On the Conditioning of Robustness Problems[1]

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Abstract

This focal point of this paper is the "conditioning" of robustness problems with performance specifications depending nonlinearly on uncertain parameters. Beginning with a rather general class of such problems, we define our so-called underlying *conditioner* θ. For large classes of robustness problems characterized by the requirement that a nonlinear function $f(x)$ be negative on a prescribed set $\mathcal{X}$ in $\mathbf{R}^n$, it is seen that the associated conditioner θ serves as a natural measure of the degree of difficulty which one might expect to encounter in order to certify that the *volume of violation* in parameter space is below some acceptable level. Be it a random sampling algorithm or a direct analytical method, some sort of certification is typically sought. In this paper, a number of properties of the conditioner θ are described. Most notably, it is shown how this conditioner can be estimated "on the fly" within the context of a robustness computation. For the large class of robustness problems with $f(x)$ being a multivariable polynomial and $\mathcal{X}$ being a hypercube, a convergent sequence of estimates for θ is obtained.

1. Introduction

For the case of robustness analysis with a so-called *linear uncertainty structure*, over the last two decades, researchers have made considerable advancement; e.g., see [1–3] and their bibliographies for an overview of results to date. However, when the uncertain parameters enter nonlinearly into the system, existing results typically apply to rather special cases obtained via the imposition of strong assumptions; e.g., for the case of one or two uncertain parameters entering polynomially into the closed loop, one might consider the results of [4]. Another special case, the so-called *multilinear uncertainty structure*, is addressed in the body of literature based on the Mapping Theorem; e.g., see [5].

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Motivated by the paucity of strong results for the general case of nonlinear parameter dependence, we see a number of papers in the literature proving that various robustness problems are NP-hard; e.g., see [6]. Consistent with this literature and other recent work by the authors, see [7] and [8], the objective of this paper is to quantify the "conditioning" of robustness problems and the extent to which a well-conditioned problem assures computational tractability. Since our problem formulation involves the assurance of negativity of a nonlinear function over a given domain, for the special case of polynomial dependence on parameters, this work also relates to the body of literature which includes methods such as those based on the Tarski decision calculus, Bernstein polynomials and quantifier elimination; e.g., see [9–12]. In contrast to these papers and others with a similar flavor, one of the main contentions in the current work is that by taking the conditioning of a robustness problem into account, it often becomes possible to avoid the exponential growth of computation which is associated with current methods.

1.1 Formulation: To capture large classes of robustness problems, in the sequel, we consider given data consisting of a continuous function $f : \mathbf{R}^n \to \mathbf{R}$, a compact pathwise connected set $\mathcal{X} \subset \mathbf{R}^n$ and a requirement to determine if the inequality

$$f(x) \; < \; 0$$

is satisfied for all $x \in \mathcal{X}$. This being the case, the pair $(f, \mathcal{X})$ is said to be *strictly robustly feasible*. To avoid trivialities in the sequel, without loss of generality, it is also assumed that there exists at least one $\overline{x} \in \mathcal{X}$ such that

$$f(\overline{x}) \; < \; 0.$$

As an illustration of this formulation, consider a family of polynomials with coefficients depending nonlinearly on parameters $x \in \mathcal{X}$. Then with associated Hurwitz matrix $\mathcal{H}(x)$ and $f(x) = -\det \mathcal{H}(x)$, strict robust feasibility of the pair $(f, \mathcal{X})$ is equivalent to robust stability.

1.2 Role of Conditioner: Motivated by our initial work in [8], in Section 2, we define a so-called underlying *conditioner* θ and describe a number of its properties. We argue that for large classes of robustness problems, the value of θ serves as an indicator of the degree of difficulty which one might expect in the assessment of constraint violation. That is, with

$$\mathcal{X}_{bad} \doteq \{x \in \mathcal{X} : f(x) \geq 0\}$$

denoting the "bad set" with respect to the specification, it is possible to generate a sequence of estimates ϵ_k satisfying

$$\frac{\mathrm{Vol}(\mathcal{X}_{bad})}{\mathrm{Vol}(\mathcal{X})} \leq \epsilon_k \leq \theta^k.$$

