GAUSSIAN MEASURES AS LIMITS ON IRREDUCIBLE SYMMETRIC SPACES AND CONES

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Abstract. We prove central limit theorems of Lindeberg-Lévy and Lindeberg-Feller type for any K-invariant random variables on all irreducible symmetric spaces and irreducible symmetric cones, completing in this way the numerous partial results known before. In all cases the limit measures turn out to be Gaussian and being such a limit characterizes these measures. On the other hand we show that other classical characterizations of Gaussian measures on $\mathbb{R}^n$, like 2-stability and the Bernstein theorem, are not true on symmetric spaces.

Introduction. — The important role of Gaussian measures on symmetric spaces of non-compact type is justified by the fact that they can be characterized as limit measures in central limit theorems on these spaces.

Symmetric cones are reducible symmetric spaces playing an important role in multivariate statistics(see eg. [L],[LM], [M],[MN] and the references in [L]). Studying central limit theorems on symmetric cones requires Jordan algebra techniques as well as methods of classical harmonic analysis and probability on symmetric spaces.

Different central limit theorems on symmetric spaces have been studied intensely and proved in recent years ([G1],[G2],[R],[S],[T],[Z]).

The central limit theorems of Lindeberg-Lévy type give the asymptotic behaviour of normalized products of independent identically distributed random variables. Such theorems were proved for K-invariant random variables with absolutely continuous distributions on symmetric cones ([R], [T],[Z]) and on the space $SO_o(p,q)/SO(p)\times SO(q)$([S]), in the first case without identifying limit measures as Gaussian ones. In this paper we prove them for any K-invariant random variables

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on any irreducible symmetric space or irreducible symmetric cone. Note that the central limit theorems on symmetric cones do not follow from central limit theorems on irreducible symmetric spaces. However the main ideas of their proofs are similar.

The central limit theorems of Lindeberg-Feller type deal with infinitezimal arrays of K-invariant random variables and are based on the knowledge of dispersions of these random variables, defined as expected values of some generalized quadratic forms on symmetric spaces. Such theorems were proved on irreducible symmetric spaces([G2]) and on the space $GL(n,\mathbb{R})/O(n)$([G1]). In this article we extend these results to all irreducible symmetric cones.

The basic tool for all our work is a Taylor expansion of spherical functions obtained using invariant differential operators.

We present our results on irreducible symmetric spaces in Section 1. In Section 2 we show that classical characterizations of Gaussian measures on $\mathbb{R}^n$ as 2-stable measures or as measures verifying the Bernstein property (independence of $X+Y$ and $X-Y$ when X,Y are independent random variables with the considered law) both fail on symmetric spaces. We discuss both a stability property related to the Cartan decomposition of the symmetric space and the stability in the sense of Tortrat. The symmetric cone case is studied in Section 3 which we wanted to make self-contained and accessible for readers familiar with symmetric cones but not with the general theory of symmetric spaces.

1. Lindeberg-Lévy central limit theorem on irreducible symmetric spaces. — Let G be a semisimple non-compact Lie group with finite center and let K be a maximal compact subgroup of G. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of the group G. The Riemannian symmetric space $X=G/K$ is *irreducible* if Ad_K acts irreducibly on $\mathfrak{p}$. Let $\mathfrak{a}\cong\mathbb{R}^r$ be a Cartan subspace of $\mathfrak{p}$. r is the rank of the symmetric space G/K. The dimension of $X=G/K$ is equal to $n=\dim\mathfrak{p}$.

Recall the Cartan decomposition $G=K(\exp\overline{\mathfrak{a}^+})K$ where $\mathfrak{a}^+$ is the positive Weyl chamber. We will denote by $x^+\in\overline{\mathfrak{a}^+}$ the middle component of $x\in G$ in this decomposition

$$x=k_1\exp(x^+)k_2.$$

On the level of the symmetric space $X=G/K$ we write $y=k\exp(y^+).x_o$, $y\in X$, $x_o=eK$.

In the sequel we identify the spaces $M^\natural(G)$ of K-biinvariant measures on G and $M^\natural(X)$ of K-invariant measures on $X=G/K$ (see e.g. [Ga]). In particular the convolution of K-invariant bounded measures on X is defined as the canonical projection on G/K of the convolution of their K-biinvariant counterparts on G.

The spherical functions on G are defined as K-biinvariant functions which are eigenfunctions of all G-invariant differential operators on G and are normalized by

$\phi(e) = 1$. They are all given by the Harish-Chandra formula

$$\phi_\lambda(x) = \int_K e^{\langle i\lambda-\rho, \mathcal{H}(xk)\rangle} dk, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^* \tag{1.1}$$

where $g = k \exp \mathcal{H}(g) n \in KAN$ is the Iwasawa decomposition of $g \in G$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$.

The *spherical Fourier transform* of a bounded measure $\mu \in M^\natural(G)$ is defined by

$$\hat{\mu}(\lambda) = \int_G \phi_\lambda(x^{-1}) d\mu(x)$$

for $\lambda \in \mathfrak{a}_{\mathbb{R}}^* + iC(\rho)$ where $C(\rho)$ is the convex hull of the orbit $W\rho$. The knowledge of $\hat{\mu}(\lambda)$ for $H \in \mathfrak{a}_{\mathbb{R}}^*$ is sufficient for the probabilistic applications of this paper. In particular the spherical Fourier transforms of two different K-biinvariant measures on G differ on $\mathfrak{a}_{\mathbb{R}}^*$.

