On heredity of various type of proximal actions

C. R. E. Raja

Abstract

We prove that action of a group T on compact metric space X by homeomorphisms is proximal if and only if T action on $\mathcal{P}(X)$ is strongly proximal and we obtain the same result for actions on exponential Lie groups by automorphisms. We also prove that for any $\mu \in \mathcal{P}(T)$, X is μ-proximal or X is mean proximal if and only if so is $\mathcal{P}(X)$.

KEYWORDS Proximal and strongly proximal actions, μ-proximal, mean proximal and probability measures.

Let X be a complete separable metric space. Let T be a semigroup acting on X by continuous selfmaps.

Definition A system (X, T) is a pair consiting of a complete separable metric space X and a semigroup T acting on X by continuous selfmaps. In such a situation X is a called a T-space.

Definition Two points x and y in a T-space X are said to be *proximal* if there exists a sequence (t_n) in T such that

$$\lim t_n x = \lim t_n y.$$

Definition We say that a system (X, T) is *proximal* or the action of T on X is *proximal* if any two points x and y in X are proximal.

It is easy to see that group of special linear automrophisms on $\mathbb{R}^n$ action on $\mathbb{R}^n$ is proximal and the compact group actions are not proximal.

Let $\mathcal{P}(X)$ be the space of all regular Borel probability measures on X, equipped with the weak* topology with respect to all continuous bounded functions. It may be seen that $\mathcal{P}(X)$ equipped with the weak* topology is a complete separable metric space (see [P]). The map $x \mapsto \delta_x$, maps X homeomorphically onto a closed subset δ_X, say of $\mathcal{P}(X)$ (see [P]) where δ_x is the measure concentrated at the point x. Suppose a semigroup T acts on X by continuous selfmaps. Then the action of T on X extends to an action on $\mathcal{P}(X)$ in the following natural way, for any $\lambda \in \mathcal{P}(X)$ and any $t \in T$

$$t\lambda(E) = \lambda(t^{-1}E)$$

for any Borel subset E of X.

Definition We say that a system (X, T) is *strongly proximal* or the action of T on X is *strongly proximal* if for any $\lambda \in \mathcal{P}(X)$, there exists a sequence $(t_n) \subset T$ such that

$$t_n\lambda \to \delta_x$$

for some $x \in X$.

By considering $\frac{1}{2}(\delta_x + \delta_y)$ for any $x, y \in X$, it is easy to see that any strongly proximal system is proximal; see [F1], [G] and [M] for more details on proximal and strongly proximal systems. But not all proximal systems are strongly proximal. The action of the special Linear group $SL(V)$ on V is proximal but it is not strongly proximal.

Remark 1 Suppose there exists a sequence (t_n) in T such that

$$t_n(x) \to x_0$$

for all $x \in X$ and for some $x_0 \in X$. We now prove that the T action on X is strongly proximal. Let f be any bounded continuous function on X. Then

$$f \odot t_n(x) \to f(x_0)$$

for all $x \in X$. Now by Bounded Convergence Theorem,

$$\mu(f \odot t_n) \to \delta_{x_0}(f)$$

for all $\mu \in \mathcal{P}(X)$. Thus,

$$t_n\mu \to \delta_{x_0}$$

for all $\mu \in \mathcal{P}(X)$. This prove that the action T on X is strongly proximal.

We now prove the following interesting result which is needed in the sequel.

Proposition 1 *Let T be a semigroup acting on a complete separable metric space X by continuous selfmaps. Then the action of T on X is strongly proximal if and only if the action of T on $\mathcal{P}(X)$ is proximal.*

Proof Suppose the action of T on X is strongly proximal. Let λ_1 and λ_2 be in $\mathcal{P}(X)$. Then there exists a sequence (τ_n) in T such that $t_n(\frac{1}{2}(\lambda_1 + \lambda_2)) \to \delta_x$ for some $x \in X$. Then for given $1 > \epsilon > 0$ there exists a compact subset K of X such that

$$t_n\lambda_1(K) + t_n\lambda_2(K) > 2 - \epsilon \qquad (i)$$

for all $n \geq 1$. Suppose for some $i = 1, 2$ and for some $m \geq 1$, $t_m\lambda_i(K) \leq 1 - \epsilon$. Then since λ_1 and λ_2 are probability measures, we get that

$$t_m\lambda_1(K) + t_m\lambda_2(K) \leq 2 - \epsilon$$

for some $m \geq 1$ which is a contradiction to (i). Thus,

$$t_n\lambda_i(K) > 1 - \epsilon$$

for $i = 1, 2$ and for all $n \geq 1$. By Prohorov's theorem (see [B] or [P]), the sequences $(t_n\lambda_1)$ and $(t_n\lambda_2)$ are relatively compact in $\mathcal{P}(X)$. Let μ_1 be a limit point of $(t_n\lambda_1)$. Then there exists a $\mu_2 \in \mathcal{P}(X)$ such that

$$\frac{1}{2}(\mu_1 + \mu_2) = \delta_x$$

and hence $\mu_1 = \delta_x$. This implies that

$$\lim t_n\lambda_1 = \delta_x = \lim t_n\lambda_2.$$

Thus, the action of T on $\mathcal{P}(X)$ is proximal.