That is, in view of the "volume certification" above, we associate low values of θ with a well-conditioned problem, and high values of θ with an ill-conditioned problem. In addition, since it can readily be shown that

$$0 \leq \theta < 1$$

corresponds to strict robust feasibility, for such cases, it follows that $\lim_{k\to\infty} \epsilon_k = 0$ and therefore, the volume estimate above eventually shrinks to zero as k increases. Moreover, since the rate at which the volume bound tends to zero is governed by θ, it follows that for a well-conditioned pair $(f, \mathcal{X})$, we can work with low values of k to obtain an acceptable level of certification on the volume of violation. As seen in the sequel, this in turn implies that the computational effort to compute ϵ_k will be correspondingly small. In summary, our claim in this line of research is that well-conditioned robustness problems, even with a high degree of nonlinearity, can be solved in a "practical sense" with reasonable computational effort. That is, we obtain a practical solution in the sense that the inequality $f(x) < 0$ is guaranteed to be satisfied for all $x \in \mathcal{X}$ except possibly on a set of small volume. To illustrate, for a well-conditioned pair $(f, \mathcal{X})$ with $\theta = 0.3$ and $k = 4$, the bound above leads to

$$\frac{\mathrm{Vol}(\mathcal{X}_{bad})}{\mathrm{Vol}(\mathcal{X})} \leq 0.01.$$

That is, with $k = 4$, we certify that the set $\mathcal{X}_{bad}$ comprises less than one percent of the total volume of $\mathcal{X}$.

This way of thinking about computational difficulty of a problem in terms of its conditioning is not unprecedented. For example, in the theory of mathematical programming, the rate of convergence of various iterative algorithms is governed by underlying Lipschitz constants, the

Hessian and the like; e.g., see [13] and [14]. While it is true that various classes of optimization algorithms have high computational complexity, the actual performance for certain problem "instances" may be quite good when the underlying conditioners, such as a Lipschitz constant or eigenvalues of the Hessian, are at appropriate levels.

2. The Conditioner and its Properties

In this section, we take the pair $(f, \mathcal{X})$ as given and define the underlying conditioner θ which will be studied. As previously stated, for robustly feasible pairs, this conditioner defines a guaranteed rate at which our estimate ϵ_k of the relative volume of violation decreases to zero as the index k increases. Specifically, the estimate ϵ_k is defined as

$$\epsilon_k \doteq \frac{1}{\mathrm{Vol}(\mathcal{X})} \min_{\alpha \geq 0} \int_{\mathcal{X}} \Big(1 + \alpha f(x)\Big)^k dx,$$

with k being a non-negative even integer; see [8] for details.

2.1 Definition of Underlying Conditioner: For a given pair $(f, \mathcal{X})$, the *underlying conditioner* θ is defined as the maximum percentage variation of $f(x)$ about the midpoint of its range. That is, with

$$f(\mathcal{X}) \doteq \{f(x) : x \in \mathcal{X}\} \doteq [f_{\min}, f_{\max}]$$

corresponding to minimization and maximization of $f(x)$, we obtain

$$\theta \doteq \frac{\sigma}{|f_0|},$$

where

$$\sigma \doteq \frac{1}{2}\Big(f_{\max} - f_{\min}\Big)$$

is the *spread* of $f(\mathcal{X})$ and

$$f_0 \doteq \frac{1}{2}\Big(f_{\max} + f_{\min}\Big)$$

is the *midpoint* of $f(\mathcal{X})$. Equivalently,

$$\theta = \frac{f_{\max} - f_{\min}}{|f_{\max} + f_{\min}|}.$$

Note that for a non-robust pair $(f,\, \mathcal{X})$, the conditioner θ may be arbitrarily large or possibly infinite. With the definition above, in addition to the defined non-negativity of θ, the following two fundamental lemmas, minor extensions of results given in [8], serve as a basis for much of the analysis in the sections to follow.

2.2 Lemma: *The inequality* $0 \leq \theta < 1$ *holds if and only if the pair* $(f,\, \mathcal{X})$ *is strictly robustly feasible.*

2.3 Lemma: *For all non-negative even integers* k*, it follows that*

$$\mathrm{Vol}(\mathcal{X}_{bad}) \;\leq\; \epsilon_k \, \mathrm{Vol}(\mathcal{X}) \;\leq\; \theta^k \, \mathrm{Vol}(\mathcal{X}).$$

2.4 Remarks (Radius Dependent Conditioner): It is also important to draw attention to the case when the hypercube $\mathcal{X} = \mathcal{X}_r$ is parameterized by its radius $r \geq 0$. In this case, with conditioner $\theta = \theta_r$, the first point to note is that at $r = 0$, θ_r begins at $\theta_0 = 0$ and eventually increases to $\theta_r = 1$ as r approaches the so-called *robustness radius*

$$r_{\max} \;=\; \sup\{r : \; f(x) < 0 \;\; \text{for all} \;\; x \in \mathcal{X}_r\}.$$

In view of the fact that one of the objectives of our volume-based theory is "recovery" of classical robustness results, it is of interest to know how quickly, with respect to increases in r, a robustness problem becomes ill-conditioned; e.g., if θ_r remains small even when r is close to $r_{\max}$, we expect our volume-based approach to perform quite well in the sense that for such radii, the estimate ϵ_k in Section 2 will rapidly converge; i.e., the k value required to certify a small volume of violation will be correspondingly small.

