Core Stability of Minimum Coloring Games

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Abstract

In cooperative game theory, a characterization of games with stable cores is known as one of the most notorious open problems. We study this problem for a special case of the minimum coloring games, introduced by Deng, Ibaraki & Nagamochi, which arises from a cost allocation problem when the players are involved in conflict. In this paper, we show that the minimum coloring game on a perfect graph has a stable core if and only if every vertex of the graph belongs to a maximum clique. We also consider the problem on the core largeness, the extendability, and the exactness of minimum coloring games. As a consequence, we show that it is coNP-complete to decide whether a given game has a large core, is extendable, or is exact.

1 Introduction

One of the scopes of cooperative game theory is to establish the criterion of how to distribute a given revenue or cost among the agents, when they work in cooperation, in a fair manner. Since the effect of cooperation is usually non-linear and non-additive, the proportional division might not be considered fair. Several criteria, called solutions, are proposed by many researchers. When game theory was founded, von Neumann & Morgenstern [29] proposed a solution called a stable set, which turned out to be very useful for the analysis of a lot of bargaining situations but also turned out to be too difficult to reveal some fundamental properties. Much easier to investigate is the core, due to Gillies [15]. So, people are interested in when the core and the stable set coincide, namely when the core is stable. This question is known as one of the most notorious problems. So far, there are some necessary or sufficient conditions known (see, e.g., [28]), but they are far from a characterization of cooperative games with stable cores. From the computational point of view, the problem around stable sets is also eccentric. Deng & Papadimitriou [11] pointed out that determining the existence of a stable set for a given cooperative game is not known to be computable, and it is still unsolved.

Since combinatorial optimization problems can be found in several real-world situations, naturally they also raise some revenue/cost allocation problems. A *combinatorial optimization game* is a cooperative game which arises from a combinatorial optimization problem. There are many kinds of combinatorial optimization games proposed and studied, according to the underlying combinatorial optimization problems [8]. However, as far as the core stability is concerned, almost nothing is studied. The only exception is a work by Solymosi & Raghavan [27] who studied the core stability of an assignment game.

In this paper, we study core stability of minimum coloring games introduced by Deng, Ibaraki & Nagamochi [9], which arise from cost allocation problems when the agents are involved in conflict [23]. The reason that we restrict to perfect graphs is that it is NP-complete to decide whether a given graph yields a minimum coloring game with a nonempty core [9] (meaning that there seems no good characterization of minimum coloring games with nonempty cores; the readers unfamiliar with the notion of NP-completeness should refer to a textbook, for example, by Garey & Johnson

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[14].) and that a graph G is perfect if and only if the minimum coloring game on G is totally balanced [10], where the total balancedness is a quite nice property. We prove that the minimum coloring game on a perfect graph has a stable core if and only if every vertex belongs to a maximum clique. We also consider the problem on the extendability, the largeness, and the exactness of cores, which are concepts related to core stability. We prove that these three concepts are equivalent for the minimum coloring game on a perfect graph, and also equivalent to that every clique is contained in a maximum clique.

Armed with our characterizations, we also study algorithmic aspects of these properties. First we give a polynomial-time algorithm to determine whether a given perfect graph yields a minimum coloring game with stable core or not. On the other hand, we prove that it is hard (or coNP-complete, technically speaking) to determine whether a given perfect graph yields a minimum coloring game which is extendable, exact or with large core. To the best of our knowledge, this is the first computational intractability result for extendability, exactness and core largeness of cooperative games.

2 Preliminaries

2.1 Notation

Throughout the paper, for a vector $\boldsymbol{x} \in \mathbb{R}^N$ and $S \subseteq N$, we write $\boldsymbol{x}(S) := \sum\{x_i \mid i \in S\}$. When $S = \emptyset$, set $\boldsymbol{x}(S) := 0$. For a subset $S \subseteq N$ of a finite set N, the *characteristic vector* of S is a vector $\mathbf{1}_S \in \{0,1\}^N$ defined as $(\mathbf{1}_S)_i = 1$ if $i \in S$ and $(\mathbf{1}_S)_i = 0$ otherwise. Note that for $S, T \subseteq N$ it holds that $\mathbf{1}_S(T) = \sum\{(\mathbf{1}_S)_i \mid i \in T\} = |S \cap T|$. We use the notation $A \subset B$ to mean that "A is a proper subset of B."

2.2 Graphs

A *graph* G is a pair $G = (V, E)$ of a finite set V and a set $E \subseteq \binom{V}{2}$ of 2-element subsets of V. An element of V is called a *vertex* and V itself is called the *vertex set* of G; an element of E is called an *edge* and E itself is called the *edge set* of G. For $U \subseteq V$, the subgraph of G induced by U is denoted by $G[U]$, where the vertices of $G[U]$ are the elements of U and the edges of $G[U]$ are the edges of G which are also 2-element subsets of U. The *complement* of $G = (V, E)$ is a graph with vertex set V and edge set the complement of E.

A *clique* is a vertex subset inducing a graph with every pair being an edge (such a graph is called *complete*). A clique is *maximal* if none of its proper supersets is a clique. A clique is *maximum* if it has a maximum size among all cliques. The size of a maximum clique of G is denoted by $\omega(G)$. An *independent set* is a vertex subset inducing a graph with no edge. A *maximal* independent set and a *maximum* independent set are defined analogously to a clique. A *coloring* of $G = (V, E)$ is a mapping $c : V \to \mathbb{N}$ such that $c(u) \neq c(v)$ for every $\{u, v\} \in E$. A *minimum coloring* of G is a coloring with minimum possible $|c(V)|$. The *chromatic number* of G is $|c(V)|$ of a minimum coloring c of G and denoted by $\chi(G)$. Conventionally, the chromatic number of a graph with no vertex is defined to be zero. A graph $G = (V, E)$ is *perfect* if $\omega(G[U]) = \chi(G[U])$ for every $U \subseteq V$. A prominent example of non-perfect graphs is a cycle with five vertices.

2.3 Cooperative games

A *cooperative game* (or simply a *game*) is a pair (N, γ) of a nonempty finite set N and a function $\gamma : 2^N \to \mathbb{R}$ satisfying $\gamma(\emptyset) = 0$. An element of N is called a *player* of the game, and γ is called the *characteristic function* of the game. Furthermore, each subset $S \subseteq N$ is called a *coalition*. Literally, for $S \subseteq N$ the value $\gamma(S)$ is interpreted as the total profit (or the total cost) for the players in

S when they work in cooperation. In particular, $\gamma(N)$ represents the total profit (or cost) for the whole players when they all agree on working together. When γ represents a profit, we call the game a *profit game*. On the other hand, when γ represents a cost, we call the game a *cost game*. (Thus, the terms "profit game" and "cost game" are not mathematically determined. They are just determined by the interpretation of a game.) In this paper, we will mainly consider a certain class of cost games.

One of the aims of cooperative game theory is to provide a concept of "fairness," namely, how to allocate the total cost (or profit) $\gamma(N)$ to each player in a "fair" manner when we take all the $\gamma(S)$'s into account. Now, we concentrate on cost games, and define some cost allocations which are considered fair in cooperative game theory. Formally, a cost allocation is defined as a preimputation in the terminology of cooperative game theory. A *preimputation* of a cost game (N, γ) is a vector $\boldsymbol{x} \in \mathbb{R}^N$ satisfying $\boldsymbol{x}(N) = \gamma(N)$. Each component x_i expresses how much the player $i \in N$ should owe according to the cost allocation $\boldsymbol{x}$.

