KNOT ADJACENCY, GENUS AND ESSENTIAL TORI

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Abstract. A knot K is called n-adjacent to another knot K', if K admits a projection containing n "generalized crossings" such that changing any $0 < m \leq n$ of them yields a projection of K'. We apply techniques from the theory of sutured 3-manifolds, Dehn surgery and the theory of geometric structures of 3-manifolds to answer the question of the extent to which non-isotopic knots can be adjacent to each other. A consequence of our main result is that if K is n-adjacent to K' for all $n \in \mathbf{N}$, then K and K' are isotopic. This provides a partial verification of the conjecture of V. Vassiliev that the finite type knot invariants distinguish all knots. We also show that if no twist about a crossing link L of a knot K changes the isotopy class of K, then L bounds a disc in the complement of K. This gives rise to a characterization of nugatory crossings of a knot.

1. Introduction

In low dimensional topology, it is common to relate different topological objects by a finite sequence of "elementary moves". The focus in such an approach is usually put on the understanding of the extent to which these elementary moves will alter the topology of the objects involved. For example, any two closed orientable 3-manifolds can be related by a finite sequence of Dehn surgeries, and the problem of which topological properties are preserved under Dehn surgeries has been studied extensively and fruitfully in the last couple of decades. Traditionally, in knot theory, one relates two knots by a sequence of crossing changes. The development of the theory of finite type knot invariants has led to the study of a multiplex relation between two knots. This is the notion of n-equivalence introduced by M.

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Gussarov. In this notion, roughly speaking, one requires that two knots be related by $2^n - 1$ different sequences of *multiple crossing changes*. This certainly imposes strong restrictions on the set of knots that can be n-equivalent. In fact, it is a theorem of Gussarov that two knots are n-equivalent iff all of their finite type invariants of orders $< n$ are the same [G]. Since n-equivalence is defined in terms of local moves on knots, it is always challenging to understand its effect on various intrinsic topological knot properties and invariants. For example, one would like to know whether the genera of two n-equivalent knots can be related.

In the current paper, we consider a strengthened version of n-equivalence where only the simplest multiple crossing changes are used. Roughly speaking, we say that a knot K is n-adjacent to another knot K' if there are $2^n - 1$ different sequences of *generalized crossing changes*, all of which change K to K'. We apply techniques from the theory of sutured 3-manifolds, Dehn surgery and the theory of geometric structures on 3-manifolds to study the interplay between knot genus, toroidal decompositions of knot complements and n-adjacency. Our main result, which answers the question of the extent to which non-isotopic knots can be n-adjacent to each other, shows that the knot genus imposes constraints on how large n can be. As a corollary, we show that if K is n-adjacent to K' for all $n \in \mathbf{N}$, then K must be isotopic to K'. This can be considered as a partial verification of the conjecture of Vassiliev that two knots indistinguishable by finite type invariants are necessarily isotopic. Although the current paper does not settle Vassiliev's conjecture, the techniques used have the potential to be useful in handling more general situations than the ones discussed here. We hope to explore this direction in the future. With this said, perhaps the main interest for the reader of the current work lies in seeing a variety of techniques from classical low dimensional topology arrayed against a novel problem originated in the realm of quantum topology. Thus, it is our hope that this work would be of interest to a broad spectrum of people working in low dimensional topology.

Let us now give a more detailed description of the results of the paper. Let K be a knot in $\mathbf{S}^3$ and let $q \in \mathbf{Z}$. A generalized crossing of order q on a projection of K is a set C of $|q|$ twist crossings on two strings that inherit opposite orientations from any orientation of K. If K' is obtained from K by changing all the crossings in C simultaneously, we will say that K' is obtained from K by a generalized crossing change of order q (see Figure 1). Note that if $|q| = 1$, K and K_1 differ by an ordinary crossing change while if $q = 0$ we have $K = K'$.