To make results of the sort above more meaningful, we normalize our calculations and study the behavior of θ versus the radius of $\mathcal{X}$ expressed as a percentage of $r_{\max}$; i.e., with

$$\mu \;\doteq\; \frac{r}{r_{\max}},$$

we study the behavior of the conditioner $\theta = \theta(\mu)$ for $0 \leq \mu < 1$. It can also be seen that an interpretation can be attached to $\theta(\mu)$ when $\mu \geq 1$. In such cases, even though $(f,\, \mathcal{X})$ does not satisfy the strict robust feasibility condition, it often turns out to be the case that the actual volume of violation is much lower than the bound provided by Lemma 2.3; see [8] for details.

2.5 Examples (Linear, Quadratic and Multilinear Forms): We first consider the special case when $f(x) = a^{\mathrm{T}}x - \gamma,\ \gamma > 0$, is a linear function and $\mathcal{X}$ is described via a norm bound $\|x\| \leq r$. Then, via a straightforward calculation, we obtain $\theta(\mu) = \mu$ as the conditioner.

As a second example, we consider the case when the specification is that a quadratic form $x^{\mathrm{T}}Ax$ not exceed some prescribed level $\gamma > 0$ for all x satisfying norm requirement $\|x\| \leq r$. Then, with $f(x) = x^{\mathrm{T}}Ax - \gamma$, a straightforward computation leads to conditioner

$$\theta(\mu) \;=\; \frac{\mu^2}{|\mu^2 - 2|}.$$

Finally, as a third benchmark example, we consider a special class of multilinear functions for which $\theta(\mu)$ is computable in closed form. Namely, with n even, $\gamma > 1$,

$$f(x) \;=\; -\gamma + \prod_{i=1}^{n}(1 - x_i),$$

and uncertainty bound $0 \leq x_i \leq r$, a lengthy but straightforward analysis leads to

$$\theta(\mu) \;=\; \begin{cases} \dfrac{1 - [1 - \mu(1+\gamma^{1/n})]^n}{2\gamma - 1 - [1 - \mu(1+\gamma^{1/n})]^n}, & 0 \leq \mu < \frac{1}{1+\gamma^{1/n}}; \\[2ex] \dfrac{\mu(1+\gamma^{1/n})}{2\gamma - 2 + \mu(1+\gamma^{1/n})}, & \frac{1}{1+\gamma^{1/n}} \leq \mu < \frac{2}{1+\gamma^{1/n}}; \\[2ex] \dfrac{[\mu(1+\gamma^{1/n}) - 1]^n + [\mu(1+\gamma^{1/n}) - 1]^{n-1}}{2\gamma - [\mu(1+\gamma^{1/n}) - 1]^n + [\mu(1+\gamma^{1/n}) - 1]^{n-1}}, & \frac{2}{1+\gamma^{1/n}} \leq \mu \leq 1. \end{cases}$$

This formula indicates that conditioning improves as the nominal $f(0) = 1 - \gamma$ decreases or as the dimension n of x increases. For instance, from Figure 1, it is seen that with $\gamma = 100$ and $n = 20$ we have $\theta(\mu) \leq 0.01$ up to about 90% of $r_{\max}$.

2.6 Example (Poorly Conditioned Case): A few versions of Ackermann's track-guided bus problem have been studied by many authors; e.g., see [1] or [15] for representative data. For