Suppose the action of T on $\mathcal{P}(X)$ is proximal. Let $\lambda \in \mathcal{P}(X)$. Now for any $x \in X$, there exists a sequence (t_n) in T such that

$$\lim t_n\lambda = \lim t_n\delta_x. \qquad (ii)$$

For any $n \geq 1$, $t_n x \in \delta_X$ which is a closed T-invariant set and hence $\lim t_n x \in \delta_X$. Thus, (ii) implies that

$$t_n\lambda \to \delta_y$$

for some $y \in X$. □

Let (X, T) be a dynamical system where X is a complete separable metric space and T is a locally compact group. Let us now consider the map $\Psi: \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ defined as

$$\Psi(\rho) = \int_{\mathcal{P}(X)} y\, d\rho(y) \in \mathcal{P}(X)$$

for any $\rho \in \mathcal{P}(\mathcal{P}(X))$.

We first establish the following properties of Ψ.

Proposition 2 *Let X, T and Ψ be as above. Then*

1. *Ψ is a continuous T-equivariant map,*
2. *$\Psi(\delta_y) = y$ for all $y \in \mathcal{P}(X)$,*
3. *for $\rho \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\rho) = \delta_x$ for some $x \in X$ implies ρ is a point mass concentrated at the point x.*
4. *suppose X is a semigroup, then Ψ is a semigroup homomorphism.*

Proof Since X, $\mathcal{P}(X)$ and $\mathcal{P}(\mathcal{P}(X))$ are all metrizable, it is enough to prove sequential continuity of Ψ. Let (ρ_n) be a sequence in $\mathcal{P}(\mathcal{P}(X))$ such that $\rho_n \to \rho \in \mathcal{P}(\mathcal{P}(X))$. Let $\nu_n = \Psi(\rho_n)$ for all n and $\Psi(\rho) = \nu$. Let f be a bounded continuous funtion on X. Then the function $y \mapsto y(f)$ is a continuous bounded funtion on $\mathcal{P}(X)$ and hence since $\rho_n \to \rho$ in $\mathcal{P}(\mathcal{P}(X))$, we have

$$\nu_n(f) = \int y(f) d\rho_n(y) \to \int y(f) d\rho(y) = \nu(f).$$

Thus, $\nu_n \to \nu$ in $\mathcal{P}(X)$. This proves that Ψ is continuous. Since the action of T on X is by homeomorphisms, we have

$$t\nu = \int ty d\rho(y) = \int y d(t\rho)(y),$$

that is $t\Psi(\rho) = \Psi(t\rho)$. Thus, verifying property (1) of the map Ψ. It is easy to verify property (2) of the map Ψ.

Suppose for $\rho \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\rho) = \delta_x = \nu$, say for some $x \in X$. Then for any $\epsilon > 0$ and any bounded continuous function f on X such that $f \geq 0$ and $f(x) = 0$, let

$$\sigma(f, \epsilon) = \{y \in \mathcal{P}(X) \mid y(f) > \epsilon\}.$$

Then

$$0 = \nu(f) \geq \int_{y \in \sigma(f,\epsilon)} y(f) d\rho(y) \geq \epsilon \rho(\sigma(f, \epsilon))$$

and hence $\rho(\sigma(f, \epsilon)) = 0$. It is easy to see that $\sigma(f, \epsilon)$ is an open set for all continuous bounded f and $\epsilon > 0$. Now let W be the set of all nonnegative bounded continuous functions f on X which vanish at x and

$$B = \cup_{f \in W} \cup_{n=1}^{\infty} \sigma(f, \frac{1}{n}).$$

Then B is an open set.

We now claim that $B \cup x = \mathcal{P}(X)$ and $x \notin B$. Let $\lambda (\neq \delta_x) \in \mathcal{P}(X)$. Then choose a compact set K such that $\lambda(K) > \frac{1}{n}$ for some n and $x \notin K$. Since $x \notin K$, there exists a continuous function f on X such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(y) = 1$ for all $y \in K$. Then $\lambda(f) \geq \frac{1}{n}$ and hence $\lambda \in \sigma(f, \frac{1}{n}) \subset B$. Thus, $\mathcal{P}(X) = B \cup x$ and it is easy to see that $x \notin B$.