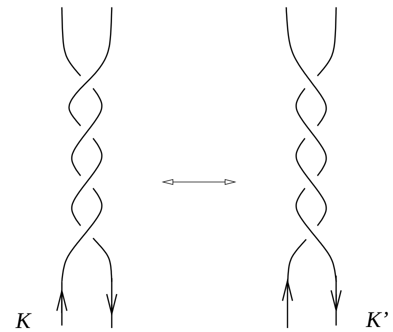

FIGURE 1. The knots K and K' differ by a generalized crossing change of order $q = -4$.

Definition 1.1. *We will say that K is n-adjacent to K', for some $n \in \mathbf{N}$, if K admits a projection containing n generalized crossings such that changing any $0 < m \leq n$ of them yields a projection of K'. We will write $K \xrightarrow{n} K'$.*

Our main result is the following:

Theorem 1.2. *Suppose that K and K' are non-isotopic knots. There exists a constant $C(K,\, K')$ such that if $K \xrightarrow{n} K'$, then $n \leq C(K,\, K')$.*

Let $g(K)$ and $g(K')$ denote the genera of K and K', respectively and let $g := \max\{\, g(K),\, g(K')\,\}$. The constant $C(K,\, K')$ is shown to encode information about the relative size of $g(K)$, $g(K')$ and the behavior of the satellite structures of K and K' under the Dehn surgeries imposed by knot adjacency. In many cases $C(K,\, K')$ can be made explicit. For example, when $g(K) > g(K')$ we have $C(K,\, K') = 6g-3$. Thus, in this case, Theorem 1.2 can be restated as follows:

Theorem 1.3. *Suppose that K, K' are knots with $g(K) > g(K')$. If $K \xrightarrow{n} K'$, then $n \leq 6g(K) - 3$.*

Another interesting case of Theorem 1.2 occurs when K' is a fibred knot. Then we have the following:

Theorem 1.4. *Suppose that K' is a fibered knot and that K is a knot such that $K \xrightarrow{n} K'$, for some $n > 1$. Then, either $g(K) > g(K')$ or K is isotopic to K'.*

The techniques used in the proof of Theorem 1.2 and Theorem 1.4 have applications to the question of whether a crossing change that doesn't change the isotopy class of the underlying knot is *nugatory*. As a corollary of the proof of Theorem 1.2 we obtain the following characterization of nugatory crossings:

Corollary 1.5. *For a crossing of a knot K, with crossing disc D, let $K(r)$ denote the knot obtained by a twist of order r along D. The crossing is nugatory if and only if $K(r)$ is isotopic to K for all $r \in \mathbf{Z}$.*

For fibred knots, we have the following:

Corollary 1.6. *Let K be a fibred knot. A crossing change in K yields a knot isotopic to K if and only if the crossing is nugatory.*

In the case that K' is the trivial knot Theorem 1.3 was proven by H. Howards and J. Luecke in [HL]. The main ingredient in the proof of the result of [HL] is a theorem of D. Gabai ([Ga]) that describes the behavior of the knot genus under Dehn filling. This result is also used in the proof of Theorem 1.2 in order to understand the interplay between knot genus and multiple crossing changes. However, in the presence of two non-trivial knots, the argument becomes significantly different and the use of new techniques is required. In particular, to prove Theorem 1.2 in the case that $g(K) \leq g(K')$ we need to understand the role played by the existence of essential tori in the complements of K and K'. For this we employ results from Dehn surgery and the theory of geometric structures on 3-manifolds. The first main ingredient for this part of the argument is provided by a result of C. Gordon ([Go]) that describes the circumstances under which Dehn filling of an atoroidal 3-manifold produces essential tori. Another ingredient is a result of D. Cooper and M. Lackenby ([CoLa]) which (loosely speaking) asserts that a given 3-manifold can be obtained by at most finitely many hyperbolic Dehn fillings with "long" slopes. We also use a result of D. McCullough that describes homeomorphisms of 3-manifolds that are Dehn twists on the boundary. The proof of this result, which uses the theory of the characteristic submanifold of Jaco-Shalen and Johannson ([Jo]), is given in the appendix of the paper.

