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# Anatomy of an ARM: The Interest Rate Risk of Adjustable Rate Mortgages

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**ABSTRACT**

This article analyzes the dynamics of the commonly used indices for Adjustable Rate Mortgages, and systematically compares the effects of their time series properties on the interest rate sensitivity of adjustable rate mortgages. Our ARM valuation methodology allows us simultaneously to capture the effects of index dynamics, discrete coupon adjustment, mortgage prepayment, and both lifetime and periodic caps and floors. We can, moreover, either calculate an optimal prepayment strategy for mortgage holders, or use an empirical prepayment function. We find that the different dynamics of the major ARM indices lead to significant variation in the interest rate sensitivities of loans based on different indices. We also find that changing assumptions about contract features, such as loan caps and coupon reset frequency, has a significant, and in some cases unexpected, impact on our results.

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# 1 Introduction

Recent surveys of major thrifts and mortgage bankers (see, for example, *Inside Mortgage Finance*) indicate that, while there are many different indices underlying adjustable rate mortgages in the U.S., four indices dominate the market:

1. The one year constant maturity Treasury yield,
2. One year LIBOR,
3. The Eleventh District Cost-of-Funds Index (EDCOFI),
4. The Federal Housing Finance Board (FHFB) national average contract interest rate.

The results of the Ott [18] ARM duration study, and numerous recent studies of the time series properties of EDCOFI,[2] suggest that only the first of these indices adjusts instantaneously to changes in contemporaneous Treasury rates. The others adjust with a lag. Ott [18] uses a classical duration approach to show that this lag can have a significant impact on the interest rate sensitivity of ARMs. However, without explicitly modeling term structure dynamics, he cannot address the impact of mortgage prepayment, or additional common contract features such as interest rate caps. On the other hand, most recent ARM valuation models, using a contingent claims approach with a richer specification of interest rate dynamics, can analyze the impact of interest rate caps and prepayment, but they ignore the lag in the adjustment of the ARM coupon to the contemporaneous term structure (see, for example, Kau et al. [10], Schwartz and Torous [23], and McConnell and Singh [13]). No previous study simultaneously analyzes the interacting effects of both the time series properties of the index *and* prepayment/interest rate caps on the interest rate risk of ARMs.

This article analyzes the time series dynamics of the most commonly used ARM indices, and includes these dynamics in an ARM valuation model that builds upon techniques developed by Kau et al. [10], Kishimoto [11], and Stanton and Wallace [24]. Our model can simultaneously take into account the dynamics of the index, a realistic specification for interest rate dynamics, and nearly all standard ARM contract features, including both lifetime and periodic interest rate caps. Another major advantage of our approach is that it allows us either to determine endogenously the optimal prepayment policy for mortgage holders, or to use an empirically derived prepayment function.

We use our valuation model to compute the interest rate sensitivity for ARMs based on different indices, and with differing contract features. We find that the interest rate sensitiv-

[2] See, for example, Cornell [2], Crockett et al. [4], Hayre et al. [8], Nothaft [16], Nothaft and Wang [17], Passmore [19], Roll [22], Stanton and Wallace [24].

ity of an ARM is highly dependent on the dynamics of the index, prepayment behavior, term structure dynamics, and rate caps, with some surprising differences in the relative interest rate sensitivity of different contracts. In particular, we find that for *uncapped* ARMs, the slower the speed of adjustment of the index, the more interest rate sensitive is an ARM based on that index (since it more closely resembles a fixed rate loan). However, for otherwise identical ARMs with a lifetime maximum coupon rate cap, this ordering can paradoxically be *reversed*. We also find that changing the coupon reset frequency has a significant impact, and that using the wrong model for the index (ignoring the slow speed of adjustment) yields misleading results for the interest rate sensitivity of an ARM. The more often the coupon rate is reset, the worse the problem.

The article is in three sections. The empirical specification for ARM indices is discussed in section 2. Section 3 discusses the valuation methodology, and analyzes the effects of index dynamics, caps and margins on the interest rate sensitivity of ARM contracts. Section 4 concludes the article.

## 2 Dynamics of ARM Indices

The ARM indices that dominate the market are:

1. **The one year constant maturity Treasury yield:** This reflects the average yield of all existing Treasury securities with one year of maturity remaining. The yield is determined from the closing market bids on actively traded Treasury Securities in the over-the-counter market, as disclosed by the five leading U.S. government securities dealers. The index is computed as a weekly average, and the Federal Reserve Board publishes this yield in its weekly H-15 statistical release.

2. **One year LIBOR:** The London Interbank Offer Rate. The rate is quoted daily by five London money center banks for loans of one year. The five quotes are averaged, and rounded to the nearest 1/16th to arrive at the index rate. The index is computed as a weekly average, and reported by the Board of Governors of the Federal Reserve.

3. **The Federal Housing Finance Board (FHFB) national average contract interest rate:** The weighted average of initial mortgage interest rates paid by home buyers for loans originated during the first five business days of every month. The weights are determined by the type, size and location of the lender. The index is constructed by the Federal Housing Finance Board and reported on a monthly basis.

4. **The Eleventh District Cost-of-Funds Index (EDCOFI):** This index is computed from the book values of liabilities for all insured savings and loan (S&L) institutions in the Eleventh District (institutions in California, Nevada, and Arizona). The index is the ratio of the month-end total interest expenses on savings accounts, advances, and purchased funds to the average book value of these liabilities from the beginning to the end of the month. The ratio is adjusted with an annualizing factor so that the interest expenses are comparable across months.

The historical values of these ARM indices from July 1981 through May 1993 are plotted in Figure 1.[3] The plot shows that EDCOFI and the FHFB average contract rate display considerably less volatility than the Treasury and LIBOR series. EDCOFI appears to lag the Treasury series by several months. This should be expected, given that it is based on book yields, which can only change when a liability matures. The FHFB average contract rate looks rather like EDCOFI, with a spread of approximately 200 basis points. The FHFB average contract also lags the Treasury rates.