Let (N, γ) be a cost game. A vector $\boldsymbol{x} \in \mathbb{R}^N$ is called an *imputation* if $\boldsymbol{x}$ satisfies the following conditions: $\boldsymbol{x}$ is a preimputation of (N, γ) and $x_i \leq \gamma(\{i\})$ for every $i \in N$. The set of all imputations of (N, γ) is denoted by $\mathsf{Imp}(N, \gamma)$. A vector $\boldsymbol{x} \in \mathbb{R}^N$ is called a *core allocation* if $\boldsymbol{x}$ satisfies the following conditions: $\boldsymbol{x}$ is an imputation of (N, γ) and $\boldsymbol{x}(S) \leq \gamma(S)$ for all $S \subseteq N$. The set of all core allocations of (N, γ) is called the *core* of (N, γ) and denote by $\mathsf{Core}(N, \gamma)$. The core was introduced by Gillies [15].

Note that $\mathsf{Core}(N, \gamma) \subseteq \mathsf{Imp}(N, \gamma)$ and both can be empty. Therefore, a cost game with a nonempty core is especially interesting, and such a cost game is called *balanced*. Moreover, we call a cost game *totally balanced* if each of the subgames is balanced. (Here, a *subgame* of a cost game (N, γ) is a cost game $(T, \gamma^{(T)})$ for some nonempty $T \subseteq N$ defined as $\gamma^{(T)}(S) = \gamma(S)$ for each $S \subseteq T$.) Naturally, a totally balanced game is also balanced. A special subclass of the totally balanced games consists of submodular games (Shapley [25]), where a cost game (N, γ) is called *submodular* (or *concave*) if it satisfies the following condition: $\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$ for all $S, T \subseteq N$. Therefore, we have a chain of implications: submodularity $\Rightarrow$ total balancedness $\Rightarrow$ balancedness. These implications are fundamental in cooperative game theory.

Let (N, γ) be a balanced cost game. The core $\mathsf{Core}(N, \gamma)$ is called *stable* if for every $\boldsymbol{y} \in \mathsf{Imp}(N, \gamma) \setminus \mathsf{Core}(N, \gamma)$ there exist a core allocation $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ and a nonempty coalition $S \subset N$ such that $\boldsymbol{x}(S) = \gamma(S)$ and $x_i < y_i$ for each $i \in S$. (The concept of stability is due to von Neumann & Morgenstern [29].) The core $\mathsf{Core}(N, \gamma)$ is called *large* if for every $\boldsymbol{y} \in \mathbb{R}^N$ satisfying that $\boldsymbol{y}(S) \leq \gamma(S)$ for all $S \subseteq N$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $\boldsymbol{y} \leq \boldsymbol{x}$. (The largeness was introduced by Sharkey [26].) The game (N, γ) is called *extendable* if for every nonempty $S \subseteq N$ and every $\boldsymbol{y} \in \mathsf{Core}(S, \gamma^{(S)})$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $x_i = y_i$ for all $i \in S$. (The extendability was introduced by Kikuta & Shapley [17], and the name was given by van Gellekom, Potters & Reijnierse [28].) The game (N, γ) is called *exact* if for every $S \subset N$ there exists $\boldsymbol{x} \in \mathsf{Core}(N, \gamma)$ such that $\boldsymbol{x}(S) = \gamma(S)$. (The exactness was first defined by Schmeidler [24].) Note that an exact game is always totally balanced.

Here, we summarize the known relationships among these classes of games. See also Figure 1. Sharkey [26] showed that if a game is submodular then it has a large core. Kikuta & Shapley [17] showed that if a balanced game has a large core then it is extendable, and if a balanced game is extendable then it has a stable core. Sharkey [26] showed that if a totally balanced game has a large core then it is exact. Biswas, Parthasarathy, Potters & Voorneveld [3] pointed out that he actually proved that extendability implies exactness. The reverse directions in Figure 1 do not hold in general. (Some of them are explained by van Gellekom, Potters and Reijnierse [28].)

3 Minimum coloring games

Let $G = (V, E)$ be a graph. The *minimum coloring game* on G is a cost game (V, χ_G) where $\chi_G : 2^V \to \mathbb{R}$ is defined as $\chi_G(S) := \chi(G[S])$ for all $S \subseteq V$. Furthermore, we always assume that $V \neq \emptyset$

submodularity			$\nearrow$	exactness	$\searrow$	
$\Downarrow$						
large core $\wedge$ total balancedness	$\Rightarrow$	extendability $\wedge$ total balancedness	$\Rightarrow$	stable core $\wedge$ total balancedness	$\Rightarrow$	total balancedness
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
large core $\wedge$ balancedness	$\Rightarrow$	extendability $\wedge$ balancedness	$\Rightarrow$	stable core $\wedge$ balancedness	$\Rightarrow$	balancedness

Figure 1: Implication relationship. The symbol "$\wedge$" represents "and."

when we consider the minimum coloring game, so that the minimum coloring game meets the definition of a cooperative game.

First, let us make some easy observations.

LEMMA 3.1 *Let $G = (V, E)$ be a graph and (V, χ_G) be the minimum coloring game on G.*

(a) *For every $S \subseteq T \subseteq V$, it holds that $\chi_G(S) \leq \chi_G(T)$.*

(b) *For every nonempty independent set $I \subseteq V$ of G it holds that $\chi_G(I) = 1$. In particular, $\chi_G(\{v\}) = 1$ for each $v \in V$.*

(c) *If $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$, then it holds that $0 \leq x_v \leq 1$ for every $v \in V$.*

PROOF. (a) Since $S \subseteq T$, we have $\chi(G[S]) \leq \chi(G[T])$. The claim follows from the definition of χ_G.

(b) For a nonempty independent set I, we have $\chi(G[I]) = 1$.

(c) Let $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. By the definition of the core and the part (b), we have that $x_v \leq \chi_G(\{v\}) = 1$. Suppose that $x_v < 0$ for contradiction. Then, it holds that $\chi_G(V) < \chi_G(V) - x_v$. Furthermore, by part (a) we have $\chi_G(V \setminus \{v\}) \leq \chi_G(V)$, and also we have $\boldsymbol{x}(V) = \chi_G(V)$ since $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. Therefore, we obtain $\chi_G(V \setminus \{v\}) \leq \chi_G(V) < \chi_G(V) - x_v = \boldsymbol{x}(V) - x_v = \boldsymbol{x}(V \setminus \{v\})$. This is a contradiction to $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$. □

Deng, Nagamochi & Ibaraki [9] proved that it is NP-complete to decide whether the minimum coloring game on a given graph is balanced. Subsequently, Deng, Ibaraki, Nagamochi & Zang [10] showed that the minimum coloring game on a graph G is totally balanced if and only if G is perfect. So the decision problem on the total balancedness of a minimum coloring game is as hard as recognizing perfect graphs, which was found to be solved in polynomial time [4, 7]. Furthermore, Okamoto [22] showed that the minimum coloring game on a graph G is submodular if and only if G is complete multipartite. So we can decide whether a given graph yields a submodular minimum coloring game in polynomial time. The following proposition due to Okamoto [23] characterizes the core of the minimum coloring game on a perfect graph. This will be used nicely in a later investigation.