As X is irreducible, the only K-invariant Gaussian semigroup on X is given by the heat semigroup $(\kappa_t)_{t>0}$ which is generated by the Laplace-Beltrami operator L on X(see [G2]). We have

$$\hat{\kappa}_t(\lambda) = e^{-t(\|\lambda\|^2+\|\rho\|^2)} = e^{t\gamma_L(\lambda)}$$

where $\gamma_L(\lambda) = -(\|\lambda\|^2 + \|\rho\|^2)$ is the eigenvalue of L on ϕ_λ: $L\phi_\lambda = \gamma_L(\lambda)\phi_\lambda$.

For all details concerning spherical harmonic analysis on symmetric spaces see the monograph [H]. We systematically use in this section the notations and definitions of this book.

Theorem 1.1. — *The Taylor expansion of order 2 of the spherical function $\phi_\lambda(\exp H)$ at $H = 0$ is*

$$\phi_\lambda(\exp H) = 1 + \frac{\gamma_L(\lambda)}{2n}\|H\|^2 + \sum f_\beta(\lambda) P_\beta(H) \tag{1.2}$$

where n is the dimension of X and P_β are W-invariant polynomials on $\mathfrak{a}$, homogeneous of order greater or equal to 3.

Proof. — In [G2] it was shown that

$$\phi_\lambda(\exp H) = 1 + b(\lambda)\|H\|^2 + \sum f_\beta(\lambda) P_\beta(H)$$

with $b(\lambda) = \dfrac{\gamma_L(\lambda)}{L(\|H\|^2)|_{H=0}}$. Using the well known formula for the radial part of the Laplace-Beltrami operator acting on a K-invariant function f([H, p. 267]):

$$Lf = L_{\mathfrak{a}}\tilde{f} + \sum_{\alpha\in\Sigma^+} m_\alpha \coth(\alpha(H))\langle \alpha, \operatorname{grad}\tilde{f}\rangle$$

where $\tilde{f}(H) = f(\exp H)$ we get $L(\|H\|^2)|_{H=0} = 2r + 2n_o$ where $n_o = \sum_{\alpha\in\Sigma^+} m_\alpha$ is the dimension of the nilpotent group N in the Iwasawa decomposition $G = KAN$. As $r + n_o = n$, the proof is finished. ∎

Let Y and Z be two independent K-invariant random variables with values in X and probability distributions μ_Y and μ_Z. Their composition $Y \circ Z$ is defined as any K-invariant random variable on X having the distribution $\mu_Y * \mu_Z$.

THEOREM 1.2. — *Let $(Y_m)_{m\geq 1}$ be a sequence of independent identically distributed K-invariant random variables on an irreducible symmetric space of non-compact type X. Suppose that*

$$\mathbb{E}(\|Y_1^+\|^2) = t > 0.$$

Let $S_m = Y_1 \circ Y_2 \circ \ldots \circ Y_m$ and $Y_j = k_j \exp(Y_j^+).x_o$. We normalize S_m by setting

$$S_m^\sharp = (k_1 \exp(\frac{Y_1^+}{\sqrt{m}}).x_o) \circ (k_2 \exp(\frac{Y_2^+}{\sqrt{m}}).x_o) \circ \ldots \circ (k_m \exp(\frac{Y_m^+}{\sqrt{m}}).x_o).$$

Then

$$S_m^\sharp \Rightarrow \kappa_{\frac{t}{2n}}, \qquad m \to \infty.$$

Proof. — We apply (1.2) in the study of the asymptotic behaviour of the spherical Fourier transform of $S_m^\sharp$ which is equal to

$$\Big(\int_G \phi_\lambda(\exp(-\frac{x^+}{\sqrt{m}}))d\mu(x)\Big)^m$$

where μ is the K-biinvariant probability measure on G corresponding to the probability distribution of the random variable Y_1. We easily see that this expression tends to $\exp(t\frac{\gamma_L(\lambda)}{2n})$ when $m \to \infty$. The proof is finished after an application of the Lévy continuity theorem on symmetric spaces([Ga]). ∎

Example 1.3. — Let $X = SO_o(p,q)/SO(p) \times SO(q)$ with $p \leq q$. The dimension of X is equal to pq. Theorem 1.2 implies that if $\mathbb{E}(\|Y_1^+\|^2) = t > 0$ then

$$S_m^\sharp \Rightarrow \kappa_{\frac{t}{2pq}}.$$

In particular, when $t = p$ what is equivalent to

$$\mathbb{E}(Y_1^+)_j^2 = 1$$

for all $j = 1, \ldots, p$ then the limit measure is $\kappa_{\frac{1}{2q}}$.

This is the result of [S] obtained there for random variables with densities (our normalization $\mathbb{E}(\|Y_1^+\|^2) = p$ boils down to that of [S] for absolutely continuous measures).

Example 1.4. — If $X = SL(k,\mathbb{R})/SO(k)$ then the dimension n of X is equal to the dimension of the space $\mathfrak{p}$ of anti-symmetric $k \times k$ matrices with zero trace, so $2n = k^2 + k - 2$. It follows from Theorem 1.2 that if $\mathbb{E}(\|Y_1^+\|^2) = t > 0$ then

$$S_m^\sharp \Rightarrow \kappa_{\frac{t}{k^2+k-2}}$$

and in particular, if $t = r = k-1$ then $S_m^\sharp \Rightarrow \kappa_{\frac{1}{k+2}}$.

Remark 1.5. — Theorem 1.2 can be expressed equivalently for K-biinvariant random variables on G. The composition of two such random variables ξ and ζ is simply given by their group product $\xi\zeta$.