Considering the construction of EDCOFI and the FHFB average contract rate, and our plots of these indices relative to market rates, a partial adjustment model[4] is a reasonable representation for the movements of EDCOFI, the FHFB average contract rate, and one year LIBOR. For a given index, $I_t$, the model can be written as

$$I_t = \alpha + \beta\, r_t + \gamma\, I_{t-1} + \epsilon_t, \tag{1}$$

where $r_t$ is an instantaneous spot rate, and $\epsilon_t$ is an error term. The coefficient $\beta$ indicates the effect of the spot rate on the index each period, and $\gamma$ indicates the speed at which the index adjusts. The extremes in the adjustment dynamics would be $\beta = 0$, where the index does not move at all with market rates, and $\gamma = 0$, where the index moves perfectly with the spot rate (the usual implicit assumption).[5]

Ignoring the error term, if the index starts at a value $I_0$, and the interest rate remains at a constant level $r$, the value of the index at any later time is given by[6]

$$I_t = \left(1 - \gamma^t\right) \frac{\alpha + \beta r}{1 - \gamma} + \gamma^t I_0. \tag{2}$$

[3]All of the data series, except EDCOFI, were obtained from CITIBASE. The EDCOFI series data were obtained from the Office of Thrift Supervision.

[4]See Ott [18], Cornell [2], Passmore [19], Roll [22] and Stanton and Wallace [24] for further discussion and justification of this specification.

[5]See Kau et al. [10], and Schwartz and Torous [23].

[6]This solution can be verified by inserting it into equation 1, with $\epsilon_t$ set to zero.

This is a weighted average of the long run value of $I_t$ and its initial value. The speed of convergence is governed by the value of $\gamma$. The half-life, the number of periods required to reach half way between the two values, is the solution to

$$\gamma^{t_{1/2}} = \frac{1}{2}, \tag{3}$$

which is

$$t_{\frac{1}{2}} = -\frac{\log(2)}{\log(\gamma)}. \tag{4}$$

Note that substituting $\gamma = 0$ into equation 2 yields the correct result for instantaneous adjustment,

$$I_t = \alpha + \beta r. \tag{5}$$

Because we are interested in the adjustment of observed ARM indices to the instantaneous spot rate, we estimate the partial adjustment models using the three month Treasury rate as a proxy for the instantaneous spot rate.[7] The estimation results are reported in Table 1. All the indices are estimated in levels.[8] We estimate the partial adjustment model for EDCOFI using dummies for January and February to account for seasonality. Because of problems with both serial correlation and heteroscedasticity, we estimate the partial adjustment model for EDCOFI using the Newey and West [15] instrumental variable procedure to obtain a heteroscedasticity and autocorrelation consistent covariance matrix. The partial adjustment model for the FHFB average contract rate was estimated using OLS, because neither heteroscedasticity nor serial correlation violations was observed. The one year LIBOR partial adjustment model was estimated using instruments for the first lag of one year LIBOR, and then using Yule-Walker estimation methods and an AR(2) specification for second stage estimation of the model. The $R^2$, Breusch-Pagan [1] tests for heteroscedasticity,

[7]This choice was made because the three month Treasury rates offered the shortest term rate with reasonable large trading volume. The one month Treasury rates reflect very erratic trading volume over our period of analysis

[8]Augmented Dickey-Fuller [5] tests of the form

$$\Delta x_t = \mu + \gamma^* x_{t-1} + \sum_{j=1}^{p-1} \phi_j \, \Delta x_{t-j} + \epsilon_t$$

were performed on all series using twelve lagged differences to control for possible seasonality. We were unable to reject the null hypothesis that there are unit roots in market rates and in the indices. Phillips and Perron [21] nonparametric unit root testing procedures were also applied, with the same result. Tests for the cointegration of the indices and Treasury rates, using Johansen [9], showed that they are not cointegrated. However, because the series are relatively short, and it is well known that the low power of standard unit roots tests often leads to acceptance of the null hypothesis of a unit root in many economic time series (Kwiatkowski et al. [12], Faust [7]), we rely on our strong priors that our interest rate series are mean reverting rather than explosive, and undertake all our estimation in levels of interest rates.

and the Durbin tests for AR(1) errors are also reported.

EDCOFI responds a little more quickly to movements in the three month Treasury rate than does the FHFB average contract rate. One year LIBOR responds faster than either, keeping very close track of the three month Treasury rate. This is shown in Figure 2, which shows the effect of an instantaneous 1% shift in the riskless interest rate on each of the three indices. Each index starts at its long run level (the level it would reach if $r$ stayed at 7.5% for ever), and the graph shows what happens when $r$ jumps from 7.5% to 8.5%. Besides the obvious lags in EDCOFI and the FHFB rate, one other interesting feature of the graph is the difference between the levels of the three series. EDCOFI is approximately 0.6% higher than $r$ in this region; LIBOR is approximately 1.2% higher, and the FHFB rate is almost 3% higher.

# 3 Implementation

This section develops an algorithm for valuing adjustable rate mortgages. The algorithm can handle all of the important features of the contract, including the partial adjustment model for index dynamics developed in section 2. We can either use an empirical prepayment function (as commonly used in Wall Street valuation models), or derive endogenously the optimal prepayment strategy for mortgage holders. The latter strategy has the advantage that it is robust to possible changes in the economic environment, such as changes in the interest rate process, which would have an unquantifiable effect on an empirical prepayment function. The algorithm is based on techniques developed by Kau et al. [10], Kishimoto [11], and Stanton and Wallace [24].

## 3.1 Main Features of an ARM Contract

The most common features of an ARM contract are:

**Coupon rate,** $C_t$. The coupon rate on an ARM changes at each "reset date". The coupon determines the monthly cash flows on the mortgage until the next reset date. The monthly cash flow equals that on a fixed rate mortgage with the same time to maturity, same remaining principal balance, and same coupon rate as the ARM.

**Index,** $I_t$. The adjustment rule for the coupon rate specifies a particular index to which the rate is tied.

**Margin,** $m$. At each coupon reset date, the new rate is set by adding a margin, $m$ (e.g. 2%), to the prevailing level of the index (subject to certain caps, discussed below).

**Initial rate,** $C_0$. It is common for the initial coupon rate to be lower than the fully indexed rate, given by adding the margin to the initial level of the index. The initial rate, $C_0$, is often referred to as a "teaser" rate.

**Periodic cap,** $\Delta$. ARM contracts usually specify a maximum adjustment in the coupon rate at each reset period (e.g. 2% per year).

**Lifetime caps,** $\overline{C}$ and $\underline{C}$. ARM contracts usually specify an overall maximum coupon rate over the life of the loan, $\overline{C}$ (e.g. the initial rate plus 6%), and a minimum coupon rate over the life of the loan, $\underline{C}$.