As already discussed, the general framework to examine the notion of knot n-adjacency is via the notion of n-equivalence. This notion was introduced in [G], where it is shown that two knots are n-equivalent precisely when all of their finite type invariants of orders $< n$ are the same. In fact, strictly speaking, our notion of n-adjacency is a special case of the notion of n-similarity. This last notion was defined independently by K. Taniyama and Y. Ohyama ([Oh]) and shown by K. Ng and T. Stanford ([NS]) to be equivalent to Gussarov's n-equivalence. By definition, if $K \xrightarrow{n} K'$ then K and K' are n-similar and thus n-equivalent. V. Vassiliev ([V]) has conjectured that if two oriented knots have all of their finite type invariants the same then they are isotopic. In the light of the aforementioned result of Gussarov, Vassiliev's conjecture can be reformulated as follows:

Conjecture 1.7. *Suppose that K and K' are knots that are n-equivalent for all $n \in \mathbf{N}$. Then K is isotopic to K'.*

To that respect, Theorem 1.2 implies the following corollary that provides a partial verification to Vassiliev's conjecture:

Corollary 1.8. *If $K \stackrel{n}{\longrightarrow} K'$, for all $n \in \mathbf{N}$, then K is isotopic to K'.*

Throughout the entire paper we work in the PL or the smooth category.

The results of this paper are used in [KLi1] to develop criteria for detecting non-fibred knots and 3-manifolds. Theorem 1.4 has a generalization to the case that K' is a knot that admits finitely many non-isotopic minimum genus Seifert surfaces. The techniques used for that generalization are of more elementary nature than the ones used in this paper and they will be presented in [KLi].

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2. Preliminaries

2.1. **Definitions.** A *crossing disc* for a knot $K \subset S^3$ is an embedded disc $D \subset S^3$ such that K intersects $\text{int}(D)$ twice with zero algebraic number. Let $q \in \mathbf{Z}$. Performing $\frac{1}{q}$-surgery on $L_1 := \partial D_1$, changes K to another knot $K^{'} \subset S^3$. Clearly $K^{'}$ is obtained from K by a generalized crossing change of order q.

An *n-collection* for a knot K is a pair $(\mathcal{D}, \mathbf{q})$, such that:

i) $\mathcal{D} := \{D_1, \dots, D_n\}$ is a set of n mutually disjoint crossing discs for K;

ii) $\mathbf{q} := \{\frac{1}{q_1}, \dots, \frac{1}{q_n}\}$, with $q_i \in \mathbf{Z} - \{0\}$;

iii) the knots $L_1, \dots, L_n$ are labeled by $\frac{1}{q_1}, \dots, \frac{1}{q_n}$, respectively. Here, $L_i := \partial D_i$. The link $L := \cup_{i=1}^n L_i$ is called the *crossing link* associated to $(\mathcal{D}, \mathbf{q})$.

Given a knot K and an n-collection $(\mathcal{D}, \mathbf{q})$, for $j = 1, \dots, n$, let $i_j \in \{0,\ 1\}$ and

$$\mathbf{i} := (i_1, \dots, i_n).$$

We denote $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

For every $\mathbf{i}$, we will denote by $K(\mathbf{i})$ the knot obtained from K by a surgery modification of order q_i (resp. 0), along each L_j for which $i_j = 1$ (resp. $i_j = 0$). We now restate Definition 1.1 in this language:

Definition 2.1. *We will say that K is n-adjacent to K' if there exists an n-collection $(\mathcal{D}, \mathbf{q})$ for K, such that the knot $K(\mathbf{i})$ is isotopic to K' for every $\mathbf{i} \neq \mathbf{0}$. We will say that $(\mathcal{D}, \mathbf{q})$ transforms K to K'.*

Remark 2.2. It follows from the definition that if $K \xrightarrow{n} K'$, then K and K' are n-equivalent in the sense of [G]. We should however mention that although n-equivalence is shown by Gussarov to be an equivalence relation on the set of knots, n-adjacency is not an equivalence relation. For example, the trefoil knot is 2-adjacent to the unknot. On the other hand, since the trefoil is a fibered knot, by Corollary 1.4 the unknot is not 2-adjacent to the trefoil.