For any random variable $Z = k\exp(Z^+).x_o$ and $c > 0$ let us denote by $Z_c^\flat$ the random variable $k\exp(\frac{Z^+}{c}).x_o$. Then

$$S_m^\sharp = (Y_1)^\flat_{\sqrt{m}} \circ (Y_2)^\flat_{\sqrt{m}} \circ \ldots \circ (Y_m)^\flat_{\sqrt{m}}.$$

Remark 1.6. — Observe that $(S_m)^\flat_{\sqrt{m}} \neq S_m^\sharp$, i.e. in order to obtain the normalized product $S_m^\sharp$ one must first normalize the random variables Y_j and then compose them and not the other way round. This may be deduced from the fact that $\mathbb{E}f(Y \circ Z) = \mathbb{E}f(Y)\mathbb{E}f(Z)$ for all independent K-invariant random variables Y and Z if and only if f is a bounded spherical function. On the other hand, if $c \neq 1$ then the function $H \to \phi_\lambda(\frac{H}{c})$ is not spherical when ϕ_λ is not constant.

Remark 1.7. — Let D_j be a dispersion as defined in [G2]. We are not able to compute $D_j(Y^\flat_{\sqrt{m}})$ knowing $D_j(Y)$. This is related to the impossibility of expressing of $\phi_\lambda(\exp(\frac{H}{c}))$ in terms of $\phi_{\lambda'}(\exp H)$. That is why Theorem 1.2 cannot be deduced from the central limit theorem for infinitesimal arrays with the dispersion D_j equal to t proved in [G2].

Conversely, it follows from both theorems that if $\mathbb{E}(\|Y_1^+\|^2) = t$ then

$$D_j(S_m^\sharp) = mD_j((Y_1)^\flat_{\sqrt{m}}) \to \frac{t}{2n}, \quad m \to \infty.$$

Remark 1.8. — On any irreducible symmetric space $D_j\kappa_t = t$ but the computation of $\mathbb{E}(\|W_t^+\|^2)$ when the random variable W_t has the distribution κ_t does not seem to be feasible in general.

If $X = SL(2,\mathbb{C})/SU(2)$ the density of the heat kernel in the Cartan coordinates is $\kappa_t(dh) = (4\pi t)^{-3/2}\frac{h}{\mathrm{sh} h}\exp(-t - \frac{h^2}{4t})J(h)dh$, $h \in \mathbb{R}$, $J(h) = 2\pi \mathrm{sh}^2 h$ and we can compute

$$\mathbb{E}(\|W_t^+\|^2) = \int_{\mathbb{R}} h^2 \kappa_t(dh) = 6t + 4t^2.$$

2. Gaussian measures and stability on symmetric spaces. — The Lindeberg-Lévy central limit theorem on $\mathbb{R}^n$ trivializes for independent random variables $Y_1, Y_2, \dots$ having all the same Gaussian law $N(0,\Sigma)$. Their normalized sum $\dfrac{Y_1+\dots+Y_m}{\sqrt{m}}$ has then the same law $N(0,\Sigma)$.

This is not the case on symmetric spaces of non-compact type, what we show in this section.

The above-mentioned property of Gaussian measures on $\mathbb{R}^n$ is called 2-stability.

It is natural to define an analogue of this property for K-invariant measures on symmetric spaces *via* the Cartan decomposition, used in the previous section in the normalization of sums of independent random variables.

We will say that a measure $\mu \in M^{\natural}(X)$ is *stable in a Cartan decomposition sense* if, for any $m \in \mathbb{N}$ and m independent random variables $Y_1, \dots, Y_m$ with the distribution μ, there exists a constant $c > 0$ such that

$$(Y_1)^\flat_c \circ (Y_2)^\flat_c \circ \dots \circ (Y_m)^\flat_c \overset{\mathrm{d}}{=} Y_1$$

where $\overset{\mathrm{d}}{=}$ denotes equality in law.

Proposition 2.1. — *If X is an irreducible symmetric space of non-compact type then the K-invariant Gaussian measures $(\kappa_t)_{t>0}$ are not stable in the above defined Cartan sense.*

Proof. — Suppose that $Y_1, Y_2, \dots$ are independent random variables with the same law κ_t. By the Cramér theorem on symmetric spaces([G3]), if the random variable product $(Y_1)^\flat_c \circ (Y_2)^\flat_c \circ \dots \circ (Y_m)^\flat_c$ had a Gaussian law then each random variable $(Y_1)^\flat_c$ would be Gaussian, too. It is then sufficient to show that if Y has the law κ_t then $Y^\flat_c$ is not Gaussian when $c \neq 1$.

Recall that the integration of a K-biinvariant function f on G can be written in the Cartan decomposition

$$\int_G f(x)dx = \int_{\mathfrak{a}^+} f(\exp H) J(H) dH$$

where $J(H) = \mathrm{vol}(K/M) \prod_{\alpha\in\Sigma^+} \mathrm{sh}^{m_\alpha}\langle\alpha, H\rangle$. As usual M denotes here the centralizer of $A = \exp\mathfrak{a}$ in K and $\alpha \in \Sigma^+$ are positive roots with multiplicity m_α.

Let us call $h_t(x), x \in G$ the density of the measure κ_t and $h_t^{(c)}$ the density of the probability distribution of $Y^\flat_c$. We have then

$$h_t^{(c)}(\exp H) = c^r \frac{J(cH)}{J(H)} h_t(\exp(cH)), \quad H \in \mathfrak{a}^+.$$

Let us suppose that $Y^\flat_c$ is Gaussian with distribution κ_s. Then $h_t^{(c)} = h_s$. By [AJ, Th. 3.7(i)], if we fix $H \in \mathfrak{a}^+$ and consider uH, $u \to \infty$, we have the estimates

$$h_t^{(c)}(\exp(uH)) \asymp P_t(ucH) e^{u(c-2)\langle\rho,H\rangle} e^{-u^2c^2\|H\|^2/4t}$$

$$h_s(\exp(uH)) \asymp P_s(uH) e^{-u\langle\rho,H\rangle} e^{-u^2\|H\|^2/4s}$$

where

$\asymp$ means that the quotient of two expressions on both sides of $\asymp$ has a lower bound $C_1 > 0$ and an upper bound $C_2 < +\infty$(for u sufficiently big)

$P_t(H) = \prod_{\alpha\in\Sigma^{++}}\langle\alpha, H\rangle(t + \langle\alpha, H\rangle)^{[(m_\alpha+m_{2\alpha})/2]-1}$ with Σ^{++} denoting the set of positive indivisible roots.