**Reset Frequency.** The coupon rate on an ARM contract adjusts at prespecified intervals. This interval is usually every 6 months or one year. In this article, we usually assume yearly adjustment. If month $t$ is a coupon reset date, the new coupon rate is given by

$$C_t = \max\left[\underline{C},\, C_{t-1} - \Delta,\, \min\left[I_t + m,\, C_{t-1} + \Delta,\, \overline{C}\right]\right]. \tag{6}$$

Some ARMs have other, less common, features such as payment caps and negative amortization. Our model can easily incorporate these additional features with only minimal modifications.

## 3.2 Interest Rates

To value the mortgage, we need to make assumptions about the process governing interest rate movements. We use the Cox, Ingersoll and Ross [3] one-factor model. In this model, the instantaneous risk-free interest rate, $r_t$, satisfies the stochastic differential equation

$$dr_t = \kappa(\mu - r_t)\,dt + \sigma\sqrt{r_t}\,dz_t. \tag{7}$$

This equation says that, on average, the interest rate $r$ converges toward the value $\mu$. The parameter $\kappa$ governs the rate of this convergence. The volatility of interest rates is $\sigma\sqrt{r_t}$. One further parameter, $\lambda$, which summarizes risk preferences of the representative individual, is needed to price interest rate dependent assets.

The parameter values used here are those estimated by Pearson and Sun [20], using data from 1979–1986. These values are

$$\begin{aligned} \kappa &= 0.29368, \\ \mu &= 0.07935, \\ \sigma &= 0.11425, \\ \lambda &= -0.12165. \end{aligned}$$

The long run mean interest rate is 7.9%. Ignoring volatility, the time required for the interest rate to drift half way from its current level to the long run mean is $\ln(1/2)/(-\kappa) \approx 2.4$ years.

### 3.3 Valuing an ARM

The value of an ARM depends not only on the current interest rate, $r_t$, but on the whole path of interest rates since its issue. This determines the current coupon rate, $C_t$, the current level of the index, $I_t$ (which in turn determines future movements in the coupon rate), and the current remaining principal balance, $F_t$. These three variables summarize all relevant information about the history of interest rates. By adding these as extra state variables, we return to a Markov setting where all prices can be written as a function only of the current values of a set of underlying state variables.

Write $B_t$ for the value of a non-callable bond which makes payments equal to the promised payments on the ARM. The mortgage holder's position can be decomposed into a short position in $B_t$ (the scheduled payments on the mortgage) plus a long position in a call option on $B_t$, with (time varying) exercise price $F_t$. Writing $M_t$ for the market value of the mortgage, and $O_t$ for the value of the prepayment option, we have

$$M_t = B_t - O_t \tag{8}$$

Since $B_t$ does not depend on the mortgage holder's prepayment decision, minimizing his or her liability value is equivalent to maximizing the value of the prepayment option, $O_t$. Write

$$\begin{aligned} B_t &\equiv B(r_t, I_t, C_t, F_t, t), && (9) \\ O_t &\equiv O(r_t, I_t, C_t, F_t, t). && (10) \end{aligned}$$

All values are homogeneous of degree one in the current remaining principal amount, $F_t$. Thus, if each month we value a mortgage with \$1 remaining principal, we can scale up or down as necessary for different principal amounts. Define normalized asset values (values per \$1 of remaining principal) by

$$\begin{aligned} \widehat{B}_t &= B_t/F_t, && (11) \\ &\equiv \widehat{B}(r_t, I_t, C_t, t). \\ \widehat{O}_t &= O_t/F_t, && (12) \\ &\equiv \widehat{O}(r_t, I_t, C_t, t). \end{aligned}$$

To hedge fluctuations in the market value of the mortgage, we need a measure of interest rate sensitivity. We use the *effective duration*,

$$D_{\text{eff}} = -\frac{1}{M_t}\frac{\partial M_t}{\partial r},$$

the percentage change in the market value of the mortgage for an instantaneous (10,000) basis point shift in the short rate. A simple interpretation of this quantity is that an asset or liability with an effective duration of $n$ has approximately the same interest rate sensitivity as an $n$ year zero coupon bond.[9]

## 3.4 Implementation with a Single State Variable

Given the interest rate model defined by equation 7, write $V(r, t)$ for the value of an asset whose value depends only on the current level of $r_t$ and time, and which pays coupons or dividends at some rate $\delta(r_t, t)$. This value satisfies the partial differential equation[10]

$$\frac{1}{2}\sigma^2 r V_{rr} + [\kappa\mu - (\kappa + \lambda)r]\, V_r + V_t - rV + \delta = 0, \tag{13}$$

which can be solved for $V$, subject to appropriate boundary conditions.

Natural boundaries for the interest rate, $r$, are 0 and $\infty$. Rather than working directly with $r$, define the variable $y$ by

$$y = \frac{1}{1 + \gamma r}. \tag{14}$$

for some constant $\gamma > 0$,[11] The infinite range $[0, \infty)$ for $r$ maps onto the finite range $[0, 1]$ for $y$. The inverse transformation is

$$r = \frac{1 - y}{\gamma y}. \tag{15}$$

Equation 14 says that $y = 0$ corresponds to "$r = \infty$" and $y = 1$ to $r = 0$. Next, rewrite

[9]The *Macaulay* duration of a zero coupon bond is *exactly* equal to its maturity. However, Macaulay duration is calculated assuming a flat term structure which makes only parallel shifts. The measure we are calculating here takes into account both the fact that the term structure is not flat, and that shifts are not entirely parallel, leading to slight differences between maturity and duration.

[10]We need to assume some technical smoothness and integrability conditions (see, for example, Duffie [6]).