PROPOSITION 3.1 (OKAMOTO [23]) *Let $G = (V, E)$ be a perfect graph. Then, the core of the minimum coloring game (V, χ_G) is the convex hull of the characteristic vectors of maximum cliques of G.*

4 Results

4.1 Core stability

The following theorem characterizes totally balanced minimum coloring games with stable cores.

THEOREM 4.1 *Let $G = (V, E)$ be a perfect graph. Then, the minimum coloring game (V, χ_G) has a stable core if and only if every vertex $v \in V$ belongs to a maximum clique of G.*

First we prove the only-if part of the theorem. The proof uses the following lemma.

LEMMA 4.1 *Let $G = (V, E)$ be a graph such that the minimum coloring game (V, χ_G) is balanced. If (V, χ_G) has a stable core, then for every $v \in V$ there exists a core allocation $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $x_v \neq 0$.*

PROOF. Assume that $\mathsf{Core}(V, \chi_G)$ is stable, and suppose, for the contradiction, there exists a vertex $v \in V$ such that

(1) $\quad x_v = 0 \quad$ for all $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$.

(Particularly $V \neq \emptyset$.) Take such a vertex v. Let $\hat{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)$ be an arbitrary core allocation. Since $V \neq \emptyset$, it holds that $\chi_G(V) > 0$. So, there exists $w \in V$ such that $\hat{x}_w > 0$. Now, define $\boldsymbol{y} \in \mathbb{R}^V$ as

$$y_u := \begin{cases} \hat{x}_u & \text{if } u \notin \{v, w\}, \\ \hat{x}_w & \text{if } u = v, \\ 0 & \text{if } u = w. \end{cases}$$

Namely, $\boldsymbol{y}$ is obtained from $\hat{\boldsymbol{x}}$ by interchanging the v-th component and the w-th component. Then, $\boldsymbol{y}$ is an imputation of (V, χ_G). Since $y_v = \hat{x}_w > 0$, due to (1), we can see that $\boldsymbol{y}$ is not a core allocation. Hence, $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$.

Since $\mathsf{Core}(V, \chi_G)$ is stable, there exist a nonempty set $S \subset V$ and a core allocation $\overline{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)$ such that $\overline{\boldsymbol{x}}(S) = \chi_G(S)$ and $\overline{x}_u < y_u$ for every $u \in S$. Now we claim that $S \setminus \{v\} \neq \emptyset$. To show this, suppose not, i.e., $S \setminus \{v\} = \emptyset$. Since $S \neq \emptyset$, we have that $S = \{v\}$. Then, it follows that

$$\begin{aligned} \chi_G(\{v\}) &= \overline{x}_v && \text{(since } \chi_G(S) = \overline{\boldsymbol{x}}(S)\text{)} \\ &< y_v && \text{(since } \overline{x}_u < y_u \text{ for every } u \in S\text{)} \\ &\leq \chi_G(\{v\}) && \text{(since } \boldsymbol{y} \in \mathsf{Imp}(V, \chi_G)\text{).} \end{aligned}$$

This is a contradiction, hence the claim follows.

Going back to the proof of Lemma 4.1, we obtain

$$\begin{aligned} \chi_G(S) &= \overline{\boldsymbol{x}}(S) \\ &= \overline{\boldsymbol{x}}(S \setminus \{v\}) && \text{(by (1))} \\ &< \boldsymbol{y}(S \setminus \{v\}) && \text{(by the choice of } \overline{\boldsymbol{x}} \text{ and Claim above)} \\ &\leq \hat{\boldsymbol{x}}(S \setminus \{v\}) && \text{(by the construction of } \boldsymbol{y}\text{)} \\ &\leq \chi_G(S \setminus \{v\}) && \text{(since } \hat{\boldsymbol{x}} \in \mathsf{Core}(V, \chi_G)\text{)} \\ &\leq \chi_G(S) && \text{(by Lemma 3.1(a)).} \end{aligned}$$

This is a contradiction. □

Then, let us prove the only-if part of the theorem.

PROOF OF THE ONLY-IF PART OF THEOREM 4.1. Assume that (V, χ_G) has a stable core. By Lemma 4.1, for every $v \in V$ there exists a core allocation $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $x_v > 0$. On the other hand, by Proposition 3.1, $\boldsymbol{x}$ is a convex combination of the characteristic vectors of maximum cliques of G. Therefore, at least one maximum clique of G must contain v. □

In order to prove the if part, we need some more lemmas.

LEMMA 4.2 *Let $G = (V, E)$ be a graph with $\chi(G) = \omega(G)$. Then, there exists a nonempty independent set $I \subseteq V$ such that $K \cap I \neq \emptyset$ for every maximum clique K of G.*

PROOF. Consider a minimum coloring of G and take the vertices colored by an identical color. Denote by I the set of these vertices. By the construction, I is an independent set. On the other hand, in each maximum clique K of G all colors used to color G can be found since $\chi(G) = \omega(G) = |K|$. Namely, every maximum clique intersects I. Thus, I is a desired independent set. □

Here is another lemma.

LEMMA 4.3 *Let $G = (V, E)$ be a perfect graph, and consider the minimum coloring game (V, χ_G). Then, for every $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$ there exists a nonempty independent set $I \subseteq V$ such that $\boldsymbol{y}(I) > \chi_G(I)$ and $y_v > 0$ for every $v \in I$.*

PROOF. Fix $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$ arbitrary, and define $\mathcal{S} := \{S \subseteq V \mid \boldsymbol{y}(S) > \chi_G(S)$ and $y_v > 0$ for every $v \in S\}$.

First, note that $\mathcal{S} \neq \emptyset$. To see this, since $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$, there exists $T' \subseteq V$ such that $\boldsymbol{y}(T') > \chi_G(T')$. Let $T := T' \setminus \{v \in T' \mid y_v \leq 0\}$. Then, it holds that $\boldsymbol{y}(T) \geq \boldsymbol{y}(T') > \chi_G(T') \geq \chi_G(T)$. (The last inequality is due to $T \subseteq T'$ and Lemma 3.1(a).) Since $y_v > 0$ for each $v \in T$, it follows that $T \in \mathcal{S}$. This implies that $\mathcal{S}$ is nonempty.

Choose $S \in \mathcal{S}$ of minimum size. Since G is perfect, we have that $\chi(G[S]) = \omega(G[S])$. By Lemma 4.2, there exists a nonempty independent set $I \subseteq S$ such that for every maximum clique K of $G[S]$ we have $K \cap I \neq \emptyset$. Now, we claim that $I \in \mathcal{S}$. (This proves the lemma.) First of all, since $I \subseteq S$ it holds that $y_v > 0$ for every $v \in I$. So it suffices to show that $\boldsymbol{y}(I) > \chi_G(I)$.