For a link $\bar{L} \subset S^3$ we will use $\eta(\bar{L})$ to denote a regular neighborhood of $\bar{L}$. For a knot $K \subset S^3$ and an n-collection $(\mathcal{D}, \mathbf{q})$, let

$$M_L := S^3 \setminus \eta(K \cup L),$$

where L is the crossing link associated to $(\mathcal{D}, \mathbf{q})$. The following lemma will be useful to us in the subsequent sections.

Lemma 2.3. *Suppose that K, K' are knots such that $K \xrightarrow{n} K'$, for some $n \geq 1$. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' and let the notation be as above. If M_L is reducible then a component of L bounds an embedded disc in the complement of K. Thus, in particular, K is isotopic to K'.*

Proof: Let Σ be an essential 2-sphere in M_L. Assume that Σ has been isotoped so that the intersection $I := \Sigma \cap (\cup_{i=1}^{n} D_i)$ is minimal. Notice that we must have $I \neq \emptyset$ since otherwise Σ would bound a 3-ball in M_L. Let $c \in (\Sigma \cap D_i)$ denote a component of I that is innermost on Σ; that is c bounds a disc $E \subset \Sigma$ such that $\mathrm{int}(E) \cap (\cup_{i=1}^{n} D_i) = \emptyset$. Since Σ is separating in M_L, E can't contain just one point of $K \cap D_i$. E can't be disjoint from K or c could be removed by isotopy. Hence E contains both points of $K \cap D_i$ and so $c = \partial E$ is parallel to ∂D_i in $D_i \setminus K$. It follows that L_i bounds an embedded disc in the complement of K. Since $\frac{1}{q_i}$-surgery on L_i turns K into K', we conclude that K is isotopic to K'. $\square$

2.2. **Outline of the proof of the main results.** We now describe the contents of the paper and the idea of the proof of the main theorem in more detail. Let K be a knot and let $(\mathcal{D}, \mathbf{q})$ be a n-collection with associated crossing link L. Since the linking number of K and every component of L is zero, K bounds a Seifert surface in the complement of L. Thus, we can define the genus of K in the complement of L, say $g_L^n(K)$. In Section 3 we study the question of the extent to which a Seifert surface of K that is of minimal genus in the complement of L remains of minimal genus under various surgery modifications along the components of L. Using a result of [Ga] we show that if $K \xrightarrow{n} K'$, and $(\mathcal{D}, \mathbf{q})$ is an n-collection that transfers K to K' then $g_L^n(K) = g := \max\{\, g(K),\ g(K')\,\}$, where $g(K), g(K')$ denotes the genus of K, K' respectively. This is done in Theorem 3.1.

In Section 4, we prove Theorem 1.3. In Section 5, we finish the proof of Theorem 1.2: We begin by defining a notion of m-adjacency between knots K, K' with respect to a one component crossing link L_1 of K (see Definition 5.1). To describe our approach in more detail, set $N := S^3 \setminus \eta(K \cup L_1)$, and let $\tau(N)$ denote the number of disjoint, pairwise non-parallel, essential embedded tori in N. Using results of [CoLa] and [Go] and inducting on $\tau(N)$ we show the following: Given knots K, K', there exists a constant $b(K,\ K') \in \mathbf{N}$ such that if K is m-adjacent to K' with respect to a crossing link L_1 then either $m \leq b(K,\ K')$ or L_1 bounds an embedded disc in the complement of K. This is done in Theorem 5.3. Theorem 3.1 implies that if $K \xrightarrow{n} K'$ and $n > m(6g-3)$, then an n-collection that transforms K to K' gives rise to a crossing link L_1 such that K is m-adjacent to K' with respect to L_1. Combining this with Theorem 5.3 yields Theorem 1.2.