[11]The larger the value of $\gamma$, the more points on a given $y$ grid correspond to values of $r$ less than, say, 20%. Conversely, the smaller the value of $\gamma$, the more points on a given $y$ grid correspond to values of $r$ greater than, say, 4%. We are most interested in values of $r$ in an intermediate range. Therefore, as a compromise between these two competing objectives, we choose $\gamma = 12.5$. The middle of the range, $y = 0.5$, then corresponds to $r = 8\%$.

equation 13 using the substitutions

$$\begin{aligned} U(y,t) &\equiv V(r(y),t), \qquad \text{so} \quad &(16)\\ V_r &= U_y \frac{dy}{dr}, &(17)\\ V_{rr} &= U_y \frac{d^2y}{dr^2} + U_{yy}\left(\frac{dy}{dr}\right)^2, &(18) \end{aligned}$$

to obtain

$$\frac{1}{2}\gamma^2 y^4 \sigma^2 r(y) U_{yy} + \left(-\gamma y^2\left[\kappa\mu - (\kappa+\lambda) r(y)\right] + \gamma^2 y^3 \sigma^2 r(y)\right) U_y + U_t - r(y)U + \delta = 0. \quad (19)$$

We can solve equation 19 using a finite difference algorithm. Finite difference algorithms replace derivatives with differences, and approximate the solution to the original partial differential equation by solving the set of difference equations that arise. We use the Crank-Nicholson algorithm.[12]

Represent the function $U(y,t)$ by its values on the finite set of points,

$$\begin{aligned} y_j &= j\,\Delta y, &(20)\\ t_k &= k\,\Delta t, &(21) \end{aligned}$$

for $j = 0, 1, \ldots, J$, and for $k = 0, 1, \ldots, K$. $\Delta y$ and $\Delta t$ are the grid spacings in the $y$ and $t$ dimensions respectively. $\Delta y = 1/J$, and $\Delta t$ is chosen for convenience to be one month, making a total of 360 intervals in the time dimension. Write

$$U_{j,k} \equiv U(y_j, t_k), \quad (22)$$

for each $(j,k)$ pair. The Crank-Nicholson algorithm rewrites equation 19 in the form

$$MU_k = D_k, \quad (23)$$

where $M$ is a tridiagonal matrix, $U_k$ is the vector $\{U_{0,k}, U_{1,k}, \ldots, U_{I,k}\}$, and $D_k$ is a vector whose elements are functions of $U_{j,k+1}$. This system of equations relates the values of the asset for different values of $y$ at time $t_k$ to its possible values at time $t_{k+1}$. To perform the valuation, we start at the final time period, when all values are known, and solve equation 23 repeatedly, working backwards one period at a time.

[12] See, for example, McCracken and Dorn [14].

### 3.5 Extension to multiple state variables

In general, when asset prices depend on more than one state variable plus time, solution of the resultant partial differential equation becomes numerically burdensome. In this case, the additional variables, $I_t$ and $C_t$, are functions of the path of interest rates, and so they introduce no additional risk premia. This allows us to extend the Crank-Nicholson finite difference algorithm to handle the multiple state variable case. The extensions required are:

1. Allow values to depend on $C_t$ and $I_t$ as well as $r_t$ and $t$, allowing for dependence between the processes governing movements in these variables.

2. Scale values to correspond to \$1 remaining principal.

3. Handle caps, floors and teaser rates.

In addition to the finite sets of values for $y$ and $t$ defined above, define a finite set of values for $I$ and $C$ by

$$I_l = l\,\Delta I, \tag{24}$$

$$C_m = m\,\Delta C, \tag{25}$$

for $l = 0, 1, \ldots, L$, and for $m = 0, 1, \ldots, M$. $\Delta I$ and $\Delta C$ are the grid spacings in the $I$ and $C$ dimensions respectively. We are now solving for values on the points of a 4-dimensional grid, whose elements are indexed by the values of $(j, k, l, m)$. Write the value of an asset whose cash flows depend on these state variables as

$$U_{j,k,l,m} \equiv U(y_j, t_k, I_l, C_m), \tag{26}$$

for each $(j, k, l, m)$. $I$ and $C$ are functions of the path of interest rates. Over the next instant, the movement in $r$ completely determines the movements in both $I$ and $C$. Assume that movements in the index are described by the equation

$$I_{t+1} = g(I_t, r_{t+1}), \tag{27}$$

so that the index value this month is a deterministic function of its value last month, plus the short term riskless rate this month (the models estimated above are of this type). Define $l^*$ by

$$I_{l^*_{1,j,l,m}} \approx g(I_l, r_{j+1}), \tag{28}$$

$$I_{l^*_{0,j,l,m}} \approx g(I_l, r_j), \tag{29}$$
$$I_{l^*_{-1,j,l,m}} \approx g(I_l, r_{j-1}). \tag{30}$$

In words, $l^*$ gives the closest index to the value of $I$ next period given the current values of $r$, $I$ and $C$, and three possible values of $r$ next period (up, the same, and down). Assuming that next month is a coupon reset date (since otherwise, the coupon rate next month will just be the same as the coupon rate this month), define $m^*$ similarly, to give the index of $C$ next period given the current values of $r$, $I$ and $C$, and the value of $r$ next period. $m^*$ is determined by the interplay between the current coupon $C_t$, the index $I_t$, the margin $m$, and the caps $\overline{C}$, $\underline{C}$ and $\Delta$. Note that the effects of caps, floors and teaser rates are all automatically captured in this definition of $m^*$.

We can now generate a set of finite difference equations for each pair $(l, m)$. For example, the approximation for the time derivative now becomes

$$U_t(y_j, t_k, I_l, C_m) \approx \left(U_{j,k+1,l^*_{0,j,l,m},m} - U_{j,k,l,m}\right) / \Delta t, \tag{31}$$

if $t_{k+1}$ is *not* a coupon reset period, and

$$U_t(y_j, t_k, I_l, C_m) \approx \left(U_{j,k+1,l^*_{0,j,l,m},m^*_{0,j,l,m}} - U_{j,k,l,m}\right) / \Delta t, \tag{32}$$

if $t_{k+1}$ *is* a coupon reset period. This allows us to write down one set of equations like 23 for each $(l, m)$ pair. These equations are independent of each other, so we can solve them for each $(l, m)$ pair in turn, looping over $l$ and $m$ to calculate values at every grid point at time $t_k$. A simplified version of this is shown graphically in Figure 3. Each horizontal plane corresponds to a grid of values in $(r, t)$ space. There is a separate such grid for each value of $I_t$ (as shown in the figure), and for each value of $C_t$.[13] As in the standard Crank-Nicholson algorithm, we value the asset by solving a set of difference equations, just like equation 23, for each $(r, t)$ plane. The difference equation for the value of the asset at any point involves its values at six points, corresponding to the current time, $t$, and the following time period, $t + 1$, and interest rates $r_i, r_{i-1}$ and $r_{i+1}$. Note that the values at time $t$ all sit on the current $(r, t)$ plane, while the values for next period may be on other planes, from equations 31 and 32 (for example, in Figure 3, if the interest rate moves from $r_i$ to $r_{i+1}$ next period, the index moves from $I_j$ to $I_{j+1}$. Similarly, if the interest rate moves from $r_i$ to $r_{i-1}$ next period, the index moves from $I_j$ to $I_{j-1}$). We can solve the equations for each $(r, t)$ plane separately, rather than having to consider them all simultaneously. This is because

[13] Imagine Figure 3 repeated in the direction perpendicular to the page.

the interaction between different $(r, t)$ planes only occurs in the values at date $t+1$. By the backward nature of the solution methodology, when we are calculating values at time $t$, we can regard all values at date $t+1$ as known, so this only affects the calculation of the right hand side of equation 23.