Since I intersects with every maximum clique of $G[S]$, we can see that $\omega(G[S \setminus I]) < \omega(G[S])$. Since G is perfect, this means that

(2) $\quad \chi_G(S \setminus I) < \chi_G(S).$

Since I is nonempty, we have $|S \setminus I| < |S|$. By the minimality of S, it holds that

(3) $\quad \boldsymbol{y}(S \setminus I) \leq \chi_G(S \setminus I).$

Now, we obtain the following.

$$\begin{aligned}
\boldsymbol{y}(I) &= \boldsymbol{y}(S) - \boldsymbol{y}(S \setminus I) && (I \subseteq S) \\
&> \chi_G(S) - \chi_G(S \setminus I) && (S \in \mathcal{S} \text{ and } (3)) \\
&\geq 1 && ((2) \text{ and the integrality of } \chi_G) \\
&= \chi_G(I) && (\text{Lemma 3.1(b)}).
\end{aligned}$$

This concludes the proof. □

Now, we are ready to prove the if part of Theorem 4.1.

PROOF OF THE IF PART OF THEOREM 4.1. Let $\boldsymbol{y} \in \mathsf{Imp}(V, \chi_G) \setminus \mathsf{Core}(V, \chi_G)$. Then, by Lemma 4.3, there exists a nonempty independent set $I \subseteq V$ such that $\boldsymbol{y}(I) > \chi_G(I) = 1$ and $y_v > 0$ for every $v \in I$. Denote by $\mathcal{K}$ the set of maximum cliques of G. To every vertex $v \in I$, we assign a maximum clique $K(v) \in \mathcal{K}$ such that $v \in K(v)$, and fix this assignment. By our assumption, this assignment is well-defined. Since I is an independent set, this assignment is injective.

For every $K \in \mathcal{K}$, let

$$\lambda_K := \begin{cases} \dfrac{y_v}{\boldsymbol{y}(I)} & \text{if } K = K(v) \text{ for some } v \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Since the assignment $v \mapsto K(v)$ is injective, the value λ_K is well-deined. Then, for each $K \in \mathcal{K}$, we have that $0 \leq \lambda_K \leq 1$ (since $y_v > 0$ for every $v \in I$ and $\boldsymbol{y}(I) > 1$ by the choice of I by the choice of I with Lemma 4.3, and $y_v \leq 1$ for every $v \in I$ by Lemma 3.1(b) and the definition of an imputation). Furthermore, we can check that $\sum_{K \in \mathcal{K}} \lambda_K = 1$. Therefore, if we let $\boldsymbol{x} := \sum_{K \in \mathcal{K}} \lambda_K \mathbf{1}_K$, by Proposition 3.1, it holds that $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$.

If $v \in I$ then $x_v = \lambda_{K(v)}$. This is because $v \notin K(u)$ for $u \in I \setminus \{v\}$. Therefore, if $v \in I$, then

$$x_v = \lambda_{K(v)} = \frac{y_v}{\boldsymbol{y}(I)} < y_v,$$

since $\boldsymbol{y}(I) > 1$. Furthermore, it holds that

$$\boldsymbol{x}(I) = \sum_{u \in I} x_u = \sum_{u \in I} \lambda_{K(u)} = 1 = \chi_G(I).$$

Thus, $\boldsymbol{x}$ is an appropriate core allocation and hence the core is stable. □

4.2 Exactness, extendability, and core largeness

We prove that exactness, extendability and core largeness are equivalent for minimum coloring games on perfect graphs. This is also characterized in terms of graphs, and summarized as the following theorem.

THEOREM 4.2 *Let $G = (V, E)$ be a perfect graph. Then, the following conditions are equivalent.*

(a) *The minimum coloring game (V, χ_G) is exact.*

(b) *The minimum coloring game (V, χ_G) is extendable.*

(c) *The core $\mathsf{Core}(V, \chi_G)$ is large.*

(d) *Every clique of G is contained in a maximum clique of G.*

First remark that the implication "(c) ⇒ (b) ⇒ (a)" is true for any kinds of games [17]. It remains to prove "(a) ⇒ (d)" and "(d) ⇒ (c)."

Let us first prove "(a) ⇒ (d)."

PROOF OF (a) ⇒ (d) OF THEOREM 4.2. Let $G = (V, E)$ be a perfect graph such that (V, χ_G) is exact. Let S be a clique of G. Then, by the definition of the exactness, there exists $\boldsymbol{x} \in \mathsf{Core}(V, \chi_G)$ such that $\boldsymbol{x}(S) = \chi_G(S) = |S|$. Denoting by $\mathcal{K}$ the set of maximum cliques of G, by Proposition 3.1, we can express $\boldsymbol{x}$ as

$$\boldsymbol{x} = \sum_{K \in \mathcal{K}} \lambda_K \mathbf{1}_K, \tag{4}$$

where $\lambda_K \geq 0$ for every $K \in \mathcal{K}$ and $\sum_{K \in \mathcal{K}} \lambda_K = 1$. Then, it holds that

$$\begin{aligned} |S| = \boldsymbol{x}(S) &= \sum_{K \in \mathcal{K}} \lambda_K \mathbf{1}_K(S) && \text{(by (4))} \\ &= \sum_{K \in \mathcal{K}} \lambda_K |S \cap K| \\ &\leq \sum_{K \in \mathcal{K}} \lambda_K |S| && \text{(since } S \cap K \subseteq S) \\ &= |S| \sum_{K \in \mathcal{K}} \lambda_K = |S| && \text{(since } \textstyle\sum_{K \in \mathcal{K}} \lambda_K = 1). \end{aligned}$$

So, the equality holds throughout the expressions, meaning that $S \cap K = S$ for each $K \in \mathcal{K}$ with $\lambda_K > 0$. Thus, S is contained in a maximum clique of G. □

To show "(d) ⇒ (c)," we use some more facts. The first one is due to van Gellekom, Potters & Reijnierse [28]. For a cost game (N, γ), let

$$L(N, \gamma) := \{\boldsymbol{y} \in \mathbb{R}^N \mid \boldsymbol{y}(S) \leq \gamma(S) \text{ for every } S \subseteq N\},$$

and call it the set of *lower vectors*.

LEMMA 4.4 (VAN GELLEKOM, POTTERS & REIJNIERSE [28]) *Let (N, γ) be a balanced cost game. Then (N, γ) has a large core if and only if $\boldsymbol{y}(N) \geq \gamma(N)$ for all extreme points $\boldsymbol{y}$ of $L(N, \gamma)$.*

In order to apply Lemma 4.4 to our setting, we have to know the extreme points of $L(V, \chi_G)$ for a perfect graph G. To keep the concentration, we postpone the proof of the next lemma to Appendix A.

LEMMA 4.5 *Let $G = (V, E)$ be a perfect graph. Then, each extreme point of $L(V, \chi_G)$ is the characteristic vector of a maximal clique of G.*

Armed with Lemmas 4.4 and 4.5, we are able to show "(d) $\Rightarrow$ (c)."

PROOF OF (d) $\Rightarrow$ (c) OF THEOREM 4.2. Let G be a perfect graph such that every clique is contained in a maximum clique of G. Choose an extreme point of $L(V, \chi_G)$. By Lemma 4.5, this is the characteristic vector of some maximal clique K of G. Namely, this extreme point is $\mathbf{1}_K$. By our assumption, K is a maximum clique of G. Therefore, it holds that $\mathbf{1}_K(V) = |K| = \omega(G) = \chi_G(V)$. Hence, by Lemma 4.4, the core is large. □

This completes the whole proof of Theorem 4.2.

5 Algorithmic aspects

In this section, using the theorems we have obtained already, we discuss the algorithmic issues for minimum coloring games. The first problem we consider is the following.

Problem: CORE STABILITY FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$.
Question: Does the minimum coloring game (V, χ_G) have a stable core?

