity. We also evaluate the performance for finite blocklengths using an improved union bound.**

Despite the simple inner code, the simulated performance in AWGN and Rayleigh fading is equal to the performance of more complex systems suggested in the literature.

I. Introduction

In order to combine the extraordinary performance of turbo codes [3] with the bandwidth efficiency of Trellis-Coded Modulation (TCM) [4], several Turbo-TCM schemes have been proposed. Both Parallel Concatenated TCM (PCTCM) [5], [6] and Serial Concatenated TCM (SCTCM) [7], [8] have been shown to achieve good performance. Bit-Interleaved Coded Modulation (BICM) increases the diversity over fading channels, with only a modest performance degradation over AWGN channels [9]. BICM with iterative decoding (BICM-ID) gives almost the same performance as Turbo-TCM over AWGN channels, but at a lower complexity [10].

Inspired by the analytical tractability of Repeat-Accumulate (RA) codes [1] and their generalizations [11], we propose an SCTCM system consisting of an outer block code and an inner accumulate code, followed by a mapping to a Gray-labeled constellation. To achieve higher spectral efficiency, we use single parity check (SPC) or high-rate convolutional outer codes.

To analyze this system, we consider the distance properties of the underlying concatenated binary codes and use a mapping from Hamming distance to squared Euclidean distance (SED). Assuming Maximum-Likelihood (ML) decoding, we extend to higher-order constellations the coding theorems of [1], [2], stating that when the SNR γ exceeds a certain threshold, γ^*_{ML}, the word, or bit, error probability goes to zero as the blocklength goes to infinity. We compute numerical values for these thresholds for BPSK, 8-PSK and 16-QAM constellations for AWGN and Rayleigh fading channels.

For finite blocklengths, we use properties of the accumulate code to derive a new performance bound, which is an improvement over the union bound. We compare the simulated performance of the proposed system to similar systems. Over AWGN channels we compare to the SCTCM system reported in [8] and over fading channels to the BICM-ID system of [10].

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II. System Description

A. Encoder

The encoder is shown in Fig. 1. The outer code is a block code, formed either by terminating a rate $r_c = k/n$ convolutional code or by concatenating short block codes, such as $r_c = (n-1)/n$ SPC codes or $r_c = 1/n$ repeat codes. The interleaver is a random or S-random interleaver and acts on all bits in a block. The size of the interleaver is N and it is assumed that n divides N. The inner code is a $r - C = 1/1$ accumulate code, $G(D) = 1/(1 \oplus D)$. The memoryless mapper maps an m-tuple of bits to a constellation point $x \in \mathcal{X}$, where $\mathcal{X}$ is a Gray-labeled constellation of size $|\mathcal{X}| = M = 2^m$. Note that m is not necessarily equal to n.

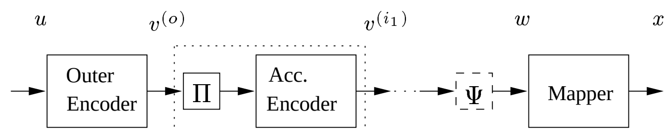

Fig. 1. The basic encoder uses a single inner accumulate code and no interleaver before the mapper. The generalized system uses multiple accumulate codes and non-trivial interleaver Ψ.

The generalization of the encoder to include multiple accumulate codes is straightforward. The dotted section in Fig. 1, the interleaver and accumulate code, is repeated the desired number of times. We can also introduce a non-trivial interleaver Ψ between the last accumulate code and the mapper. With no accumulate codes, but a non-trivial interleaver before the mapper, we get the BICM system of [9] as a special case.

B. Decoder

The decoder, shown in Fig. 2, consists of a bit metric calculator, soft-input soft-output (SISO) module(s) matched to the inner code(s) and an SISO module matched to the outer code, separated by appropriate interleavers and deinterleavers. The final, binary decision is given by a slicer.

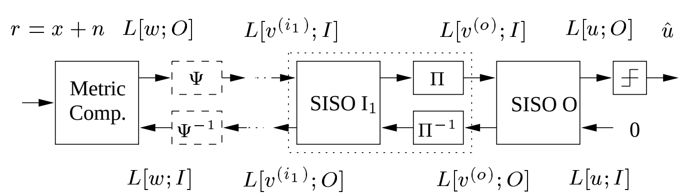

Fig. 2. The basic decoder has two SISO modules and does not feed back any a priori information to the metric computation. The general decoder can include more SISO modules and may feed back a priori information.