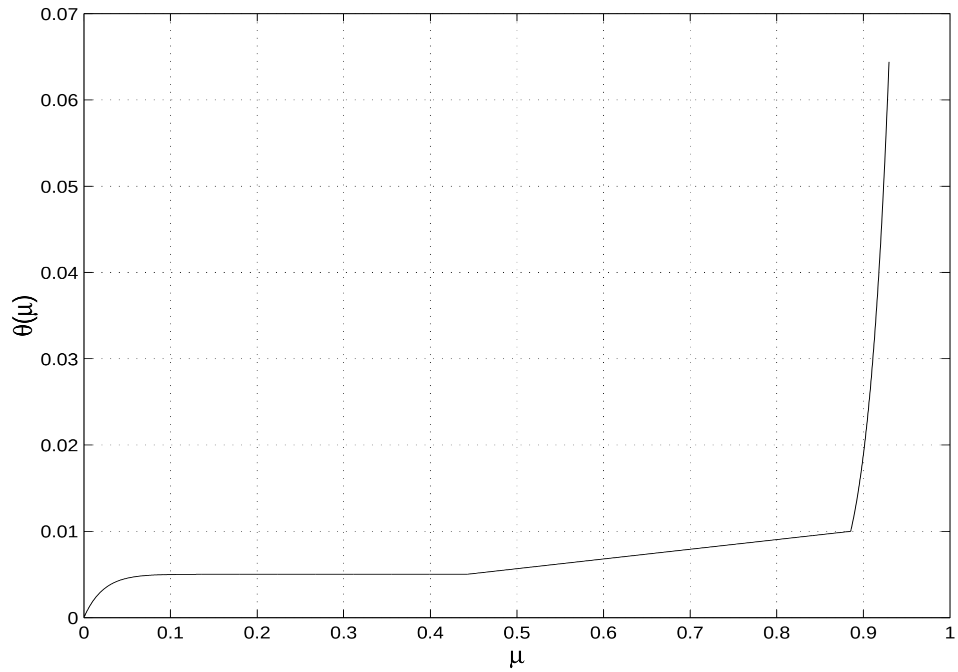

Figure 1: Plot of $\theta(\mu)$ for $n = 20,\ \gamma = 100$ in Example 2.5

this system with fifth order plant and third order compensator, the lightly damped poles of the system, even for small levels of uncertainty, make classical robust stability analysis quite challenging to perform. Accordingly, our objective in this section is to show that difficulties in robust stability analysis are reflected in $\theta(\mu)$ function for low values of the parameter μ.

To study the conditioning for this example, we start with Ackermann's characteristic polynomial

$$p(s,x) \ = \ \sum_{i=1}^{8} a_i(x)s^i$$

with uncertain parameters bounds $11.5 - r_1 \leq x_1 \leq 11.5 + r_1$ and $21 - r_2 \leq x_2 \leq 21 + r_2$ and coefficients

$$\begin{aligned}
a_0 &= 4.53 \times 10^8 x_1^2; \\
a_1 &= 5.28 \times 10^8 x_1^2 + 3.64 \times 10^9 x_1; \\
a_2 &= 5.72 \times 10^6 x_1^2 x_2 + 1.13 \times 10^8 x_1 x_2 + 4.25 \times 10^9 x_1; \\
a_3 &= 6.92 \times 10^6 x_1^2 x_2 + 9.11 \times 10^8 x_1 + 4.22 \times 10^9 x_1; \\
a_4 &= 1.45 \times 10^6 x_1^2 x_2 + 1.68 \times 10^7 x_1 x_2 + 3.38 \times 10^8; \\
a_5 &= 1.56 \times 10^4 x_1^2 x_2^2 + 8.4 \times 10^2 x_1^2 x_2 + 1.35 \times 10^6 x_1 x_2 + 1.35 \times 10^7;
\end{aligned}$$

$$
\begin{aligned}
a_6 &= 1.25 \times 10^3 x_1^2 x_2^2 + 1.68 \times 10^1 x_1^2 x_2 + 5.39 \times 10^4 x_1 x_2 + 2.7 \times 10^5; \\
a_7 &= 50 x_1^2 x_2^2 + 1.08 \times 10^3 x_1 x_2; \\
a_8 &= x_1^2 x_2^2.
\end{aligned}
$$

Now, to analyze the conditioning of this robust stability problem, we work with the two radii $r_1 = 8.5$, $r_2 = 11$ corresponding to a region in parameter space which nearly touches the stability boundary. Hence, it is natural to expect that robust stability analysis may be difficult in this case. To see the extent to which this is reflected in the conditioner, we consider radial scaling parameter $0 \leq \mu \leq 1$ and take

$$
f(x) = -\det \mathcal{H}(x),
$$

where $\mathcal{H}(x)$ is the Hurwitz matrix. Since only two uncertain parameters are involved, we can use a brute force gridding of the (x_1, x_2) region to accurately estimate θ. To this end, the maximum and minimum of $f(x)$ are estimated as μ varies between zero and one. This leads to the estimate of $\theta(\mu)$ given in Figure 2. From the plot, a key point to note is that $\theta(\mu)$ is quite close to unity even for small values of μ. In other words, even for uncertainty levels as low as 20% of $r_{\max}$, our theory deems the pair $(f, \mathcal{X}_r)$ to be poorly conditioned.