Now, we describe an algorithm which shows the following theorem.

THEOREM 5.1 *The problem* CORE STABILITY FOR PERFECT GRAPHS *can be solved in polynomial time.*

PROOF. Consider the following algorithm.

Algorithm: A polynomial-time algorithm for CORE STABILITY FOR PERFECT GRAPHS
Input: a perfect graph $G = (V, E)$.
Output: "Yes" if $(V, \chi(G))$ has a stable core; "No" otherwise.
Step 1: $\omega(G) \leftarrow$ the weight of a maximum clique in G;
Step 2: $M \leftarrow |V|$;
Step 3: foreach vertex $v \in V$
  Step 3.1: set a weight vector $\boldsymbol{w} \in \mathbb{R}^V$ as $w_v = M$ and $w_u = 1$ $(u \in V \setminus \{v\})$;
  Step 3.2: $\omega(G, \boldsymbol{w}) \leftarrow$ the maximum weight of a clique in G with respect to $\boldsymbol{w}$;
  Step 3.3: if $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$ **then return** "No";
Step 4: return "Yes."

Let us prove that the algorithm above is correct. The first observation is that in each "for-each" loop we compute a clique K_v of maximum size which contains v. That is just because M is huge. (So, it is not important for M to be exactly $|V|$. It can be larger but must be polynomially bounded.) Now, if $|K_v| < \omega(G)$, then we can see that a maximum clique containing v is not a maximum clique of G. Namely, v is not contained in any maximum clique of G. Then, by Theorem 4.1, the game does not have a stable core. Therefore, we have to check that $|K_v| < \omega(G)$ if and only if $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$ (i.e., the condition in Step 3.3 is true). First of all, we can see that

$|K_v| = \omega(G, \boldsymbol{w}) - M + 1$. So, we have that $|K_v| - \omega(G) = \omega(G, \boldsymbol{w}) - \omega(G) + M - 1$. Hence, $|K_v| < \omega(G)$ holds if and only if $\omega(G, \boldsymbol{w}) - \omega(G) < M - 1$. This completes the proof of the correctness.

Now, we discuss the running time of the algorithm. Computing a maximum weight clique in a perfect graph can be done in polynomial time [16]. So, Steps 1 and 3.2 can be executed in polynomial time. Step 2 is also fine. In the "for-each" loop, Step 3.1 can be done swiftly. The condition check in Step 3.3 is easy. The number of iterations of the for-each loop is at most $|V|$. Hence, the overall running time is polynomial in the size of input. □

Next, we discuss the following three problems.

Problem: EXTENDABILITY FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$.
Question: Is the minimum coloring game (V, χ_G) extendable?

Problem: EXACTNESS FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$.
Question: Is the minimum coloring game (V, χ_G) exact?

Problem: CORE LARGENESS FOR PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$.
Question: Does the minimum coloring game (V, χ_G) have a large core?

Thanks to Theorem 4.2, these problems are equivalent to the following problem.

Problem: SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS
Instance: A perfect graph $G = (V, E)$.
Question: Do a maximum clique and a minimum maximal clique in G have the same size?

This problem turns out to be coNP-complete.

THEOREM 5.2 *The problem* SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS *is coNP-complete. Consequently,* EXTENDABILITY FOR PERFECT GRAPHS*,* EXACTNESS FOR PERFECT GRAPHS *and* CORE LARGENESS FOR PERFECT GRAPHS *are coNP-complete.*

PROOF. Let $G = (V, E)$ be a given perfect graph. To prove the membership in coNP, we give a polynomial-time checkable certificate for perfect graphs with "No" answer. The certificate consists of a pair of a maximum clique K and a maximal clique K' of G such that $|K| > |K'|$. We can check that K is really a maximum clique in polynomial time (since a maximum clique can be found in polynomial time for a perfect graph [16]) and that K' is a maximal clique in polynomial time (by testing $K' \cup \{v\}$ is not a clique for every $v \in V \setminus K'$). Since the size of a minimum maximal clique is at most $|K'|$, they form a certificate of "No" answer.

The coNP-hardness follows from the proof of a result by Zverovich [31]. For readers' convenience, we include the proof in Appendix B. □

Theorem 5.2 deals with perfect graphs in general. Now, let us discuss some special cases for which the problem can be solved in polynomial time. Observe that, due to Theorem 4.2, it suffices to compute a minimum maximal clique in a given perfect graph. If it is also a maximum clique in the graph, then all maximal cliques are maximum cliques. Then, the condition (d) in Theorem 4.2 holds. If not, then this maximal clique is not contained in a maximum clique, meaning that the condition (d) is violated. Namely, we consider the following optimization problem.

Problem: MINIMUM MAXIMAL CLIQUE
Instance: A graph G.
Feasible solution: A maximal clique K of G.
Objective: Minimize $|K|$.

There are some classes of perfect graphs for which we can solve MINIMUM MAXIMAL CLIQUE in polynomial time. They include the bipartite graphs (easy), the comparability graphs [19], the chordal graphs [13], and the complements of chordal graphs [12]. (See also an article by Kratsch [18].) For these classes of graphs, as we already observed, we can conclude the following.

THEOREM 5.3 *Consider a class of perfect graphs for which* MINIMUM MAXIMAL CLIQUE *can be solved in polynomial time. For this class of graphs,* EXTENDABILITY FOR PERFECT GRAPHS*,* EXACTNESS FOR PERFECT GRAPHS *and* CORE LARGENESS FOR PERFECT GRAPHS *can be solved in polynomial time.*

On the other hand, there are some classes of perfect graphs for which MINIMUM MAXIMAL CLIQUE is NP-hard. For these classes of graphs, the idea above does not work. However, for the complements of bipartite graphs SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS can be solved in polynomial time. The rest of this section is devoted to proving this fact. Note that for the complements of bipartite graphs MINIMUM MAXIMAL CLIQUE is NP-hard [6].

To ease our notation, we formulate the problem in the setting of the complement. Let $G = (U \cup V, E)$ be a bipartite graph with U and V forming color classes of G. (Namely, all edges of G are between U and V.) Then, SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS for the complement of G is equivalent to the following problem.

Problem: SIZE EQUALITY OF A MAXIMUM INDEPENDENT SET AND A MINIMUM MAXIMAL INDEPENDENT SET IN BIPARTITE GRAPHS

Instance: A bipartite graph G.
Question: Do a maximum independent set and a minimum maximal independent set in G have the same size?

Let us make some observations. Assume that our bipartite graph $G = (U \cup V, E)$ is a YES instance of the problem above. Then, without loss of generality, we may further assume that G has no isolated vertex (i.e., a vertex with no incident edge). That is because isolated vertices are contained in any maximal independent set. After assuming that, we can see that in the graph G, the sets U and V are maximal independent sets. Then, since G is a YES instance of the problem above, it must hold that $|U| = |V|$. Now we can observe that G contains a perfect matching (Here, a *matching* is an edge subset M of G such that every vertex of G belongs to at most one edge of M. A matching is *maximum* if it has a maximum size among all matchings. A matching is *perfect* if every vertex of G belongs to exactly one of the edges.) That is because by the König–Egerváry theorem (see for example a book by West [30]), for a bipartite graph it holds that

the size of a maximum independent set $=$
the number of vertices $-$ the size of maximum matching,

and when G is a YES instance the left-hand side is $|U|$ (since U is a maximal independent set) and the right-hand side is $|U| + |V| - m = 2|U| - m$ where m is the size of a maximum matching. Therefore, m must be $|U|$, implying that there has to be a perfect matching. This way, we have derived that if a bipartite graph $G = (U \cup V, E)$ is a YES instance of SIZE EQUALITY OF A MAXIMUM INDEPENDENT SET AND A MINIMUM MAXIMAL INDEPENDENT SET IN BIPARTITE GRAPHS, then G has a perfect matching.