The output of the metric computation is the extrinsic probability that the i-th bit of the m-tuple w has the value b, given all available information except the information about the i-th bit itself. If the interleaver Ψ is non-trivial, we can assume that the bits that make up the symbol x are independent. Then the extrinsic probability is

$$P[w^i = b; O] \propto \sum_{x \in \mathcal{X}_b^i} P[r|x] \prod_{\substack{j=1 \\ j \neq i}}^{m} P[w^j = \ell^j(x); I] \quad (1)$$

where $\mathcal{X}_b^i = \{x \in \mathcal{X} | \ell^i(x) = b\}$ is the set of points in the constellation $\mathcal{X}$ such that the i-th bit, $i = 1, \ldots, m$, in the binary label of the point x has the value $b \in \{0, 1\}$. Log-likelihood ratios (LLR) are formed, $L[w; O] = \log(P[w = b; O]) - log(P[w = \bar{b}; O])$, and fed to the innermost SISO.

III. Channel Parameters

In this section we derive channel parameters z for higher-order constellations over AWGN and Rayleigh fading channels. In Section V we use the channel parameter z instead of the SNR γ in the coding theorems.

A. Gaussian Channels

For the binary-input AWGN channel, the Bhattacharayya bound gives the channel parameter z as $z = e^{-r_c E_b/N_0}$, where r_c is the code rate and E_b/N_0 is the bit SNR.

For higher-order, Gray-labeled constellations, let d_h^2 denote the minimum SED between symbols whose labels differ in h positions. From the Law of Cosines it follows that for PSK and QAM constellations

$$d_h^2 \geq h \cdot d_1^2 = h \cdot d_{\min}^2. \quad (2)$$

By first lower bounding the probability of correct decision, P_c, and then applying (2), we get the probability of h bit errors, P_{Eh} as

$$P_{Eh} \leqslant e^{-\frac{d_h^2}{8\sigma^2}} \leqslant e^{-\frac{h \cdot d_{\min}^2}{8\sigma^2}} = \left(e^{-\frac{d_{\min}^2}{8\sigma^2}}\right)^h. \quad (3)$$

Since $d_{\min}^2$ depends on E_b/N_0, the code rate r_c, and the number of bits per symbol m, we get

$$z = e^{-\frac{r_c m E_b}{4N_0} d_{\min}^2} \quad (4)$$

for any 2-dimensional Gray-labeled constellation.

B. Rayleigh Fading Channels

For an independently fading Rayleigh channel, consider two codewords differing in h positions and let η denote the index set for the differing positions. Then the channel parameter for BPSK, conditioned on the fading power, is

$$z_\alpha^h = \prod_{i \in \eta} e^{-\alpha_i r_c E_b/N_0} = e^{-\sum_{i \in \eta} \alpha_i r_c E_b/N_0} \quad (5)$$

where α_i is the fading power for the i-th differing bit. By integrating over the fading power density, we get the unconditional channel parameter as

$$z = \frac{1}{1 + r_c E_b/N_0}. \quad (6)$$

For higher-order constellations we get a similar expression. Since m bits are transmitted per channel symbol, the diversity order L may be lower than for BPSK. We bound the diversity order, normalized with respect to h, as $1/m \leqslant L \leqslant 1$, and get

$$z = \left[\frac{1}{1 + \frac{r_c m d_{\min}^2}{4} \frac{E_b}{N_0}}\right]^L. \quad (7)$$

Assuming that the performance is determined by codewords of low weight h, we set $L = 1$.

IV. Code Weight Enumerators

For serial concatenation of two codes through a uniform interleaver, the Input Output Weight Enumerator (IOWE) is given by

$$A_{w,h}^{(c)} = \sum_{h_o=0}^{n_o} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{n_o}{h_o}} \quad (8)$$

where $A_{w,h_o}^{(o)}$ and $A_{h_o,h}^{(i)}$ are the IOWE coefficients for the outer and inner codes. The overall codeword length is N, and n is the length of the "constituent" codewords when the code is made up of shorter block codes, and w and h are the input and output weights, respectively. Note that to compute the WE for a concatenated code, it suffices to know the WE of the outer code.