We now claim that $\rho(B) = 0$. Let K be any compact set contained B. Then there exists a finite number nonnegative continuous function $f_1, f_2, \cdots, f_k$ and a finite set of integers $n_1, n_2, \cdots, n_k$ such that $K \subset \cup \sigma(f_i, \frac{1}{n_i})$. Since $\rho(\sigma(f, \epsilon)) = 0$ for all $f \in W$ and $\epsilon > 0$, we have $\rho(K) = 0$ and hence since K is any arbitrary compact subset contained in B, we have $\rho(B) = 0$. Thus, ρ is the mass concentrated at the point $x \in X$. Thus, verifying property (3) of the map Ψ.

Suppose X is a semigroup. The Ψ is a semigroup homomorphism follows from the facts

1. Ψ is affine, that is for any $0 \leq \alpha_1, \alpha_2, \cdots, \alpha_n \leq 1$ with $\sum \alpha_i = 1$ and for $\rho_1, \rho_2, \cdots, \rho_n \in \mathcal{P}(\mathcal{P}(X))$, $\Psi(\sum \alpha_i \rho_i) = \sum \alpha_i \Psi(\alpha_i)$,
2. the set of measures with finite supports in $\mathcal{P}(X)$ is dense in $\mathcal{P}(\mathcal{P}(X))$ and
3. Ψ is continuous.

□

We now prove that strongly proximal actions on compact metric spaces is hereditary in the following sense.

Theorem 1 *Let X be a compact metric space and T be a locally compact group acting on X. Then the following are equivalent:*

1. *T action on X is strongly proximal;*
2. *the action of T on $\mathcal{P}(X)$ is strongly proximal.*

Proof Suppose (X, T) is strongly proximal. Let $\rho \in \mathcal{P}(\mathcal{P}(X))$, and let $\nu = \Psi(\rho) \in \mathcal{P}(X)$. Then since the action of T on X is strongly proximal, there exists a sequence (t_n) in T such that $t_n\nu \to \delta_x$, for some $x \in X$. Since for each n, $t_n\rho \in \mathcal{P}(\mathcal{P}(X))$ which is a compact metrizable space, the sequence $(t_n\rho)$ is a relatively compact sequence. Now let $\rho_o \in \mathcal{P}(\mathcal{P}(X))$ be a limit point of $(t_n\rho)$. Since $\mathcal{P}(\mathcal{P}(X))$ is a metrizable space, there exists a subsequence (t_{k_n}) of (t_n) such that $t_{k_n}\rho \to \rho_0$ in $\mathcal{P}(\mathcal{P}(X))$. Let $\nu_0 = \Psi(\rho_0) \in \mathcal{P}(X)$. Then by Proposition 2,

$$t_{k_n}\nu = t_{k_n}\Psi(\rho) = \Psi(t_{k_n}\rho) \to \Psi(\rho_0) = \nu_0$$

in $\mathcal{P}(X)$ and hence since $t_n\nu \to \delta_x$ in $\mathcal{P}(X)$ which is a metric space, we get that $\nu_0 = \delta_x$. Again by Proposition 2, ρ_0 is the mass concentrated at $x \in X$. Thus, the relatively compact sequence $(t_n\rho)$ has a unique limit point and hence it converges to the point mass concentrated at $x \in X$. This proves that (1) implies (2). That (2) implies (1) follows from Proposition 1 and from the remark that strongly proximal actions are proximal. □

As a consequence we have the following corollary for affine actions: *affine action* of a group T on a closed convex subset X of a locally convex vector space is an action of T on X such that $t(ax + (1-a)y) = at(x) + (1-a)t(y)$ for all $x, y \in X$ and all $0 \geq a \geq 1$.