However, the existence of a perfect matching is not sufficient for G to be a YES instance of our problem. (Consider a cycle with six vertices.) Actually, the following condition is proved to characterize a YES instance. To state the theorem, we need some more definitions. A bipartite graph $G = (U \cup V, E)$ is *complete bipartite* if every pair of vertices $u \in U$ and $v \in V$ forms an edge (i.e., $\{u, v\} \in E$). For a vertex v, we denote by $N(v)$ the set of vertices which are adjacent to v and call it the *neighborhood* of v.

THEOREM 5.4 *Let $G = (U \cup V, E)$ be a bipartite graph with a perfect matching. Choose an arbitrary perfect matching $M \subseteq E$ of G. Then, every maximal independent set in G is a maximum independent set of G if and only if for every edge $\{u, v\} \in M$, the graph induced by $N(u) \cup N(v)$ is complete bipartite.*

This theorem is due to Ambühl, Paffenholz, Schurr and Welzl [1].

PROOF. First we prove the only-if part. Suppose that there exists an edge $\{u, v\} \in M$ such that $N(u) \cup N(v)$ does not induce a complete bipartite graph. This means that there exists a pair of vertices $x \in N(u)$ and $y \in N(v)$ such that $\{x, y\} \notin E$. Let $z \in U$ be a unique vertex such that $\{z, y\} \in M$. We claim that $W := (U \setminus \{u, z\}) \cup \{y\}$ is a maximal independent set of G. To show that, first observe that neither $W \cup \{u\}$ nor $W \cup \{z\}$ is independent since y is adjacent to both u and z. Next, $W \cup \{v\}$ is not independent since v is adjacent to x and x belongs to W. Finally, for any $t \in V \setminus \{v, y\}$, the set $W \cup \{t\}$ is not independent since there exists an edge $\{s, t\} \in M$ and $s \in W$. This completes the proof of the claim. Now, it follows that $|W| = (|U| - 2) + 1 = |U| - 1 < |U|$. Since G has a perfect matching, U is a maximum independent set. Therefore, the maximal independent set W is not a maximum independent set. Thus, the only-if part is done.

For the if part, assume that $N(u) \cup N(v)$ induces a complete bipartite graph for every $\{u, v\} \in M$, and choose an arbitrary maximal independent set X of G. We want to show that X is a maximum independent set of G. Let $X_U := X \cap U$ and $X_V := X \cap V$. If $X_U = \emptyset$, then X must be V since X is a maximal independent set. Since G has a perfect matching, V is a maximum independent set, and so is X. The same holds when $X_V = \emptyset$. Therefore, without loss of generality, we may assume that X_U and X_V are nonempty. Since X is independent and M is a perfect matching of G, for each vertex $u \in X_U$ there exists a unique vertex $v \in V \setminus X_V$ such that $\{u, v\} \in M$. This implies that $|V \setminus X_V| \geq |X_U|$. If $|V \setminus X_V| = |X_U|$, then it follows that $|X| = |X_U| + |X_V| = |V \setminus X_V| + |X_V| = |V|$. Therefore X is a maximum independent set. Similarly, since it holds that $|U \setminus X_U| \geq |X_V|$, if $|U \setminus X_U| = |X_V|$, then X is a maximum independent set. Hence, it suffices to show that $|V \setminus X_V| = |X_U|$ or $|U \setminus X_U| = |X_V|$. To show that by contradiction, suppose that $|V \setminus X_V| > |X_U|$ and $|U \setminus X_U| > |X_V|$. This means that there exist vertices $a \in U \setminus X_U$ and $b \in V \setminus X_V$ such that $\{a, b\} \in M$. Since X is a maximal independent set, $X \cup \{a\}$ is not independent. Therefore, there must exist a vertex $b' \in X_V \cap N(a)$. Similarly, there must exist a vertex $a' \in X_U \cap N(b)$. However, by our first assumption, $N(a)$ and $N(b)$ induce a complete bipartite graph. This means that $\{a', b'\}$ is an edge of G, which contradicts the fact that X is independent and $a', b' \in X$. □

Since a maximum matching of a bipartite graph can be found in polynomial time, we obtain the following corollary of Theorem 5.4.

COROLLARY 5.1 *The problem* SIZE EQUALITY OF A MAXIMUM INDEPENDENT SET AND MINIMUM MAXIMAL INDEPENDENT SET IN BIPARTITE GRAPHS *can be solved in polynomial time. As a consequence, for the complements of bipartite graphs,* EXTENDABILITY FOR PERFECT GRAPHS, EXACTNESS FOR PERFECT GRAPHS *and* CORE LARGENESS FOR PERFECT GRAPHS *can be solved in polynomial time.*

6 Summary

We discussed the core stability problem for minimum coloring games, introduced by Deng, Ibaraki & Nagamochi [9], of perfect graphs. We obtained a good characterization for a minimum coloring game with stable core (Theorem 4.1), and this led us to a polynomial-time algorithm for the corresponding decision problem (Theorem 5.1). We also discussed the extendability, the exactness and the core largeness for minimum coloring games of perfect graphs, and characterized them in terms of a property of graphs (Theorem 4.2). With this characterization, we showed that it is coNP-complete to determine whether a given perfect graph yields the minimum coloring game which is extendable, exact, or with large core (Theorem 5.2). For some subclasses of perfect graphs, we know that there exists a polynomial-time algorithm for this problem (Theorem 5.3 and Corollary 5.1).

Interestingly, the perspective from combinatorial optimization games enables us to give the first computational intractability result for extendability, exactness and core largeness. This approach should be useful for other properties of cooperative games since graph structures in combinatorial optimization games possess potential to make many hidden properties more transparent to us so that we can connect them with hardness results more easily.

Little is known about core stability of cooperative games. We hope that this paper expanded the knowledge of this problem and gave rise to some algorithmic perspectives.

A Proof of Lemma 4.5

Here in the appendix, we include the postponed proof of Lemma 4.5.

Let $G = (V, E)$ be a perfect graph. First, let us remember the definition of $L(V, \chi_G)$:

$$L(V, \chi_G) := \{\boldsymbol{y} \in \mathbb{R}^V \mid \boldsymbol{y}(S) \leq \chi_G(S) \text{ for every } S \subseteq V\}.$$

We look at a similar polyhedron defined as follows:

$$L'(V, \chi_G) := \{\boldsymbol{y} \in \mathbb{R}^V \mid \boldsymbol{y}(I) \leq 1 \text{ for every independent set } I \subseteq V \text{ of } G\}.$$

Now, we claim the following.