The final step in the process is to deal with the normalization of asset prices to correspond to a remaining principal balance of $1. This is possible because, at any time, we know exactly how much principal will be repaid over the next one month. Given a coupon rate $C_t$ and a current remaining principal $F_t$, the usual amortization formula tells us the value of $F_{t+1}$, regardless of any possible movements in $r_t$, $I_t$ or $C_t$. The values stored in the grid for next period correspond to $1 in remaining principal *next* period. These need only to be multiplied by $F_{t+1}/F_t$ (a function only of $C_t$) to make them correspond to $1 of remaining principal.

## 3.6 Results

The extended Crank-Nicholson algorithm described above was used to calculate (numerically) the effective duration of 30 year ARMs with different combinations of underlying index and other contract terms. Starting in month 360, the algorithm works backward to solve equation 23 one month at a time, calculating the bond value, $B_t$. The same process is used to give the option value, conditional on remaining unexercised for the next month. This value must then be compared with the option's intrinsic value ($\max[0, B_t - F_t]$), to determine whether prepayment is optimal. $O_t$ is set to the higher of these two values, and the mortgage value is calculated from the relationship

$$M_t = B_t - O_t.$$

Figures 4–5 and Tables 2–3 show the effective duration, $D_{\text{eff}}$,[14] of mortgages with different underlying indices and different contract terms. Figure 4 plots the effective durations of mortgages based on EDCOFI, FHFB, LIBOR and the 1yr. T-Bill rate, against the current short term riskless interest rate, $r$. Each loan shown has 30 years to maturity, and an initial coupon rate of 10.5%. The coupon rate on each loan adjusts every 12 months (with *no* caps) to the current value of the index, plus a margin equal to either 2% (for EDCOFI, LIBOR and 1yr. T-Bill) or 0% (for FHFB).[15] The graph shows several interesting features. First, despite the differences in dynamics of LIBOR and the 1yr. T-Bill rate, the interest rate sensitivities of loans based on these two indices are almost identical (it is impossible to separate the two lines on the graph). Second, it is very clear that, throughout the range of

[14]Obtained by numerical differentiation.

[15]The average value of the FHFB rate is approximately 2% higher than that of the other indices shown.

possible interest rates, the interest rate sensitivity of loans based on FHFB is the highest, followed by EDCOFI loans, then loans based on LIBOR and the 1yr. T-Bill rate. This can be understood by looking at the models for the indices estimated in Table 1. The coefficient on the lagged index in the regression model is highest for FHFB, followed by EDCOFI, then LIBOR. Thus FHFB moves slower as interest rates move than does EDCOFI, and loans based on this index look more like fixed rate loans, i.e. they have a higher effective duration. Third, while the interest rate sensitivity of each loan does not change much with the level of interest rates for high interest rates, this is not true at lower rates. As rates continue to drop, the effective duration of each loan eventually starts to fall rapidly (negative convexity), as it becomes more and more likely that mortgage holders will find it optimal to refinance their loans in the near future. A model which did not explicitly model prepayment behavior would completely miss this behavior.

Figure 5 shows the same loans as Figure 4, with one difference. All contract terms are identical, except that the coupon rate on each loan now has a lifetime cap of 15%. Again, loans based on LIBOR and the 1yr. T-Bill rate behave almost identically, though the two lines can just be distinguished in this figure. Again (at least for low interest rates), loans based on FHFB are the most interest rate sensitive, followed by EDCOFI, followed by LIBOR/T-Bill. However, the graph differs from figure 4 in several important respects. First, the presence of the cap means that loans based on all the indices are more interest rate sensitive at higher interest rates (when the cap becomes binding). This is because, once the cap is binding, it will probably remain so for some time, so the coupon rate (once it has hit the cap) is likely to remain constant for a while, making the loan behave more like a fixed rate mortgage. Second, and rather surprisingly, while the ranking of the indices by interest rate sensitivity is the same as the no-cap case when interest rates are low, this ranking is *reversed* at higher interest rates. The intuition here is that the current level of the index is only 8.5% (EDCOFI) or 10.5% (FHFB). If interest rates move a long way above this value, the slower speed-of-adjustment of FHFB and EDCOFI, compared with LIBOR and the 1 yr. T-Bill rate, mean that it will take some time for the coupon rate to rise sufficiently for the cap to become binding, during which time interest rates will probably have fallen (due to mean-reversion). As a result, at high interest rates, the expected amount of time that the coupon rate will remain equal to the cap rate is *lowest* for the slowest-adjusting indices, with the paradoxical result that mortgages based on the *fastest* adjusting index actually behave *most* like a fixed rate mortgage in this region.

To see the impact of different reset frequencies, Table 2 compares the effective durations of mortgages indexed to EDCOFI, FHFB and the 1yr. T-Bill rate (again, LIBOR and the 1yr. T-Bill rate show almost identical results), for reset periods of length 1 month and 12 months. The ranking of the indices by interest rate sensitivity is unaltered by changing the reset period from yearly to monthly. However, the exact results change appreciably. For all three indices, loans with annual reset have longer duration than loans with monthly reset. The differences between the indices become relatively more pronounced as the coupon resets more and more frequently.

To emphasize the importance of using the right model for the dynamics of the index, Table 3 compares the effective durations of loans indexed to EDCOFI and FHFB, using the correct model (shown in Table 1), and using a model which merely regresses the index on the current 3 month T-Bill rate, thus correctly estimating the average long term relationship between the index and Treasury rates, but ignoring the lag in the index. Results are shown for both annual and monthly coupon reset. Assuming annual reset, the two models yield similar results for EDCOFI, but the true model yields substantially higher estimates of interest rate sensitivity for FHFB loans. Indeed, the ranking of the two indices in terms of interest rate sensitivity is reversed when the wrong models are used. The impact of using the wrong model is more extreme for monthly reset. The less frequently coupon reset occurs, the more significant the impact of ignoring the lags in the index.