A. Repeat Codes

For a rate-$1/n$ repeat code, the IOWE is given by [1]

$$A_{w,h} = \begin{cases} \binom{N/n}{w} & h = nw \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

B. Parity Check Codes

For a rate $r_c = (n-1)/n$ SPC code, we upper bound the WE as

$$A_h \leqslant \binom{\lfloor n/2 \rfloor \cdot N/n}{h/2} \binom{n}{\lfloor n/2 \rfloor}^{h/2}. \quad (10)$$

The second factor in (10) can be approximated by $\binom{n}{\delta n}$, where $\delta \triangleq h/N$. The $r_c = 1/2$ and $r_c = 2/3$ SPC codes are special cases in that we have exact expressions for the WEs.

C. Accumulate Codes

The IOWE for an accumulate code is given by [1]

$$A_{w,h} = \binom{n-h}{\lfloor \frac{w}{2} \rfloor} \binom{h-1}{\lceil \frac{w}{2} \rceil - 1}. \quad (11)$$

V. Coding Theorems

In this section we extend the coding theorems of [1], [2] to higher-order, Gray-labeled constellations. The theorems state that when the SNR γ exceeds a threshold γ^*_{ML}, the word error probability, P_W, goes to zero as the blocklength N goes to infinity. Equivalently, when the channel parameter z is less than the corresponding threshold z^*, $P_W \to 0$ as $N \to \infty$. For Parity-Accumulate (PA) codes, the bit error probability, $P_b \to 0$ as $N \to \infty$.

The idea is to express the union bound on P_W in the form

$$P_W \leqslant \sum_{h>0} e^{h(F(\cdot)+\ln z)} \tag{12}$$

where $F(\cdot)$ depends on the outer and inner codes. Then, if $\ln z < -F(\cdot)$, $P_W \to 0$ as $N \to \infty$.

The inequality $d_h^2 \geq h \cdot d_1^2 = h \cdot d_{\min}^2$ holds for any constellation, as long as Gray labeling is possible. Thus, the following theorems hold for higher-dimensional constellations.

A. Repeat-Accumulate codes

For an outer repeat code of rate $r_c \leqslant 1/3$, and an inner accumulate code, there exists a z^* such that for $z < z^*$, $P_W \to 0$ as $N \to \infty$.

This follows from a straightforward application of the technique in [1], combined with the channel parameters in (4), (6) and (7). The E_b/N_0-thresholds γ^*_{ML} above which $P_W \to 0$ as $N \to \infty$ are given in Table I.

TABLE I

E_b/N_0-THRESHOLDS IN dB FOR P_W FOR RA CODES.

	AWGN			Fading		
rate	BPSK	8-PSK	16-QAM	BPSK	8-PSK	16-QAM
1/3	2.20	5.77	6.18	3.46	7.03	7.44
1/4	1.93	5.50	5.91	2.80	6.37	6.78

The thresholds for a $r_c = 1/3$, RA code with 8-PSK modulation over AWGN and Rayleigh fading channels are compared to simulations in Fig. 3. The thresholds are loose, since the threshold is based on the union bound, and the derivation of the channel parameter uses a loose bound (2).

B. Parity-Accumulate codes

For an outer SPC code, and a single inner accumulate code, there exists a z^* such that for $z < z^*$, $P_b \to 0$ as $N \to \infty$.

For the outer SPC code, the minimum Hamming distance is 2, and P_W does not go to zero as N goes to infinity [12]. However, by expressing P_b as

$$P_b \leqslant \sum_{h=1}^{N} \sum_{w=1}^{rN} \frac{w}{rN} A_{w,h} z^h \leqslant \sum_{h=1}^{N} \sum_{h_o=0}^{2h} \frac{h_o}{rN} \frac{A^{(o)}_{h_o} A^{(i)}_{h_o,h}}{\binom{N}{h_o}} z^h \tag{13}$$

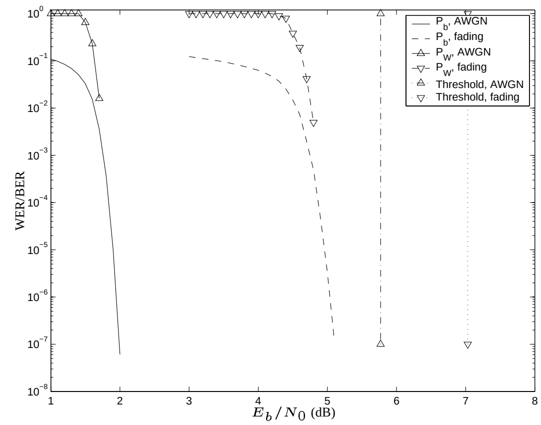

Fig. 3. Comparison of thresholds to simulations results for outer $r = 1/3$ repeat code with 8-PSK modulation. $N = 49152$.

we can show that $P_b \to 0$ as $N \to \infty$.