If $h_t^{(c)} = h_s$ than comparing the dominating factors of these estimates we get $s = t/c^2$. If this is the case, one must have $c - 2 = -1$ and $c = 1$. ∎

Corollary 2.2. — *If $Y_1, \ldots, Y_m$ are independent K-invariant random variables with a Gaussian law on X then their normalized product $S_m^\sharp$ has not a Gaussian law.*

Remark 2.3. — One could also define the Cartan stability of a measure $\mu \in M^\natural(X)$ by the existence of a constant $c > 0$ such that

$$(Y_1 \circ \ldots \circ Y_m)_c^\flat \overset{\mathrm{d}}{=} Y_1 \tag{2.1}$$

(cf. [F, p.170]). In the case of $\mathbb{R}^n$ both definitions are equivalent, but not on symmetric spaces(cf. Remark 1.2). Using the same argument as in the second part of the proof of the Theorem 2.1 we see that the Gaussian measures on X do not verify the condition (2.1), either.

The asymptotic behavior of different components of the product $Y_1 \circ \ldots \circ Y_m$ normalized like in (2.1)

$$S_m^\flat = (Y_1 \circ Y_2 \circ \ldots \circ Y_m)_{\sqrt{m}}^\flat$$

is studied in [LP] and [GR]. It does not seem possible to study it by methods of the spherical harmonic analysis.

On any topological group G one may consider probability measures which are stable in the sense of Tortrat([To]). A probability measure μ on G is strictly 2-stable in the sense of Tortrat if for any positive integer n and a random variable ξ with probability distribution μ the law of ξ^n is equal to μ^{*n^2}. In the case of $G = \mathbb{R}^n$ this definition is equivalent to the standard one(cf.[F]). Note that μ^{*n^2} is the law of the product $\xi_1 ... \xi_{n^2}$ of n^2 independent random variables with the law μ. The 2-stability of μ means that

$$\xi_1^n \overset{\mathrm{d}}{=} \xi_1 \ ... \ \xi_{n^2}.$$

Proposition 2.4. — *Let G/K be a Riemannian symmetric space of noncompact type. There exists $t_o > 0$ such that, if $\xi_1, \xi_2, \xi_3, \xi_4$ are independent K-invariant random variables with Gaussian distribution κ_t, with $t > t_o$, then*

$$\mathbb{E}\phi_o(\xi_1^2) \neq \mathbb{E}\phi_o(\xi_1\xi_2\xi_3\xi_4).$$

Proof. — We have $\mathbb{E}\phi_o(\xi_1\xi_2\xi_3\xi_4) = \hat{\kappa}_{4t}(0) = e^{-4\|\rho\|^2 t}$.

On the other hand, using the K-invariance of ϕ_o and κ_t we observe that

$$\mathbb{E}\phi_o(\xi_1^2) = \int_G \phi_o(x^2)d\kappa_t(x) = \int_K \int_G \phi_o(xkx)d\kappa_t(x) = \int_G \phi_o^2(x)d\kappa_t(x).$$

For $H \in \mathfrak{a}^+$ with $|H| < 1$ and $t > 0$ we deduce from the lower estimate of the heat kernel h_t [AJ, Th. 3.7(ii)] that

$$h_t(\exp H) \geq ct^{-n/2}(1+t)^{(m/2)-|\Sigma^{++}|}e^{-\|\rho\|^2 t}e^{-\|H\|^2/4t}$$

where $m = \dim N$ and the constant c is independent of $t > 0$. It follows that for $t \geq 1$

$$\begin{aligned}\mathbb{E}\phi_o(\xi_1^2) = &\int_{\mathfrak{a}^+} \phi_o^2(\exp H)h_t(\exp H)J(H)dH \\ &\geq c_1 t^{-n/2}(1+t)^{(m/2)-|\Sigma^{++}|}e^{-\|\rho\|^2 t}\int_{H\in\mathfrak{a}^+, |H|<1} e^{-\|H\|^2/4}J(H)dH \\ &= c_2 t^{-n/2}(1+t)^{(m/2)-|\Sigma^{++}|}e^{-\|\rho\|^2 t}\end{aligned}$$

where $c_1, c_2 > 0$. Hence, for t big enough, $\mathbb{E}\phi_o(\xi_1^2) \neq e^{-4\|\rho\|^2 t}$. ∎

Corollary 2.5. — *There exists $t_o > 0$ suth that for $t > t_o$ the Gaussian measures κ_t are not 2-stable in the sens of Tortrat.*

Remark 2.6. — Similarly, one proves that for sufficiently large t the Gaussian measures κ_t are not p-stable in the sens of Tortrat for any positive integer p.