# 4 Summary

This article analyzes the interest rate sensitivity of adjustable rate mortgages based on the four most commonly used indices,

1. The one year constant maturity Treasury yield,
2. One year LIBOR,
3. The Federal Housing Finance Board (FHFB) national average contract interest rate,
4. The Eleventh District Cost-of-Funds Index (EDCOFI).

We find that a simple partial adjustment model closely describes the behavior of EDCOFI, the FHFB average contract rate, and one year LIBOR, and we develop an ARM valuation methodology which allows us simultaneously to capture the effects of index dynamics, discrete coupon adjustment, and nearly all standard ARM contract features, including both lifetime and periodic caps and floors. Our methodology allows us either to calculate an optimal prepayment strategy for mortgage holders, or to use an empirical prepayment function.

We conduct a systematic comparison of the properties of ARMs based on the different indices, and find that the interest rate sensitivity of an ARM depends significantly on its contract terms, on the dynamics of the index underlying the mortgage, and on the prepayment behavior of the mortgage holders. Ignoring any of these interacting factors will lead to significant errors in measuring and hedging the interest rate risk of these mortgages. We find in particular that, for *uncapped* ARMs, the slower the speed of adjustment of the index, the more interest rate sensitive is an ARM based on that index (since it more closely resembles a fixed rate loan). However, for otherwise identical ARMs with a lifetime maximum coupon rate cap, this ordering can paradoxically be *reversed*. We also find that changing the coupon reset frequency has a significant impact on the interest sensitivity of an ARM, and that using the wrong model for the index (ignoring the slow speed of adjustment) yields misleading results for the interest rate sensitivity of an ARM. The more often the coupon rate is reset, the worse the problem. Our valuation methodology allows us, for the first time, to quantify the effect of all of the interacting factors which affect the interest rate sensitivity of an ARM.

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| Independent Variables | Dependent Variable | | |
|---|---|---|---|
| | EDCOFI | FHFB Average Contract Rate | One Year LIBOR (Weekly Avg.) |
| Constant | .3306** (2.065) | .366*** (6.062) | .6688** (2.068) |
| January dummy | $-.0632$** ($-2.094$) | | |
| February dummy | .1517*** (3.441) | | |
| First lag of EDCOFI | .8430*** (17.415) | | |
| First lag of FHFB Avg. Contract Rate | | .8966*** (91.979) | |
| First lag of One Year LIBOR | | | .1361** (2.622) |
| Three month T-Bill rate | .1263*** (3.539) | .0928*** (11.340) | .9148*** (16.002) |
| Autoregressive Parameters | | | $(1 - .969\,B + .214\,B^2)$ $(-11.56)$ $(2.55)$ |
| $R^2$ | .994 | .997 | .857 |
| Breusch-Pagan test for heteroscedasticity, $\chi^2_m$ | 9.2 | 4.2 | 8.7 |
| Durbin test for AR(1) (t-statistic) | .684 | $-.195$ | .803 |

* Significant at the 10% level.
** Significant at the 5% level.
*** Significant at the 1% level.

Table 1: Estimates for Adjustment Models of ARM Indices, July 1981 – May 1993 (t-statistics in parentheses).

| $r$ (%) | Reset | EDCOFI | FHFB | 1yr T-Bill |
|---|---|---|---|---|
| 10% | Annual | 0.79 | 1.01 | 0.51 |
| | Monthly | 0.34 | 0.70 | 0.02 |
| 15% | Annual | 0.98 | 1.10 | 0.83 |
| | Monthly | 0.60 | 0.74 | 0.08 |
| 20% | Annual | 1.00 | 1.10 | 0.85 |
| | Monthly | 0.60 | 0.80 | 0.08 |

Table 2: Comparison of interest rate sensitivities for ARMS based on EDCOFI, FHFB rate, 1yr T-Bill rate, for different adjustment periods.

| | | EDCOFI | | FHFB | |
|---|---|---|---|---|---|
| $r$ (%) | Model | Annual | Monthly | Annual | Monthly |
| 10% | Lag | 0.79 | 0.34 | 1.01 | 0.70 |
| | No Lag | 0.79 | 0.21 | 0.60 | 0.02 |
| 15% | Lag | 0.98 | 0.60 | 1.10 | 0.74 |
| | No Lag | 1.02 | 0.45 | 0.94 | 0.14 |
| 20% | Lag | 1.00 | 0.60 | 1.10 | 0.80 |
| | No Lag | 1.00 | 0.41 | 1.03 | 0.49 |

Table 3: Comparison of interest rate sensitivities calculated for ARMS based on EDCOFI and FHFB rate, using models for the index with and without dependence on lagged index.