The numerical values for the E_b/N_0-thresholds are large. If, instead of using upper bounds, we approximate the WE in (10) and the IOWE in (11), we get smaller, approximate values. In Table II we report upper bounds and approximations for the thresholds of different rate PA codes for AWGN and Rayleigh fading channels.

TABLE II

E_b/N_0-THRESHOLDS IN dB FOR P_b FOR PA CODES, AWGN AND FADING CHANNELS.

	BPSK		8-PSK		16-QAM	
Rate	u.b.	appr.	u.b.	appr.	u.b.	appr.
	AWGN					
1/2	17.72	7.78	18.28	9.59	18.69	10.00
2/3	19.48	8.04	20.04	11.35	20.45	11.76
3/4	21.98	9.03	28.56	15.06	28.97	15.47
8/9	36.29	16.10	42.87	17.85	43.28	18.25
	Fading					
1/2		9.82		13.39		13.79
2/3		19.05		22.62		23.03
3/4		47.15		50.72		51.13
8/9		103.78		107.36		107.76

However, simulations show that the thresholds are quite loose. In Fig. 4 we compare the approximate thresholds for a $r_c = 2/3$, PA code with 8-PSK modulation to simulation results.

C. Parity-Accumulate-Accumulate codes

For an outer parity check code, and two (or more) inner accumulate codes, there exists a z^* such that for $z < z^*$, $P_W \to 0$ as $N \to \infty$.

For an outer SPC code followed by two accumulate codes,

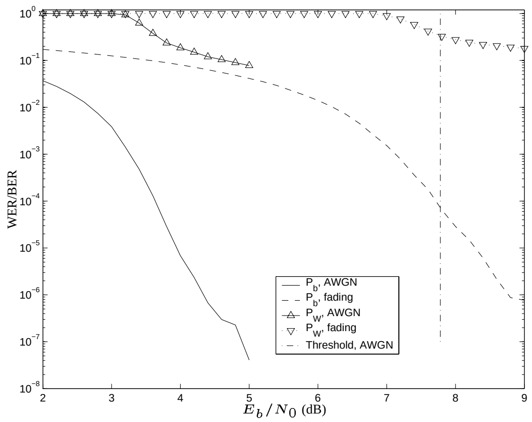

Fig. 4. Comparison of thresholds and simulations for the $r = 2/3$ parity check code and 8-PSK modulation over AWGN and Rayleigh channels.

the word error probability is given by

$$P_W \leqslant \sum_{h=1}^{N} \sum_{h_1=0}^{N} \sum_{h_2=0}^{N} A_{h_2}^{(o)} \frac{A_{h_2,h_1}^{(i_1)}}{\binom{N}{h_2}} \frac{A_{h_1,h}^{(i_2)}}{\binom{N}{h_1}}, \qquad (14)$$

which can be shown to go to zero as the blocklength grows to infinity. The thresholds derived from these bounds are so loose, however, that they are only interesting in the context of existence proofs.

VI. Finite Blocklength Analysis

In this section we derive a performance bound for finite blocklengths that improves on the union bound. The new bound is based on the simple relationship between inputs and outputs of the accumulate code, which can be used to tighten the relationship between Hamming distance and SED, as compared to (4).

The SED for a codeword of Hamming weight h is $h \cdot d_{\min}^2$ only if the codeword contains h runs of ones. If a codeword consists of t runs of 1's and $h > t$, then at least one run must be longer than one 1 and the SED may be larger. To count the number of runs, we augment the IOWE for the serial concatenation in (8) with a "run counter" denoted T,

$$A^{(c)}(W,H,T) = \sum_{w,h} \sum_{h_o}^{N} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} W^w H^h T^{\lceil \frac{h_o}{2} \rceil}. \qquad (15)$$

All runs must have at least one 1, so the remaining $h-t$ 1's are distributed among the t runs. A combinatorial argument gives the probability of a run of length $j,\ 1 \leqslant j \leqslant h-t+1$, as

$$P(j|h,t) = \frac{\binom{h-1-j}{t-2}}{\binom{h-1}{t-1}}. \qquad (16)$$

It remains to determine the SED for a run of length j. In Table III we list possible patterns and their SED for runs of length $j = 1, \ldots, 5$ for a Gray-labeled 8-PSK constellation.