We say that a probability measure μ on a group G has the Bernstein property (B) if, given two independent random variables ξ and η with values in G and having the same probability distribution μ, the random variables

$$(B) \qquad \xi\eta \text{ and } \xi\eta^{-1} \text{ are independent.}$$

Corollary 2.7. — *The Gaussian measures κ_t on a Riemannian symmetric space of non-compact type do not have the Bernstein property (B) for sufficiently large values of t.*

Proof. — If $\xi \overset{d}{=} \xi^{-1}$, what is true when the distribution of ξ is equal to κ_t, then the property (B) implies 2-stability:

$$\eta^2 = (\eta\xi^{-1})(\xi\eta)$$

and the law of $(\eta\xi^{-1})(\xi\eta) = (\xi\eta^{-1})^{-1}(\xi\eta)$ is equal to κ_{4t}. ∎

Remark 2.8. — The Gaussian measures κ_t on $X = G/K$ with K non-commutative have not the Bernstein property (B) also for sufficiently small $t > 0$. In fact, if they verified the condition (B), then by the Proposition 11[GL] and the remark after it, also the Haar measure on K would verify the condition (B). This is equivalent to

$$\int f(k^2)dk = \int f(k)dk$$

for any bounded Borel function f on K and may be shown to be false if K is a non-commutative connected compact Lie group.

Remark 2.9. — In [NRS] one can find a result contrary to the Corollary 2.7 and the Remark 2.8. The proof in [NRS] is not correct because a pair (ξ, η) of K-invariant random variables is in general not $K \times K$-invariant without the hypothesis of independence of ξ and η.

3. A reducible symmetric space case: symmetric cones. — In this section we show that our methods make it possible to obtain in a simple way central limit theorems also on Riemannian symmetric spaces which are not irreducible, provided one has some information about their structure.

An important class of such spaces is given by symmetric cones associated to simple Euclidean Jordan algebras(which is equivalent to say that the considered cones are irreducible, see [FK, III.4]).

Central limit theorems for normalized products of independent random variables on symmetric cones were studied by Terras([T]) and Richards([R]) in the case of $\mathcal{P}_r \cong GL(r, \mathbb{R})/O(r)$, the cone of positive definite real symmetric $r \times r$ matrices, and by Zhang([Z]) for all irreducible symmetric cones. Their results concern random variables whose distributions have densities.

In this section we prove this central limit theorem on irreducible symmetric cones without the assumption of absolute continuity of considered measures and with a different description of the limit measure as a convolution of two Gaussian measures. Our method of finding coefficients of the Taylor expansion of spherical functions, based on an application of invariant differential operators, is much simpler than the very computational methods of before-mentioned authors. We also prove a central limit theorem for an infinitesimal array of K-invariant measures on an irreducible symmetric cone, generalizing [G1].

For any needed information about analysis on symmetric cones we refer the reader to the monograph [FK]. Throughout this section we use the notations of this book.

We assume that V is a simple Euclidean Jordan algebra of dimension n as a real vector space. Let Ω be the associated symmetric cone:

$$\Omega = \{x^2 \mid \ x \in X \text{ invertible}\}.$$

The cone Ω is a Riemannian symmetric space G/K where G is the connected component of the identity in the automorphism group $G(\Omega) \subset GL(V)$ and $K = G \cap O(V)$.

Let $c_1, \ldots, c_r$ be a Jordan frame in V with $r = \mathrm{rank} V = \mathrm{rank} G/K$ as a symmetric space. The polar decomposition of any point $x \in \Omega$ writes

$$x = k.\sum_{j=1}^{r} e^{t_j} c_j, \qquad k \in K, t_j \in \mathbb{R}$$

which we abbreviate to

$$x = k.e^{\mathbf{t}} \ \text{ where } \ \mathbf{t} \in R = \{\mathbf{t} = \sum_{j=1}^{r} t_j c_j \mid \ t_j \in \mathbb{R}\} \cong \mathbb{R}^r.$$

Let us denote

$$R^+ = \{\mathbf{t} = \sum_{j=1}^{r} t_j c_j \mid \ t_1 < t_2 < \ldots < t_r\}.$$

Then every element $x \in \Omega$ may be represented as $x = k.e^{\mathbf{t}}$ with $\mathbf{t} \in R^+$ and such a $\mathbf{t}$ is unique. We define in this way the map

$$x \in \Omega \to \mathbf{t}(x)$$

which may be understood as the projection of Ω on R^+ via the polar decomposition; $\mathbf{t}(x)$ is the vector of logarithms of eigenvalues of x, ordered increasingly.

Let L be the Laplace-Beltrami operator on Ω and E the Euler operator on Ω:

$$Ef(x) = \frac{d}{dt} f(tx)\big|_{t=1}.$$

All the elliptic second order G-invariant differential operators on Ω are of the form

$$D = a(L - \frac{1}{r}E^2) + bE^2 + cE + d, \quad a, b > 0.$$

A K-invariant probability measure is called *Gaussian* if it belongs to the continuous semigroup of K-invariant measures $(\mu_t)_{t>0}$ generated by such an operator. It is called *Gaussian degenerate* if $a = 0$ or $b = 0$.

The semigroup $(\kappa_t)_{t>0}$ generated by L is called the *heat semigroup.*

The semigroups $(\eta_t)_{t>0}$ generated by $\frac{1}{2}E^2$ and $(\nu_t)_{t>0}$ generated by $L_1 = L - \frac{1}{r}E^2$ are Gaussian degenerate. They may be interpreted in the following way:

Let us call

$$\Omega_1 = \{x \in \Omega \mid \det(x) = 1\}.$$

Then Ω_1 is an irreducible symmetric space.

For example, if $\Omega = \mathcal{P}_r$ is the cone of positive definite symmetric matrices of dimension r, we have $\Omega_1 \cong SL(r, \mathbb{R})/SO(r)$.

We can identify in an evident way Ω with $\Omega_1 \times \mathbb{R}^+$.

The operators E and E^2 restrained to Ω_1 are equal to zero. The operator $L - \frac{1}{r}E^2$ restrained to Ω_1 is the Laplace-Beltrami operator on Ω_1.