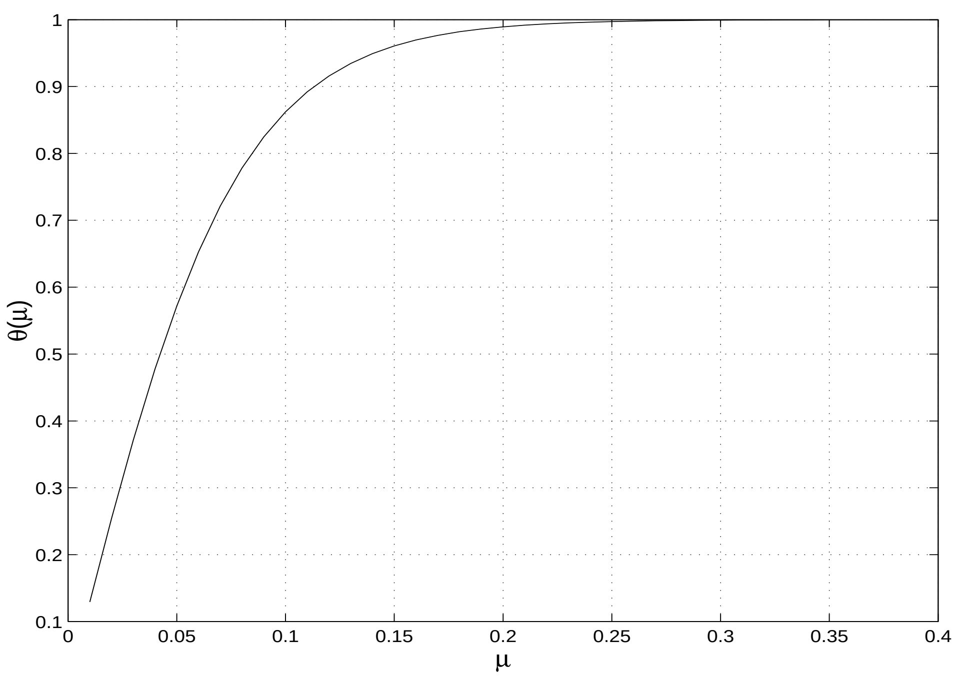

Figure 2: Plot of $\theta(\mu)$ for Track-guided Bus

3. Main Result: Conditioner Estimation and Convergence

Relating to the previous section, the theorem to follow applies for a fixed radius of uncertainty. Hence, μ is fixed and we simply refer to θ instead of $\theta(\mu)$. The main result in this paper is that the conditioner θ can be estimated via a monotonically convergent sequence of one variable minimizations of the convex functions

$$\theta_k(\alpha) \doteq \left(\frac{1}{\mathrm{Vol}(\mathcal{X})} \int\limits_{\mathcal{X}} \Big(1 + \alpha f(x)\Big)^k dx \right)^{\frac{1}{k}}$$

with k being a positive even integer and $\alpha \geq 0$.

3.1 Monotonic Convergence Theorem: *Assume the pair $(f, \mathcal{X})$ is strictly robustly feasible and define*

$$\theta_k \doteq \min_{\alpha \geq 0} \theta_k(\alpha) = \sqrt[k]{\epsilon_k}.$$

Then it follows that

$$\lim_{k \to \infty} \theta_k = \theta.$$

Alternatively, if $(f, \mathcal{X})$ is not strictly robustly feasible, then

$$\lim_{k \to \infty} \theta_k = 1.$$

Moreover, for both cases above, the sequence of estimates θ_k is non-decreasing.

Proof: Some basic properties of the functions $\theta_k(\alpha)$, which will be used later in the proof, are first noted: Every function $\theta_k(\alpha)$ is convex and corresponds to the L^k norm of $1 + \alpha f(x)$ with respect to the measure

$$\mathrm{mes}\,(\mathcal{M}) \doteq \frac{\mathrm{Vol}(\mathcal{M})}{\mathrm{Vol}(\mathcal{X})}$$

on measurable subsets $\mathcal{M}$ of $\mathcal{X}$. Second, for the case of robust feasibility, using the fact that $\theta_k(\alpha) \to \infty$ as $\alpha \to \infty$, it follows that every $\theta_k(\alpha)$ attains its minimum at a finite point $\alpha_k \geq 0$; hence, the quantities θ_k and α_k introduced above are well defined. Finally, note that $\theta_k(0) = 1$ guarantees that $\theta_k \leq 1$ is satisfied.