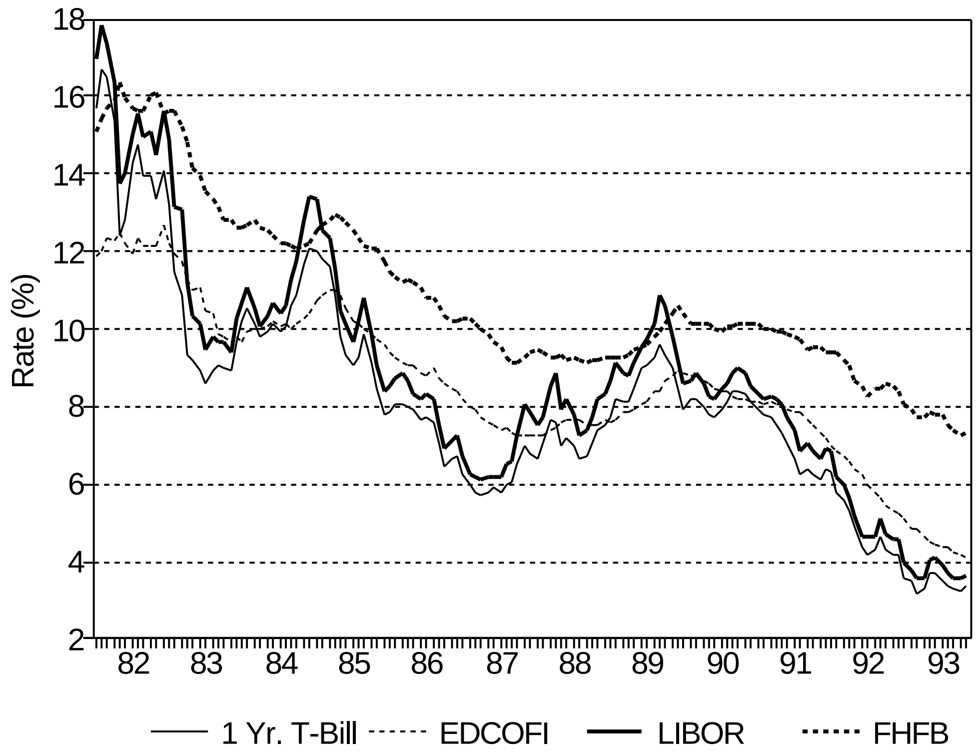


Figure 1: The main ARM indices, July 1981 – May 1993.

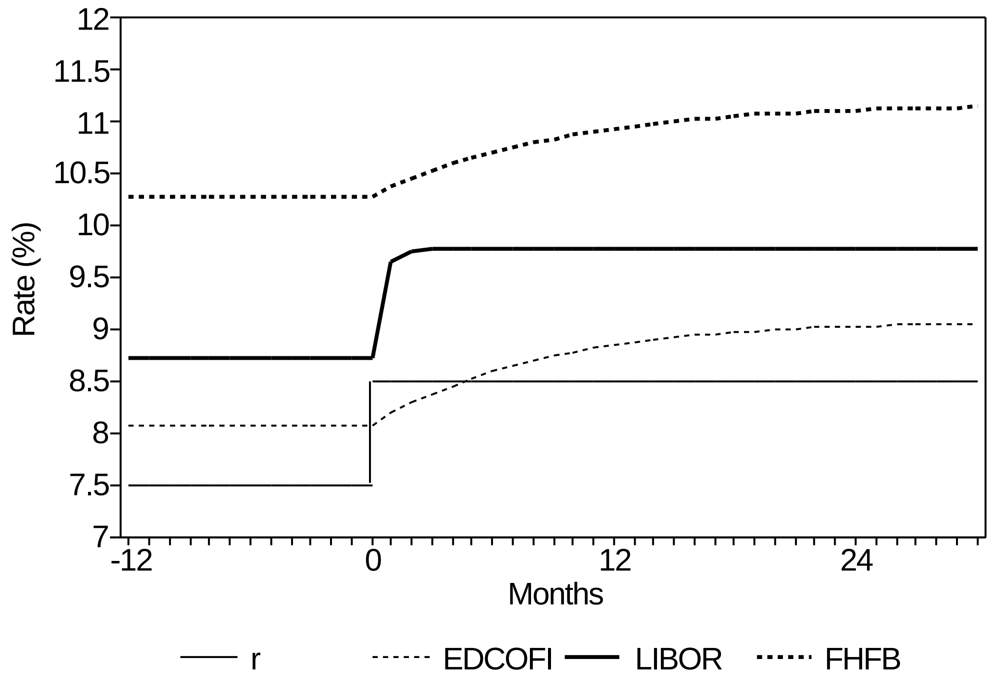


Figure 2: Example of the lags in the different indices' responses to movements in the term structure. The graph shows the movement in the different indices resulting from a jump in the short term riskless interest rate from 7.5% to 8.5%.

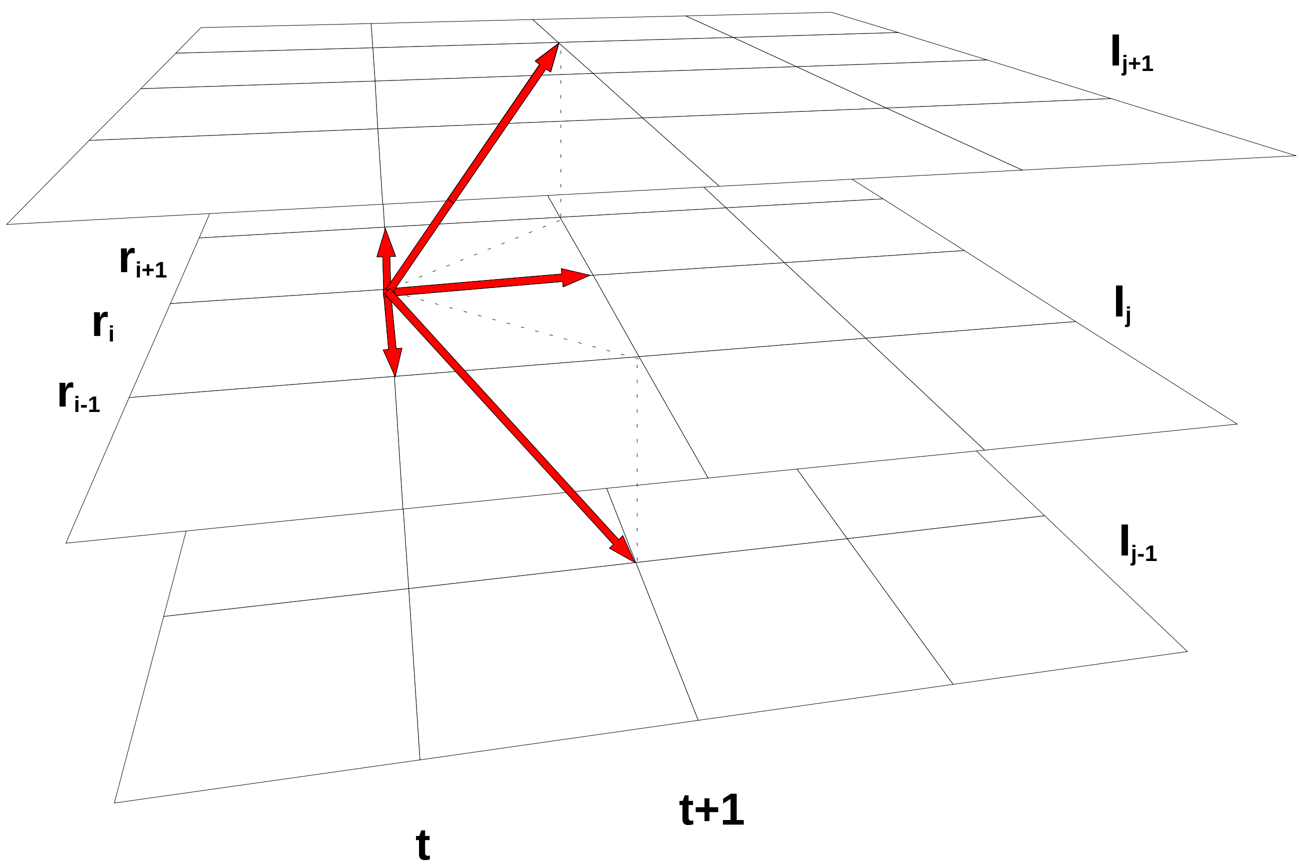


Figure 3: Extended Crank Nicholson algorithm.