Hence, the semigroups $(\eta_t)_{t>0}$ and $(\nu_t)_{t>0}$ are equal to the heat semigroups $(\eta_t^o)_{t>0}$ and $(\nu_t^o)_{t>0}$ on, respectively, $\mathbb{R}^+$ and Ω_1 extended to Ω by

$$\eta_t(B) = \eta_t^o(B \cap \mathbb{R}^+ e) \text{ and } \nu_t(B) = \nu_t^o(B \cap \Omega_1), \quad B \in \mathcal{B}_\Omega.$$

The spherical functions φ_λ on Ω ($\lambda \in \mathbb{C}^r$) which are traditionally used on symmetric cones ([FK, XIV.3]) differ slightly from spherical functions ϕ_λ given by the Harish-Chandra formula (1.1) on the symmetric space G/K:

$$\varphi_\lambda = \phi_{i\lambda}.$$

We still use systematically the approach and the notation of [FK].

If $\mu \in M^\natural(\Omega)$ then its spherical Fourier transform

$$\hat{\mu}(\lambda) = \int_\Omega \varphi_\lambda(x^{-1}) d\mu(x), \quad \lambda \in i\mathbb{R}^r$$

is always well defined and determines uniquely the measure μ .

Recall that $L\varphi_\lambda = \gamma_L(\lambda)\varphi_\lambda$ and $E\varphi_\lambda = \gamma_E(\lambda)\varphi_\lambda$ with

$$\gamma_L(\lambda) = \|\lambda\|^2 - \|\rho\|^2, \quad \gamma_E(\lambda) = \sum_{j=1}^r \lambda_j$$

where $\rho = (\rho_j)_{j=1\dots r}$, $\rho_j = \dfrac{d}{4}(2j - r - 1)$ with d equal to the degree of the Jordan algebra V.

It follows that the spherical Fourier transforms of κ_t, η_t and ν_t are:

$$\hat{\kappa}_t(\lambda) = e^{t\gamma_L(\lambda)}, \quad \hat{\eta}_t(\lambda) = e^{\frac{t}{2}\gamma_E^2(\lambda)}, \quad \hat{\nu}_t(\lambda) = e^{t(\gamma_L(\lambda) - \frac{1}{r}\gamma_E^2(\lambda))}.$$

THEOREM 3.1. — *The Taylor expansion of order 2 of the spherical function $\varphi_\lambda(e^{\mathbf{t}})$ in $\mathbf{t} = 0$ is*

$$\varphi_\lambda(e^{\mathbf{t}}) = 1 + a(\lambda)\sum_{j=1}^r t_j \; + \; b(\lambda)\sum_{j=1}^r t_j^2 + c(\lambda)(\sum_{j=1}^r t_j)^2 + R_\lambda(\mathbf{t})$$

where
$$a(\lambda) = \frac{1}{r}\gamma_E(\lambda)$$
$$b(\lambda) = \frac{r\gamma_L(\lambda) - \gamma_E^2(\lambda)}{dr(r-1)(r+2d^{-1})}$$
$$c(\lambda) = \frac{(dr - d + 2)\gamma_E^2(\lambda) - 2\gamma_L(\lambda)}{2dr(r-1)(r+2d^{-1})}$$

and $R_\lambda(\mathbf{t}) = \sum f_\beta(\lambda) P_\beta(\mathbf{t})$ *where* $P_\beta(\mathbf{t})$ *are symmetric polynomials in* $t_1, \ldots, t_r$ *homogeneous of order greater or equal to 3.*

Proof. — The proof is similar to the proof of the Theorem 3 in [G1]. We apply the operators E, E^2 and L to the Taylor expansion of $\varphi_\lambda(e^{\mathbf{t}})$ in order to determine its coefficients up to the order 2. ∎

Let Y and Z be two independent K-invariant random variables with values in Ω. Their composition $Y \circ Z$ is defined, like on any symmetric space, as a random variable with the law $\mu_Y * \mu_Z$, the convolution of K-invariant measures on Ω being defined as the projection on Ω of the convolution of the corresponding K-biinvariant measures on G.

In the symmetric cone case one may give an equivalent definition of the composition $Y \circ Z$, based on the following fact. We denote by $P(x)$, $x \in V$ the quadratic representation of V and we recall that if $x \in \Omega$ then there exists a unique $y \in \Omega$ such that $y^2 = x$. We then write $y = x^{1/2}$.

PROPOSITION 3.2. — *Let* Y *and* Z *be two independent* K*-invariant random variables on* Ω *with probability distributions* μ *and* ν*. Then the convolution* $\mu * \nu$ *is the probability distribution of the random variable* $P(Y^{1/2})Z$ *and of the random variable* $P(Z^{1/2})Y$*.*

Proof. — Let $e \in \Omega$ be fixed by the action of K. If $g \in G$ then $ge = x \in \Omega$ and there exists $k_o \in K$ such that $g = P(x^{1/2})k_o$ ([FK, III.5.1]).

Let $f(x)$ be a bounded Borel function on Ω and F a bounded right K-invariant Borel function on G.