LEMMA A.1 *If G is a perfect graph, then $L(V, \chi_G) = L'(V, \chi_G)$.*

PROOF. Since $\chi_G(I) = 1$ for every nonempty independent set I of G (by Lemma 3.1(b)), it holds that $L(V, \chi_G) \subseteq L'(V, \chi_G)$. To show the other direction of inclusion $L(V, \chi_G) \supseteq L'(V, \chi_G)$, Let $\boldsymbol{y} \in L'(V, \chi_G)$. We have to check that $\boldsymbol{y}(S) \leq \chi_G(S)$ for every $S \subseteq V$.

Assume that $\chi_G(S) = k$ and let us choose a minimum coloring of $G[S]$. Then, this coloring yields a partition of S into k nonempty independent sets $I_1, \ldots, I_k$. Now, for each $i \in \{1, \ldots, k\}$, we have that $\boldsymbol{y}(I_i) \leq 1$ since $\boldsymbol{y} \in L'(V, \chi_G)$. Therefore, it holds that

$$\boldsymbol{y}(S) = \sum_{i=1}^{k} \boldsymbol{y}(I_i) \leq \sum_{i=1}^{k} 1 = k = \chi_G(S).$$

This completes the proof. □

Now, for convenience, consider the complement $\overline{G}$. It is known that $\overline{G}$ is perfect if G is perfect [16, 21]. Then, we can write as follows.

$$L'(V, \chi_{\overline{G}}) = \{\boldsymbol{y} \in \mathbb{R}^V \mid \boldsymbol{y}(K) \leq 1 \text{ for every clique } K \subseteq V \text{ of } G\}.$$

Therefore, to prove Lemma 4.5, we only have to prove the following.

LEMMA A.2 *Let $G = (V, E)$ be a perfect graph. Then, each extreme point of $L'(V, \chi_{\overline{G}})$ is the characteristic vector of a maximal independent set of G.*

To prove A.2, we use the so-called "replication lemma" due to Lovász [21], which was used to prove the weak perfect graph theorem [21].

LEMMA A.3 (LOVÁSZ [21]) *Let $G = (V, E)$ be a perfect graph and $v \in V$. Put a new vertex v' and join it to v and to all the neighbors of v in G. (This procedure is called a* replication *of v.) Then, the resulting graph G' is also perfect.*

With the replication lemma, we are able to prove Lemma A.2.

PROOF OF LEMMA A.2. Let $H := \overline{G}$, to ease the notation. Let $\boldsymbol{y} \in L'(V, \chi_H)$ be an extreme point of $L'(V, \chi_H)$. Then, observe that every component of $\boldsymbol{y}$ is non-negative. To see that, suppose that $\boldsymbol{y}$ has a negative component. Then, define two vectors $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ as

$$(5) \qquad y_v^{(1)} := \begin{cases} y_v & \text{if } y_v \geq 0, \\ 0 & \text{if } y_v < 0, \end{cases} \qquad y_v^{(2)} := \begin{cases} y_v & \text{if } y_v \geq 0, \\ 2y_v & \text{if } y_v < 0. \end{cases}$$

CLAIM 1. The two vectors $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ belong to $L(V, \chi_H)$.

PROOF OF CLAIM 1. Choose a nonempty clique $K \subseteq V$ of G arbitrarily. Let $K' := \{v \in K \mid y_v \geq 0\}$ and $K'' := K \setminus K'$. Then, it holds that

$$\boldsymbol{y}^{(1)}(K) = \boldsymbol{y}^{(1)}(K') + \boldsymbol{y}^{(1)}(K'') = \boldsymbol{y}^{(1)}(K') = \boldsymbol{y}(K') \leq 1,$$

and

$$\boldsymbol{y}^{(2)}(K) = \boldsymbol{y}^{(2)}(K') + \boldsymbol{y}^{(2)}(K'') \leq \boldsymbol{y}^{(2)}(K') = \boldsymbol{y}(K') \leq 1.$$

This completes the proof of the claim. □

Furthermore, we can check that $\boldsymbol{y} \neq \boldsymbol{y}^{(1)}$, $\boldsymbol{y} \neq \boldsymbol{y}^{(2)}$ and $\boldsymbol{y} = (\boldsymbol{y}^{(1)} + \boldsymbol{y}^{(2)})/2$. Therefore, $\boldsymbol{y}$ is not an extreme point of $L'(V, \chi_H)$.

Now, since the constraints defining $L'(V, \chi_H)$ have integral coefficients, we can see that $\boldsymbol{y}$ is rational. Therefore, there exists a natural number m such that $\boldsymbol{z} := m\boldsymbol{y} \geq \mathbf{0}$ is an integral vector.

We construct a new graph H' by replication. More specifically, we replicate each vertex $v \in V$ into z_v vertices $v_1, v_2, \ldots, v_{z_v}$. (If $z_v = 0$, then we remove v from H.) By Lemma A.3, the resulting graph H' is also perfect. Let $U_v := \{v_1, \ldots, v_{z_v}\}$. (If $z_v = 0$, then we set $U_v = \emptyset$.) Note that U_v is a clique of H'.

Let K' be a maximum clique of H', and let $K'' := \{v \in V \mid K' \cap U_v \neq \emptyset\}$. Then K'' is a clique of H, and it holds that

$$\begin{aligned}
\chi(H') &= \omega(H') && \text{(since } H' \text{ is perfect)} \\
&= |K'| && \text{(since } K' \text{ is a maximum clique of } H'\text{)} \\
&= \sum_{v \in V} |K' \cap U_v| && \text{(since } \{U_v : v \in V\} \text{ partitions the vertex set of } H'\text{)} \\
&\leq \sum_{v \in K''} |K' \cap U_v| && \text{(by the construction of } K''\text{)} \\
&\leq \sum_{v \in K''} |U_v| && \text{(since } K' \cap U_v \subseteq U_v\text{)} \\
&= \boldsymbol{z}(K'') && \text{(by the construction of } U_v\text{)} \\
&= m\boldsymbol{y}(K'') && \text{(by the construction of } \boldsymbol{z}\text{)} \\
&\leq m \cdot 1 = m && \text{(since } \boldsymbol{y} \in L(V, H) \text{ and } K'' \text{ is a clique of } H\text{).}
\end{aligned}$$

Hence, H' can be colored by m colors. Let us color H' by a color set $\{1, \ldots, m\}$, and fix such a coloring $f : \bigcup_{v \in V} U_v \to \{1, \ldots, m\}$. For each $i \in \{1, \ldots, m\}$, let I_i be the set of vertices v of H such that U_v has a vertex colored by i, namely, $I_i := \{v \in V \mid \text{there exists a vertex } u \in U_v \text{ such that } f(u) = i\}$.

CLAIM 2. For every $i \in \{1, \ldots, m\}$, I_i is an independent set.

PROOF OF CLAIM 2. For contradiction, suppose that there exists $i \in \{1, \ldots, m\}$ such that I_i is not an independent set. By construction, this means that there exists two adjacent vertices a, b of H such that U_a and U_b have vertices colored by the same color i. However, since a and b are adjacent in H, all vertices in U_a are adjacent to all vertices in U_b in H', by the construction of H'. Therefore, the coloring is not a proper coloring. A contradiction. □

We get back to the proof of Lemma A.2. Since U_v is a clique of H, f colors all vertices in U_v by different colors. Therefore, for each $v \in V$, we have that $z_v = |U_v| = |\{i \in \{1, \ldots, m\} \mid I_i \ni v\}|$. Thus, it holds that $\boldsymbol{z} = \sum_{i=1}^{m} \mathbf{1}_{I_i}$, meaning that

$$\boldsymbol{y} = \frac{1}{m}\boldsymbol{z} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{I_i}.$$

Since $I_i \in L(V,H)$ for every $i \in \{1,\dots,m\}$, we now have $\boldsymbol{y}$ as a convex combination of vectors in $L(V,H)$. However, since $\boldsymbol{y}$ is an extreme point of $L(V,H)$, m must be 1 and thus $\boldsymbol{y} = \mathbf{1}_I$ for some independent set I of H.