At the first step of the proof, we claim that the sequence of functions $\theta_k(\alpha)$ converges pointwise to the

$$\theta(\alpha) \doteq \max_{x\in\mathcal{X}} |1+\alpha f(x)|.$$

To prove this, we fix $\alpha \geq 0$ arbitrarily and define the continuous function

$$\phi(x) \doteq \phi_\alpha(x) \doteq |1+\alpha f(x)|$$

on the compact set $\mathcal{X}$. Noting classical results for convergence of L^p norms (for example, see pg. 73 of [16]), with the fact that $\left(\mathrm{Vol}(\mathcal{X})\right)^{1/k} \to 1$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \left(\frac{1}{\mathrm{Vol}(\mathcal{X})} \int_{\mathcal{X}} \phi^k(x)dx \right)^{1/k} = \max_{x\in\mathcal{X}} \phi(x).$$

That is,

$$\lim_{k\to\infty} \theta_k(\alpha) = \theta(\alpha)$$

and the claim of pointwise convergence is established.

As the next step of the proof, we claim that the sequence of functions $\theta_k(\alpha)$ is non-decreasing:

$$\theta_k(\alpha) \leq \theta_{k+2}(\alpha)$$

for all even k. Indeed, for the non-negative continuous $\phi(x)$ introduced above, by Hölder's inequality, for example, see [16], we have

$$\int_{\mathcal{X}} \phi^k(x)dx \leq \left(\mathrm{Vol}(\mathcal{X})\right)^{\frac{2}{k+2}} \left(\int_{\mathcal{X}} \phi^{k+2}(x)dx \right)^{\frac{k}{k+2}}.$$

Now multiplying both sides by $\left(\mathrm{Vol}(\mathcal{X})\right)^{-1}$ and taking the k-th root of both sides yields

$$\theta_k(\alpha) \leq \left(\frac{1}{\mathrm{Vol}(\mathcal{X})} \int_{\mathcal{X}} \phi^{k+2}(x)dx \right)^{\frac{1}{k+2}} = \theta_{k+2}(\alpha).$$

From this inequality, it follows that the sequence $\theta_k = \min_{\alpha} \theta_k(\alpha)$ is non-decreasing.

To continue the proof, our next claim is that for every k, for the case when $(f,\, \mathcal{X})$ is strictly robustly feasible, the minimization of $\theta_k(\alpha)$ is attained with

$$0 \leq \alpha \leq \frac{2}{|f_{\max}|}.$$

That is, we have a fixed compact set $\mathcal{S}$ over which all minimizations can be restricted. To prove this, it suffices to show that with

$$\alpha \; > \; \frac{2}{|f_{\max}|},$$

we have

$$\theta_k(\alpha) \; > \; \theta_k(0).$$

In other words, such α exceeding this threshold cannot be optimal. To this end, we first observe that for such α, we have $|\alpha\, f(x)| > 2$ for all $x \in \mathcal{X}$. Hence, for such α and all $x \in \mathcal{X}$, it follows that

$$|1 + \alpha\, f(x)| \; = \; \alpha |f(x)| - 1 \; > \; 1.$$

This implies that

$$\theta_k(\alpha) \; > \; 1 \; = \; \theta_k(0)$$

which establishes the claim. In view of this claim, we now have a non-decreasing pointwise convergence of the sequence of continuous functions $\theta_k(\alpha)$ defined over the compact set $\mathcal{S}$ to a (continuous) limiting function $\theta(\alpha)$. By Dini's theorem (e.g., see [17]), it follows that the sequence of functions $\theta_k(\alpha)$ converges uniformly to $\theta(\alpha)$. Therefore, taking minimum with respect to α of both sides of the limiting relation $\lim_{k\to\infty} \theta_k(\alpha) = \theta(\alpha)$, we can change the order of the lim and min operations:

$$\min_{\alpha} \lim_{k\to\infty} \theta_k(\alpha) \; = \; \lim_{k\to\infty} \min_{\alpha} \theta_k(\alpha) \; = \; \lim_{k\to\infty} \theta_k$$

and arrive at

$$\lim_{k\to\infty} \theta_k \; = \; \min_{\alpha} \theta(\alpha).$$

The last step is to show that the right-hand side of the equality above is equal to θ if the problem is feasible, and to unity if infeasible. This can be done by representing the limiting function $\theta(\alpha)$ in the following "parametric" form:

$$\theta(\alpha) \; = \; \max_{f_{\min} \le \lambda \le f_{\max}} |1 + \alpha\, \lambda| \; = \; \max\{|1 + \alpha f_{\min}|,\; |1 + \alpha f_{\max}|\}.$$

A straightforward analysis of the piecewise linear function $\theta(\alpha)$, performed under the assumption $f_{\min} < 0$, leads to a conclusion: If $f_{\max} < 0$, then

$$\min_{\alpha} \theta(\alpha) \; = \; \frac{f_{\max} - f_{\min}}{|f_{\max} + f_{\min}|} \; = \; \theta \; < \; 1,$$

and if $f_{\max} \geq 0$, then

$$\min_{\alpha} \theta(\alpha) \;=\; 1.$$

The proof of theorem is now complete.