TABLE III

Patterns and distances for runs of length 1 through 5 for an 8-PSK Gray-labeled constellation.

j	Patterns	SED	Min. SED	$d_H \cdot d_{\min}^2$
1	001	0.5858		
	010	3.4141	0.5858	0.5858
	100	0.5858		
2	001,100	1.1716		
	011	4	1.1716	1.1716
	110	2		
3	001,110	2.5858		
	011,100	4.5858	2.5858	1.7574
	111	3.4142		
4	001,111	4		
	011,110	6	4.000	2.3431
	111,100	4		
5	001,111,100	4.5858		
	011,111	7.4142	4.5858	2.9289
	111,110	5.4142		

Note that for runs of length 5, the patterns repeat the patterns of length 2, with the addition of a symbol '111'. For a constellation of size $|\mathcal{X}| = M = 2^m$, patterns of weight $2m-1$ or greater are repetitions of patterns of lower weight, with addition of a number of symbols of m 1's.

The SED is now a function of the constellation $\mathcal{X}$ and its labeling, the codeword weight h, and the number of runs t in the codeword. We denote this distance function $f(\mathcal{X},h,t)$ and get the improved bound as

$$P_b \leqslant \sum_{h=1}^{n} \sum_{w=1}^{k} \sum_{t=1}^{\lceil \frac{h}{2} \rceil} \frac{w}{k} A_{w,h,t}^{(c)} f(\mathcal{X},h,t) \qquad (17)$$

where $A_{w,h,t}^{(c)}$ is the coefficient of the augmented IOWE.

The use of a uniform interleaver gives non-integer weights in the IOWE, since this is the average IOWE over the ensemble of interleavers. We can expurgate codewords whose average multiplicity is less than 1, since there exist codes without codewords of those weights.

VII. Simulation Results

A system with an outer memory-2 convolutional code and an interleaver size $N = 1536$ was simulated to evaluate the improved bound. In Fig. 5 we compare the simulation results to the union bound and the improved bound and we see that for moderate E_b/N_0, the simulation results and the improved bound coincide.

In Fig. 6 we show the performance in AWGN and compare to the SCTCM scheme reported in [8]. Both systems use an outer $r_c = 3/4$, memory-2 convolutional code with 16-QAM modulation for a spectral efficiency of 3 bits/s/Hz, and $N = 16384$. Our system uses a simpler inner code, but still achieves the same performance.

As BICM was initially proposed for a fading channel, we compare the performance of the proposed SCTCM system to

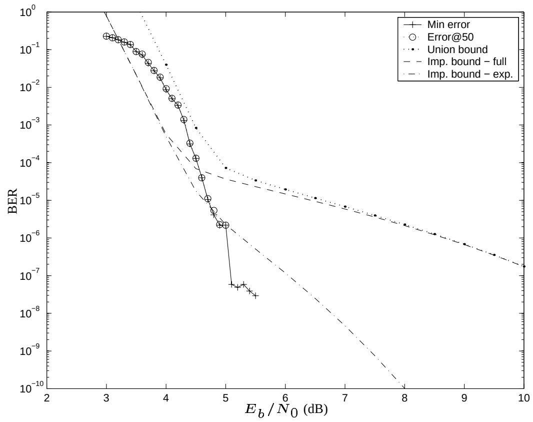

Fig. 5. Bounds for a memory-2 convolutional code and interleaver size $N = 1536$. "Full" refers to the complete IOWE, and "exp" refers to an IOWE where codewords of average weight less than 1 have been expurgated. "Min. error" is the minimum non-zero P_b seen between iterations 10 and 50.

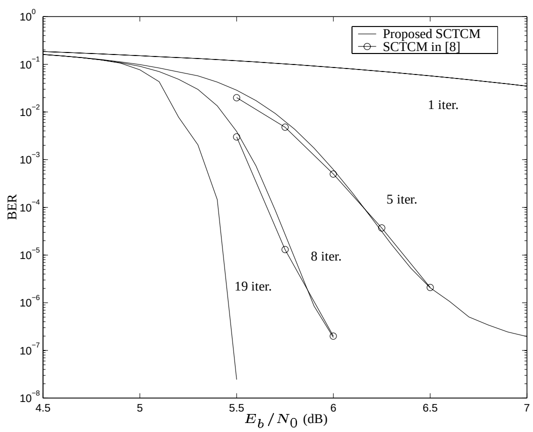

Fig. 6. Performance comparison in AWGN. 16-QAM constellation and block-length 4096 channel symbols. After 20 iterations we have not seen errors for $E_b/N_0 > 5.4$ dB.