The final thing we have to check is that such a set I is a maximal independent set. To show that, suppose not. Then, there exists a maximal independent set J such that $I \subset J$. This implies that $\mathbf{1}_I \leq \mathbf{1}_J$. Now, consider the following vector $\hat{\boldsymbol{y}} \in \mathbb{R}^V$ defined as

$$\hat{y}_v := \begin{cases} (\mathbf{1}_I)_v & \text{if } v \in I, \\ -(\mathbf{1}_I)_v & \text{if } v \notin I. \end{cases}$$

Then, we can see that $\hat{\boldsymbol{y}} \in L(V,H)$ since for each nonempty clique $K \subseteq V$ it holds that

$$\hat{\boldsymbol{y}}(K) = \hat{\boldsymbol{y}}(K \cap I) + \hat{\boldsymbol{y}}(K \setminus I) = \mathbf{1}_I(K \cap I) - \mathbf{1}_I(K \setminus I) = |K \cap I| - 0 \leq 1.$$

Furthermore, it follows that $\mathbf{1}_I = (\mathbf{1}_J + \hat{\boldsymbol{y}})/2$. Therefore, $\mathbf{1}_I$ cannot be an extreme point of $L(V,H)$. This concludes the whole proof of Lemma A.2, thus of Lemma 4.5. □

B Completion of the proof of Theorem 5.2

To prove the coNP-hardness of SIZE EQUALITY OF A MAXIMUM CLIQUE AND A MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS, we use the satisfiability problem. A *boolean variable* is a variable x which takes the value either 0 or 1. A *literal* is a boolean variable x or its negation $\neg x$. When x takes the value 0 and 1, the negation $\neg x$ takes the value 1 and 0 respectively (so the value is flipped in the negation). Let S be a set of boolean variables. A *clause* over S is a set of literals from S, e.g., $\{x, y, \neg z\}$. An assignment $\alpha : S \to \{0,1\}$ of the values *satisfies* a clause C over S if at least one of the literals in C takes the value 1.

Problem: SATISFIABILITY
Instance: A set S of boolean variables and clauses $C_1, \dots, C_m$ over S. **Question:** Does there exist an assignment $\alpha : S \to \{0,1\}$ which satisfies all clauses $C_1, \dots, C_m$?

The problem SATISFIABILITY is NP-complete [5, 20, 14].

Now, we reduce SATISFIABILITY to SIZE EQUALITY OF MAXIMUM CLIQUE AND MINIMUM MAXIMAL CLIQUE IN PERFECT GRAPHS in polynomial time. Let a set $S = \{x_1, x_2, \dots, x_n\}$ of boolean variables and a set $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ of clauses over S be given. We construct a graph G from S and $\mathcal{C}$. Let us begin with vertices. For each boolean variable $x_i \in S$, we set two vertices u_i and u'_i. We call u_i a *positive vertex* and u'_i a *negative vertex*. For each clause $C_j \in \mathcal{C}$, we set one vertex v_j, called a *clause vertex*. So, the vertex set of G is $\{u_i, u'_i \mid i \in \{1,\dots,n\}\} \cup \{v_j : j \in \{1,\dots,m\}\}$. Let us proceed to edges. For each $j,k \in \{1,\dots,m\}$ we put an edge $\{v_j, v_k\}$. (In other words, the clause vertices induce a clique in G.) These edges are called *clause edges*. For each $i \in \{1,\dots,n\}$ we put an edge $\{u_i, u'_i\}$. (In other words, a positive vertex and the corresponding negative vertex forms an edge.) These edges are called *consistency edges*. For each $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$ we put an edge $\{u_i, v_j\}$ if and only if x_i is a literal in C_j; similarly, we put an edge $\{u'_i, v_j\}$ if and only if $\neg x_i$ is a literal in C_j. These edges are called *literal edges*.

Now, without loss of generality we may assume that every boolean variable appear in at least one clause in $\mathcal{C}$ as a positive or a negative literal (since if a boolean variable never appears in any clause, then the value of such a boolean variable has nothing to do with the satisfiability of $\mathcal{C}$) and that no clause in $\mathcal{C}$ contains a positive literal and a negative literal from the same variable (since such a clause is satisfied by any assignment). Then, we can show that the constructed graph G is perfect. (The proof is omitted, which is not difficult but lengthy.) Therefore, by the weak perfect graph theorem [21], the complement of G is also perfect. For later convenience, we also assume that there exists at least one assignment on S which does not satisfy all clauses in $\mathcal{C}$. (Otherwise, it is easy to find an assignment satisfying all clauses.)

First let us observe the possible sizes of maximal independent sets of G.

LEMMA B.1 *The size of a maximal independent set of* G *is either* n *or* $n + 1$.

PROOF. Let I be a maximal independent set of G. Since the clause vertices form a clique, I can contain at most one clause vertex. Suppose I contains a clause vertex, say v_j. Then, since I is maximal, it must contain either u_i or u'_i for all $i \in \{1, \dots, n\}$ because of the assumption that every clause contains only one of x_i and $\neg x_i$, but not both because of the consistency edges. Therefore, in this case the size of I is $n + 1$. As the second case, suppose that I contains no clause vertex. Then again I must contain either u_i or u'_i for all $i \in \{1, \dots, n\}$ but not both. Therefore, in this case the size of I is n. □

Consider the following correspondence between an assignment α on S and a maximum independent set U_α of the subgraph G' of G induced by the positive vertices and the negative vertices. The set U_α contains either u_i or u'_i for every $i \in \{1, \dots, n\}$, but not both. If $\alpha(x_i) = 1$ then we include u_i into U_α. If $\alpha(x_i) = 0$ then we include u'_i into U_α. Note that U_α is indeed a maximum independent set of G', and the inverse operation is also well-defined. The proof of the lemma above shows that every maximal independent set of G contains a set U_α for some assignment α.

Assume that α satisfies all clauses in $\mathcal{C}$. Now, look at U_α. Because of the literal edges, every clause vertex is adjacent to some vertex in U_α. This means that every maximal independent set containing U_α is of size n when α is a satisfying assignment. On the other hand, assume that α does not satisfy some clause, say C_j, in $\mathcal{C}$. Then, the vertex v_j is adjacent to no vertex in U_α. Therefore, $U_\alpha \cup \{v_j\}$ is independent. This means that every maximal independent set containing U_α is of size $n + 1$ when α is not a satisfying assignment. By the assumption that there exists at least one assignment which does not satisfy all clauses, we can see that the size of a maximum independent set is $n + 1$ in such a case. Thus, we have proved the following lemma.

LEMMA B.2 *There exists an assignment satisfying all clauses in* $\mathcal{C}$ *if and only if there exists a maximal independent set in* G *which is not maximum.*

Since $\overline{G}$ is perfect, this lemma concludes the reduction, thus the whole proof of Theorem 5.2.

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