4. Examples

4.1 Linear Function: Consider the simple strictly robustly feasible pair $(f,\, \mathcal{X})$ described by

$$f(x) \;=\; x - 1.5 \quad \text{and} \quad |x| \;\leq\; 0.5.$$

Calculating $\theta \approx 0.3333$, we readily compute integrals defining $\theta_k(\alpha)$ for $k = 2, 4, \ldots, 16$. For example, for $k = 6$, we obtain

$$\theta_6(\alpha) \;=\; \sqrt[6]{\frac{1}{448}\,\alpha^6 + \frac{3}{16}\left(1 - \frac{3}{2}\,\alpha\right)^2 \alpha^4 + \frac{5}{4}\left(1 - \frac{3}{2}\,\alpha\right)^4 \alpha^2 + \left(1 - \frac{3}{2}\,\alpha\right)^6}.$$

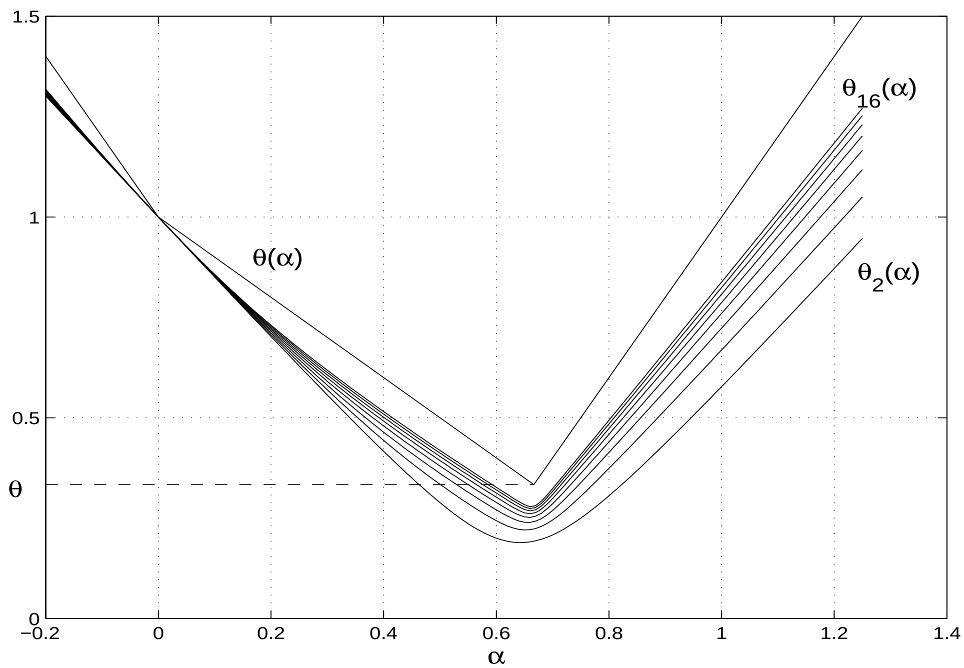

Figure 3: Plots of $\theta(\alpha)$ and $\theta_k(\alpha)$ for $k = 2, 4, \ldots, 16$ in Example 4.1

Based on Figure 3, the one-dimensional minimization defining θ_k leads to the estimates for θ given in Table 1:

k	2	4	6	8	10	12	14	16
θ_k	0.1890	0.2205	0.2394	0.2518	0.2609	0.2680	0.2738	0.2786

Table 1: Estimates of Conditioner

The numerical results are consistent with the monotonicity of θ_k predicted by the theorem. The slow rate of convergence of θ_k to θ is also noted.

4.2 Multilinear Function: We consider the strictly robustly feasible pair described by

$$f(x) \;=\; x_1 + 2x_1x_2 - 3x_1x_2x_3 - 10; \qquad |x_i| \;\leq\; 1.$$

For the multilinear function above, extreme point considerations lead us to $f_{\min} = -16$ and $f_{\max} = -4$ with corresponding $\theta = 0.6$. For this benchmark, we computed the associated integrals for k ranging from 2 to 40. Using the minima obtained from the functions $\theta_k(\alpha)$ shown in Figure 4, we obtained a slowly convergent monotonic sequence estimating θ.