Corollary 1 *Let X be a compact metric space. Let T be a group acting on $\mathcal{P}(X)$. Suppose the action T on $\mathcal{P}(X)$ is affine. Then the action of T on $\mathcal{P}(X)$ is proximal if and only if the action of T on $\mathcal{P}(X)$ is strongly proximal. Suppose X is a compact group and the T action on $\mathcal{P}(X)$ respects the semigroup operation. Then the action of T on $\mathcal{P}(X)$ is proximal if and only if X is trivial.*

Proof We now define an action of T on X and show that T action on $\mathcal{P}(X)$ is the extension of the above defined action of T on X. Now, for $t \in T$ and $x \in X$, define $tx = t\delta_x$. Suppose the support of tx contains two different elements, say z and y. Then there exists a neighbourhood U of z such that $y \notin \overline{U}$ and hence $0 < tx(U) < 1$. Let

$$\lambda(E) = tx(U)^{-1}tx(U \cap E)$$

and

$$\mu(E) = (1 - tx(U))^{-1}tx((X \setminus U) \cap E)$$

for all Borel subsets E of X. Then it is easy to see that $tx = a\lambda + (1-a)\mu$ for $a = tx(U)$. Since $0 < a < 1$, we get that $t^{-1}\lambda = \delta_x = t^{-1}\mu$. This implies that $\lambda = tx = \mu$. This is a contradiction. Thus, $tx \in \delta_X$ (that is, λ is an extreme point implies $t\lambda$ is also an extreme point). Since the action of T on $\mathcal{P}(X)$ is affine and the set of measures in $\mathcal{P}(X)$ with finite support is dense, we get that the action of T on X extends to the action of T on $\mathcal{P}(X)$. Since the action of T on $\mathcal{P}(X)$ is proximal, by Proposition 1, the action of T on X is strongly proximal. By Theorem 1, the action of T on $\mathcal{P}(X)$ is strongly proximal.

Suppose X is a compact group. Since the action of T on $\mathcal{P}(X)$ respects the semigroup operation, T acts on X by automorphisms. Let μ be the normalized Haar measure on X. Then μ is T-invariant. Suppose the T action on $\mathcal{P}(X)$ is proximal. Then the action of T on X is strongly proximal. Since μ is T-invariant, we get that $X = (e)$.

Remark 2 It should be noted that the conclusion of Theorem 1 is valid for any Polish space if the map Ψ is proper.

In general for a complete separable metric space X, it is not clear that $(\mathcal{P}(X), T)$ is strongly proximal if (X, T) is strongly proximal. However the system $(\mathcal{P}(X), T)$ does not admit a non-trival T-invariant measure in the following sense:

Corollary 2 *Let T be a locally compact group acting strongly proximaly on a complete separable metric space X. Suppose $\rho \in \mathcal{P}(\mathcal{P}(X))$ is T-invariant. Then ρ is the mass concentrated at a point $x \in X$.*

Proof Let $\rho \in \mathcal{P}(\mathcal{P}(X))$ be a T-invariant measure and let $\nu = \Psi(\rho) \in \mathcal{P}(X)$. Since ρ is T-invariant, by Proposition 2 we get that ν is T-invariant. Since T acts strongly proximaly on X, $\nu = \delta_x$ for some $x \in X$. By Proposition 2, we get that ρ is the point mass concentrated at the point $x \in X$. □

We have an analogue of Theorem 1 for exponential Lie groups.

Theorem 2 *Let G be an exponential Lie group (that is, exponential is a diffeomorphism of the Lie algebra of G onto G) and T be a group of automorphisms of G. Then the action of T on G is strongly proximal if and only if the action of T on $\mathcal{P}(G)$ is strongly proximal.*

Proof Suppose the action of T on $\mathcal{P}(G)$ is strongly proximal. Then the action of T on $\mathcal{P}(G)$ is proximal. By Proposition 1, the action of T on G is strongly proximal.

Suppose the action of T on G is strongly proximal. Let $\mathcal{G}$ be the Lie algebra of G. Then since G is an exponential Lie group, we get that exponential map is a T-equivariant homeomorphism of the T-spaces G and $\mathcal{G}$. Thus, the action of T on $\mathcal{G}$ is strongly proximal. By Proposition 1, the action of T on $\mathcal{P}(\mathcal{G})$ is proximal. Thus, there exists a sequence (t_n) in T such that

$$t_n(\frac{1}{m}\sum_{i=1}^{m}\delta_{v_i}) \to 0$$

where $\{v_1, v_2, \cdots, v_n\}$ form a base for $\mathcal{G}$. This implies that

$$t_n v \to 0$$

for all $v \in \mathcal{G}$. Since G is an exponential Lie group, we have

$$t_n g \to e$$

for all $g \in G$. This implies by Remark 1 that

$$t_n \lambda \to \delta_e$$

for all $\lambda \in \mathcal{P}(G)$ and hence once again by Remark 1

$$t_n \rho \to \delta_x$$

for all $\rho \in \mathcal{P}(\mathcal{P}(G))$ where $x \in \mathcal{P}(G)$ is the point mass at e. Thus, the action of T on $\mathcal{P}(G)$ is strongly proximal. □

Remark 3 It should be noted that Theorem 2 may be proved for algebraic groups over local fields which do not have central torus and T is a group of rational automorphisms.