The correspondence between a measure m on Ω and its right K-invariant analogue $\tilde{m}$ on G is given by

$$\int_\Omega f(x) dm(x) = \int_G f(ge) d\tilde{m}(g); \qquad \int_G F(g) d\tilde{m}(g) = \int_\Omega F(P(x^{1/2})) dm(x).$$

We get, using these equalities and the K-invariance of Y

$$\begin{aligned}
\langle \mu * \nu, f \rangle &= \int_G \int_G f(ghe) d\tilde{\mu}(g) d\tilde{\nu}(h) = \int_G \Big(\int_\Omega f(gy) d\nu(y) \Big) d\tilde{\mu}(g) = \\
&= \int_G \int_K \Big(\int_\Omega f(gky) d\nu(y) \Big) dk \, d\tilde{\mu}(g) = \int_\Omega \int_K \Big(\int_\Omega f(P(x^{1/2})ky) d\nu(y) \Big) dk \, d\mu(x) \\
&= \int_\Omega \int_\Omega f(P(x^{1/2})y) d\mu(x) d\nu(y) = \mathbb{E} f(P(Y^{1/2})Z).
\end{aligned}$$

∎

Hence, on the symmetric cones we may define

$$Y \circ Z = P(Y^{1/2}) Z.$$

In particular, on the matrix symmetric cones $\Pi_r(\mathbb{R}) = \mathcal{P}_r$, $\Pi_r(\mathbb{C}), \Pi_r(\mathbb{H})$ and $\Pi_3(\mathbb{O})$ we may put
$$Y \circ Z = Y^{\frac{1}{2}} Z Y^{\frac{1}{2}}.$$
Note that we have $P(Y^{1/2})Z \stackrel{\mathrm{d}}{=} P(Z^{1/2})Y$, but in general $P(Y^{1/2})Z \neq P(Z^{1/2})Y$.

THEOREM 3.3. — *Let $(Y_m)_{m\geq 1}$ be a sequence of independent identically distributed K-invariant random variables with values in Ω such that*

$$\mathbb{E}(\mathbf{t}(Y_1)) = \mathbf{0} \quad \text{and} \quad \mathbf{Cov}(\mathbf{t}(Y_1)) = \mathbb{E}(\mathbf{t}(Y_1)^T \mathbf{t}(Y_1)) = \mathbf{Id}.$$

Let $S_m = Y_1 \circ Y_2 \circ \ldots \circ Y_m$. Let us normalize S_m by putting

$$S_m^\sharp = (k_1.e^{\frac{\mathbf{t}(Y_1)}{\sqrt{m}}}) \circ \ldots \circ (k_m.e^{\frac{\mathbf{t}(Y_m)}{\sqrt{m}}}) \qquad \textit{when} \quad Y_j = k_j.e^{\mathbf{t}(Y_j)}.$$

$$\textit{Then} \quad S_m^\sharp \Rightarrow \kappa_{\frac{1}{r+2d^{-1}}} * \eta_{\frac{1}{r+2d^{-1}}} = \nu_{\frac{1}{r+2d^{-1}}} * \eta_{\frac{r+2}{r(r+2d^{-1})}}, \qquad m \to \infty.$$

Proof. — Let μ be the law of Y_1 and μ_m the law of $S_m^\sharp$. We study the behaviour of the spherical Fourier transform

$$\hat{\mu}_m(-\lambda) = \Big(\int_\Omega \phi_\lambda(e^{\frac{\mathbf{t}(x)}{\sqrt{m}}}) d\mu(x)\Big)^m$$

when $m \to \infty$ and $\lambda \in i\mathbb{R}^r$. By Theorem 3.1 we see that

$$\hat{\mu}_m(\lambda) \to e^{r(b(\lambda)+c(\lambda))} = \exp(\frac{1}{2(r+2d^{-1})}\gamma_E^2(\lambda) + \frac{1}{r+2d^{-1}}\gamma_L(\lambda)).$$

An application of the Levy-Gangolli continuity theorem([Ga]) ends the proof. ∎

Remark 3.4. — When the distribution μ of Y_1 has a density with respect to the G-invariant measure $d\dot{x} = (\det x)^{-n/r} dx$ on Ω then the normalization hypothesis of the Theorem 3.3 coincide with those of [R],[T] and [Z].

Let us now present briefly the Lindeberg-Feller type central limit theorem for irreducible symmetric cones, generalizing the main result of [G1].

The *dispersions* D_j, $j = 1, \ldots r-1$ having the property

$$D_j(Y) \geq 0, \qquad \text{and} \quad D_j(Y \circ Z) = D_j(Y) + D_j(Z)$$

for independent K-invariant random variables Y, Z and such that

$$D_j(\kappa_t) = D_j(\nu_t) = t, \quad t > 0$$

are defined by
$$D_j(Y) = \mathbb{E}Q_j(Y)$$

where the generalized quadratic forms Q_j of the symmetric pair (G,K) (see e.g. [FH]) are defined by

$$Q_j(x)=\frac{1}{d}(\frac{\partial}{\partial\lambda_{j+1}}-\frac{\partial}{\partial\lambda_j})\big|_{\lambda=\rho}\varphi_\lambda(x)=\frac{1}{d}(\frac{\partial}{\partial\lambda_j}-\frac{\partial}{\partial\lambda_{j+1}})\big|_{\lambda=-\rho}\varphi_\lambda(x).$$

If we denote $\partial_j=\frac{1}{d}(\frac{\partial}{\partial\lambda_{j+1}}-\frac{\partial}{\partial\lambda_j})|_{\lambda=\rho}$ and $\tilde\partial_j=\frac{1}{d}(\frac{\partial}{\partial\lambda_j}-\frac{\partial}{\partial\lambda_{j+1}})|_{\lambda=-\rho}$ then

$$D_j(Y)=\partial_j(\widehat{\check{\mu}_Y})=\tilde\partial_j(\widehat{\check{\mu}_Y})$$

where $\check f(\lambda)=f(-\lambda)$.

For example, if for $p>\frac{d}{2}(r-1)$ and $c>0$

$$\gamma_{p,ce}(dx)=\frac{c^{-rp}}{\Gamma_\Omega(p)}\,e^{-\frac{1}{c}\mathrm{tr}(x)}\det(x)^{p-\frac{n}{r}}\,dx$$

is a K-invariant Wishart distribution on Ω, then

$$\hat\gamma_{p,ce}(\lambda)=\frac{c^{-\sum_{j=1}^r\lambda_j}}{\Gamma_\Omega(p)}\Gamma_\Omega(\rho-\lambda+p)$$

pour $\mathrm{Re}\lambda_j<p-\frac{d}{4}(r-1)$ and

$$D_j(\gamma_{p,ce})=\frac{1}{d}\big[\psi(p-(j-1)\frac{d}{2})-\psi(p-j\frac{d}{2})\big]$$

where ψ is the digamma function $\psi=\Gamma'/\Gamma$.