BICM-ID [10] over a flat Rayleigh fading channel, shown in Fig. 7. Simulation results are shown for an outer $r_c = 2/3$ convolutional code with free distance 4, and $N = 49152$. At high bit error rates (BER), BICM-ID performs slightly better, but at low BER the proposed SCTCM system is advantageous. Also, BICM-ID exhibits an error floor, in fading as well as in AWGN [10].

VIII. CONCLUSIONS

An SCTCM system with an outer block code, inner accumulate code(s) and a higher-order Gray-labeled constellation has been introduced. We have derived coding theorems yielding SNR-thresholds for reliable performance as the blocklength goes to infinity. For finite blocklengths we have derived a new bound, which is an improvement of approximately 1 dB over the union bound.

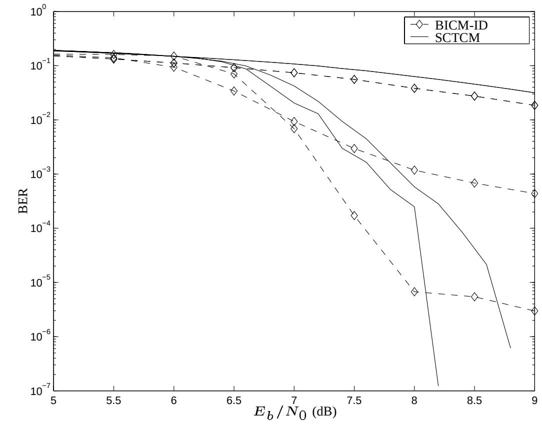

Fig. 7. Performance comparison in fading. Performance after 1, 5 and 16 iterations shown.

Though the proposed system uses a simple inner accumulate code, in AWGN it performs comparably to serial concatenation schemes reported in the literature [8] that use more complex rate-1 inner codes. Over a fading channel, the proposed SCTCM system performs better than BICM-ID at low bit error rates, since the proposed system does not exhibit an error floor.

REFERENCES

[1] D. Divsalar, H. Jin, and R. J. McEliece, "Coding theorems for "turbo–like" codes," in *Proc. 36th Annual Allerton Conf. on Commun., Control, and Comp.*, (Monticello, IL, USA), pp. 201–210, Sept. 1998.

[2] H. D. Pfister and P. H. Siegel, "Coding theorems for generalized repeat accumulate codes," in *Int. Symp. Inform. Theory and its Applications*, vol. 1, (Honolulu, HI, USA), pp. 21–25, IEEE, Nov. 2000.

[3] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo-codes," in *Proc. IEEE Int. Conf. Commun.*, vol. 2, (Geneva, Switzerland), pp. 1064–1070, IEEE, May 1993.

[4] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inform. Theory*, vol. 28, pp. 55–67, Jan. 1982.

[5] P. Robertson and T. Wörz, "Bandwidth-efficient turbo trellis-coded modulation using punctured component codes," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 201–218, Feb. 1998.

[6] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "Parallel concatenated trellis coded modulation," in *Proc. IEEE Int. Conf. Commun.*, vol. 2, (Dallas, TX, USA), pp. 974–978, IEEE, June 1996.

[7] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "Serial concatenated trellis coded modulation with iterative decoding," in *Proc. IEEE Int. Symp. Information Theory*, (Ulm, Germany), p. 8, IEEE, June – July 1997.

[8] D. Divsalar, S. Dolinar, and F. Pollara, "Serial concatenated trellis coded modulation with rate-1 inner code," in *Proc. IEEE Global Telecom. Conf.*, (San Francisco), pp. 777–782, IEEE, Nov. - Dec. 2000.

[9] E. Zehavi, "8-PSK trellis codes for a Rayleigh channel," *IEEE Trans. Commun.*, vol. 40, pp. 873–884, May 1992.

[10] X. Li and J. A. Ritcey, "Bit-interleaved coded modulation with iterative decoding using soft feedback," *Electronic Letters*, vol. 34, pp. 942–943, May 1998.

[11] H. D. Pfister and P. H. Siegel, "The serial concatenation of rate 1 codes through uniform random interleavers," in *Proc. 37th Annual Allerton Conf. on Commun., Control, and Comp.*, (Monticello, IL, USA), pp. 260–269, Sept. 1999.

[12] N. Kahale and R. Urbanke, "On the minimum distance of parallel and serially concatenated codes," in *Proc. IEEE Int. Symp. Information Theory*, (Cambridge, MA, USA), p. 31, IEEE, Aug. 1998.