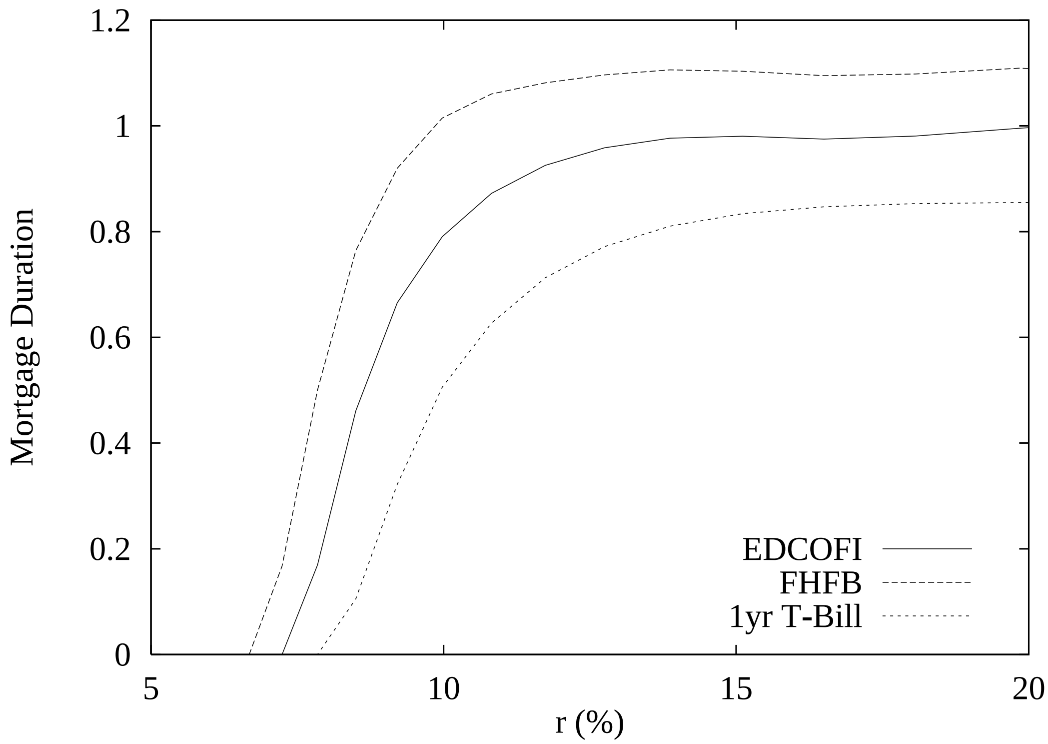


Figure 4: Interest rate sensitivities of ARMs based on different indices (results for LIBOR, which is not shown, are visually indistinguishable from those for the 1yr. T-Bill rate). Coupon is adjusted annually, and has no caps.

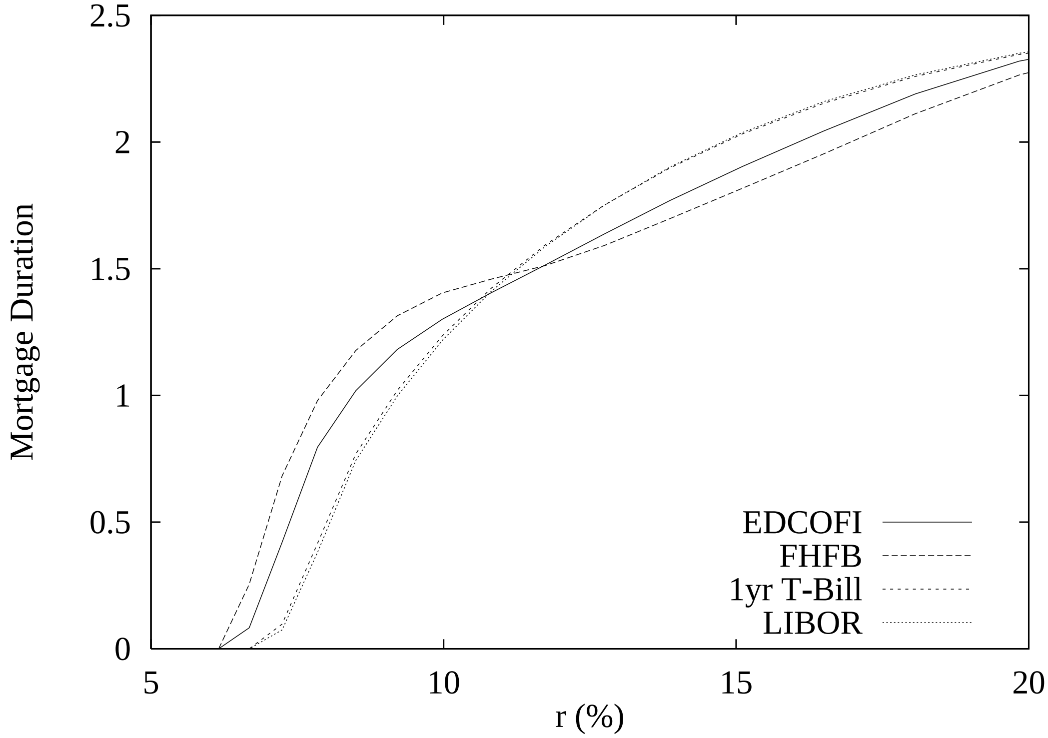


Figure 5: Interest rate sensitivities of ARMs based on different indices. Coupon is adjusted annually, and has a cap of 15%.