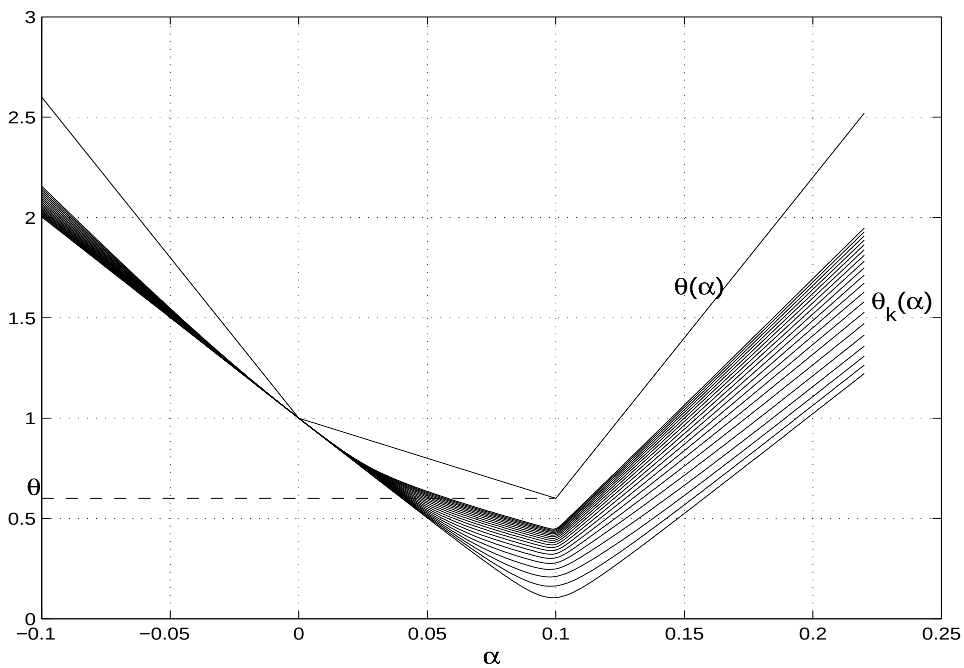

Figure 4: Plots of $\theta(\alpha)$ and $\theta_k(\alpha)$ for $k = 2, 4, \ldots, 40$ in Example 4.2

5. Conclusion

The motivation for this paper and its predecessor [8] is derived from a desire to address robustness problems with nonlinear parameter dependence. Two ideas are central to our approach: First, instead of making a categorical determination if $f(x)$ is negative for all $x \in \mathcal{X}$, we soften the problem to allow for a small, user defined, fractional volume of violation $\epsilon > 0$. Second, the ease with which one can "certify" this volume of violation is dependent on the underlying conditioner θ. For the rather general class of nonlinear problems described by $f(x)$ being a multivariable polynomial and $\mathcal{X}$ being a hypercube, a well-conditioned problem readily admits a closed form for the function $\theta_k(\alpha)$ used in the estimation of the conditioner.

By way of future research, be it the estimation of θ or the volume of violation, it would be important to develop computationally efficient schemes, which take advantage of the structure induced by a control system on $f(x)$, to compute *high order moments*

$$m_k \doteq \int_{\mathcal{X}} f^k(x)dx.$$

To illustrate, for a feedback system with uncertain parameters x entering into the Hurwitz matrix $\mathcal{H}(x)$, it can be shown that with $f(x) = -\det \mathcal{H}(x)$, the function $\theta_k(\alpha)$ used in Section 3 to estimate the conditioning reduces to

$$\theta_k(\alpha) = \left(\frac{1}{\mathrm{Vol}(\mathcal{X})} \sum_{i=0}^{k} \frac{(-1)^i}{i!} \alpha^i \int_{\mathcal{X}} [\det \mathcal{H}(x)]^i dx \right)^{1/k}.$$

Hence, when the higher order moments are readily computable, both the estimation of the conditioner and its corresponding volume of violation are facilitated.

Finally, it would also be of interest to investigate the extent to which the ideas in this paper and its companion [8] might be useful in a nonlinear programming context. To see the connection with our robust feasibility formulation, consider objective function $g(x)$ to be minimized over a constraint set $\mathcal{X}$. Now, given any real scalar γ, by taking $f(x) = \gamma - g(x)$, it is easy to see that strict robust feasibility of $(f, \mathcal{X})$ is equivalent to γ being a lower bound for for the minimum of $g(x)$.

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