Definition Let T be a locally compact group second countable group acting on a complete separable metric space X by homeomorphisms and $\mu \in \mathcal{P}(T)$.

1. A measure $\nu \in \mathcal{P}(X)$ is said to be *μ-stationery*, if

$$\mu * \nu = \int_{t\in T} t\nu d\mu(t) = \nu.$$

2. The T-space X is said to be *μ-proximal* if for each μ-stationery probability measure ν on X, with probability 1

$$g_1 g_2 \cdots g_n \nu \to \delta_x$$

for some $x \in X$ where (g_i) is a sequence of independent identically distributed T-valued random variables with common distribution μ.

3. The T-space X is said to be *mean proximal* if X is μ-proximal, for every probabilty measure μ whose support is all of G.

For more details on μ-proximal and mean proximal spaces, see [F2] and [M]. Here we claim that μ-proximality and hence mean proximality are hereditary.

Theorem 3 *Let X be a compact metric space and T be a locally compact group second countable group acting on X by homeomorphisms. Then we have the following:*

1. *for any measure $\mu \in \mathcal{P}(T)$, X is μ-proximal if and only if $\mathcal{P}(X)$ is μ-proximal;*
2. *X is mean proximal if and only if $\mathcal{P}(X)$ is mean proximal.*

Proof Suppose X is μ-proximal. Let $\rho \in \mathcal{P}(\mathcal{P}(X))$ be μ-stationery and $\nu = \Psi(\rho)$. By Proposition 2, ν is μ-stationery. Let (g_i) be a sequence of independent identically distributed random variables with common distribution μ. Then since X is μ-proximal, with probability 1,

$$g_1 g_2 \cdots g_n \nu \to \delta_x \qquad (i)$$

for some $x \in X$. By Proposition 2.4 of Chapter VI of [M], with probability 1 limit of $g_1 g_2 \cdots g_n \rho$ exists. Now by (i) and Proposition 2, this implies that with probability 1, $g_1 g_2 \cdots g_n \rho$ converges to a point measure in X. Thus, $\mathcal{P}(X)$ is μ-proximal.

Conversely, suppose $\mathcal{P}(X)$ is μ-proximal. Let $\lambda \in \mathcal{P}(X)$ be a μ-stationery measure. Define $\rho \in \mathcal{P}(\mathcal{P}(X))$ to be

$$\rho(E) = \lambda(E \cap \delta_X)$$

for every Borel subset E of $\mathcal{P}(X)$. Then ρ is a μ-stationery measure on $\mathcal{P}(\mathcal{P}(X))$. Thus, with probability 1,

$$g_1 g_2 \cdots g_n \rho \to \delta_y$$

for some $y \in \mathcal{P}(X)$ and for any sequence (g_i) of i.i.d. random variables with common distribution μ. We now claim that $y = \delta_x$. Since $\rho(\delta_X) = 1$ and X is T-invariant and by Thoerem 2.1 of [B], we have $\delta_y(\delta_X) = 1$. This implies that $y \in \delta_X$. Thus, X is μ-proximal. This proves (1) and (2) follows from (1). □

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References

[A] R. Azencott, Espaces de Poisson des groupes localement compacts, Lecture Notes in Math., Vol. 148, Springer, Berlin, 1970.

[B] P. Billingley, Convergence of Probability Measures, John Willey and Sons, New York-Tornoto, 1968.

[F1] H. Furstenberg, Poisson formula for semisimple Lie groups, Annals of Math., 77 (1963), 335-386.

[F2] H. Furstenberg, Boundary theory and stochastic process on homogeneous spaces, in: Harmonic analysis on homogeneous spaces. (Symposia on Pure and Applied Mathematics) Williamstown, Mass, 1972, Proceedings, 26 (1973), 193-229.

[G] S. Glasner, Proximal flows on Lie groups, Israel Journal of Mathematics, 45 (1983), 97-99.

[M] G. A. Margulis, Discrete subgroups of Semisimple Lie Groups, A Series of Modern Surveys in Mathematics, 3. Folge. Band 17, Springer-Verlag, Berlin, 1990.

[P] K. R. Parthasarathy, Probability Measures on Metric Spaces, Academic Press, New York-London, 1967.

C. Robinson Edward Raja,
Post-doctoral fellow of CCRRDT Pays de Loire,
Université d'Angers,
Faculté des Sciences,
Département de Mathématiques,
2, Boulevard Lavoisier ,
49045 Angers Cedex 01.
France.

E-mail: raja@tonton.univ-angers.fr