Theorem 3.5. — *Let (Y_{mj}), $m\in\mathbb{N},1\le j\le k_m$ be a triangular array of K-invariant random variables such that for each $m\in\mathbb{N}$ the variables $Y_{m1},\ldots,Y_{mk_m}$ in the m-th row of the array are independent. Define*

$$Z_m=Y_{m1}\circ Y_{m2}\circ\ldots\circ Y_{mk_m}.$$

Suppose that the random variables (Y_{mj}) satisfy

$$\mathbb{E}(\log(\det(Y_{mj}))=0 \tag{3.1}$$

$$\frac{1}{r^2}\mathbf{Var}(\log(\det(Z_m))\to u,\qquad m\to\infty \tag{3.2}$$

$$D_1(Z_m)\to t,\qquad m\to\infty. \tag{3.3}$$

If the technical condition

$$\lim_{m\to\infty}\sum_{j=1}^{k_m}\big[\mathbb{E}(\frac{\|\mathbf{t}(Y_{mj})\|^3}{1+\|\mathbf{t}(Y_{mj})\|^2})+\mathbb{E}(\log^2(\det(Y_{mj})),\|\mathbf{t}(Y_{mj})\|>1)\big]=0 \tag{3.4}$$

is verified, then

$$Z_m \Rightarrow \nu_t * \eta_u, \qquad m \to \infty.$$

Proof. — Using Theorem 3.1 we obtain the following expansion of the spherical function $\varphi_\lambda(e^{\mathbf{t}})$ in $\mathbf{t} = 0$:

$$\varphi_\lambda(e^{\mathbf{t}}) = 1 + \frac{\gamma_E(\lambda)}{r}\sum_{j=1}^{r} t_j + (\gamma_L(\lambda) - \frac{1}{r}\gamma_E^2(\lambda))Q_1(e^{\mathbf{t}})$$
$$+ \frac{1}{2r^2}\gamma_E^2(\lambda)(\sum_{j=1}^{r} t_j)^2 + \tilde{R}_\lambda(\mathbf{t})$$

with the rest $\tilde{R}_\lambda(\mathbf{t})$ of the same form as the rest $R_\lambda(\mathbf{t})$ in the Theorem 3.1.

Observe that for $x \in \Omega$

$$\log(\det x) = \sum_{j=1}^{r} t_j(x).$$

The rest of the proof proceeds along the lines of the proof of Theorem 4 in [G1]. ∎

If we consider a triangular array (Y_{mj}) as in the Theorem 3.5 with normalization conditions analogous to those of the Theorem 3.3, then one proves in a similar way as the Theorem 3.5 the following central limit theorem.

THEOREM 3.6. — *Suppose that*

$$\mathbb{E}(\mathbf{t}(Y_{mj})) = \mathbf{0} \quad \text{and} \quad \sum_{j=1}^{k_m} \mathbf{Cov}(\mathbf{t}(Y_{mj})) = \mathbf{Id}.$$

If the following technical condition

$$\lim_{m\to\infty} \sum_{j=1}^{k_m} \mathbb{E}(\frac{\|\mathbf{t}(Y_{mj})\|^4}{1 + \|\mathbf{t}(Y_{mj})\|^2}) = 0$$

is verified then

$$Z_m \Rightarrow \nu_{\frac{1}{r+2d^{-1}}} * \eta_{\frac{r+2}{r(r+2d^{-1})}}, \qquad m \to \infty.$$

Remark 3.7. — Note that Theorem 3.3 is a special case of the Theorem 3.6. In fact, if $Y_1, Y_2, \ldots$ is a sequence of independent identically distributed K-invariant random variables as in the Theorem 3.3 , set

$$Y_{mj} = k.e^{\frac{\mathbf{t}(Y_j)}{\sqrt{m}}} \quad \text{when} \quad Y_j = k.e^{\mathbf{t}(Y_j)}.$$

Then the random variables Z_m and $S^\sharp_m$ are equal.

Remark 3.8. — The technical condition of the Theorem 3.6 is stronger than the condition (3.4) of the Theorem 3.5.

Remark 3.9. — If we replace the condition (3.4) by

$$\forall \epsilon > 0 \quad \lim_{m\to\infty} \sum_{j=1}^{k_m} \mathbb{E}\big[Q_1(Y_{mj}) + \log^2(\det(Y_{mj})), \|\mathbf{t}(Y_{mj})\| > \epsilon\big] = 0 \tag{3.5}$$

then the normalization hypotheses (3.1),(3.2),(3.3) together with the condition (3.5) are sufficient and necessary for the weak convergence of Z_m to $\nu_t * \eta_u$. The condition (3.4) implies (3.5) and in the proof of the Theorem 3.5 we prove in fact the sufficiency of (3.1),(3.2),(3.3) and (3.5). The proof of necessity is similar to the classical proof of necessity in the Lindeberg-Feller Central Limit Theorem on $\mathbb{R}^n$([F]).

Remark 3.10 . — The Remarks 3.7, 3.8 and 3.9 also apply to irreducible symmetric spaces. For the irreducible symmetric spaces $SL(2,\mathbb{R})/SO(2)$ and $SL(2,\mathbb{C})/SU(2)$ the conditions (3.4) and (3.5) (simplified by the disappearence of terms with the determinant) are equivalent (see the explicit formulae for Q in these cases in [G1] and [G2]).

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