In Section 6, we present some applications of the results of Section 5 and the methods used in their proofs. For example, one of our applications is concerned with fibred knots. We show that if a crossing change of such a knot doesn't change the knot type then it is *nugatory*. This answers partially a problem in Kirby's Problem List [GT] in affirmative. More generally, we show that if no twist around a crossing link L_1 of a knot K changes the isotopy class of K, then L bounds a disc in the complement of K. Also, for every $n \in \mathbf{N}$, we construct examples of non-isotopic knots K, K' such that $K \xrightarrow{n} K'$.

In Section 7 we study the question of the extent to which a knot can be adjacent to fibred knots and we prove Theorem 1.4.

Finally there is an appendix written by D. McCullough, which deals with homeomorphisms of 3-manifolds that are Dehn twists on the boundary.

3. Taut surfaces, knot genus and multiple crossing changes

Let K be a knot and $(\mathcal{D}, \mathbf{q})$ an n-collection for K with associated crossing link L. Since the linking number of K and every component of L is zero, K bounds a

Seifert surface S in the complement of L. Define

$$g_n^L(K) := \min \{ \text{genus(S)} \mid S \text{ a Seifert surface of } K \text{ as above} \}.$$

Our main result in this section is the following:

Theorem 3.1. *Suppose that $K \xrightarrow{n} K'$, for some $n \geq 1$. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' with associated crossing link L. We have*

$$g_n^L(K) = \max \{ g(K), g(K') \}.$$

In particular, $g_n^L(K)$ is independent of L and n.

We begin by recalling the following definition:

Definition 3.2. ([Th]) *Let M be a compact, oriented 3-manifold with boundary ∂M. For a compact, connected, oriented surface $(S,\ \partial S) \subset (M,\ \partial M)$, the complexity $\chi^-(S)$ is defined by $\chi^-(S) := \max \{ 0, -\chi(S) \}$, where $\chi(S)$ denotes the Euler characteristic of S. If S is disconnected then $\chi^-(S)$ is defined to be the sum of the complexities of all the components of S. Let $\eta(\partial S)$ denote a regular neighborhood of ∂S in ∂M. The Thurston norm $x(z)$ of a homology class $z \in H_2(M,\ \eta(\partial S))$ is the minimal complexity over all oriented, embedded surfaces representing z. The surface S is called taut if it is incompressible and we have $x([S,\ \partial S]) = \chi^-(S)$; that is S is norm-minimizing.*

We will need the following lemma the proof of which follows from the definitions:

Lemma 3.3. *Let $(\mathcal{D}, \mathbf{q})$ be an n-collection for a knot K with associated crossing link L and $M_L := S^3 \setminus \eta(K \cup L)$. A compact, connected, oriented surface $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$, such that $\partial S = K$, is taut if and only if among all Seifert surfaces of K in the complement of L, S has the minimal genus.*

To continue we need to introduce some more notation. For $\mathbf{i}$ as before the statement of Definition 2.1, let $M_L(\mathbf{i})$ denote the 3-manifold obtained from M_L by performing Dehn filling on ∂M_L as follows: The slope of the filling for the components $\partial\eta(L_j)$ for which $i_j = 1$ (resp. $i_j = 0$) is $\frac{1}{q_j}$ (resp. $\infty := \frac{1}{0}$). Clearly we have $M_L(\mathbf{i}) = S^3 \setminus \eta(K(\mathbf{i}))$, where $K(\mathbf{i})$ is as in Definition 2.1. Also let $M_L^+(\mathbf{i})$ (resp. $M_L^-(\mathbf{i})$) denote the 3-manifold obtained from M_L by only performing Dehn filling with slope $\frac{1}{q_j}$ (resp. ∞) on the components $\partial\eta(L_j)$ for which $i_j = 1$.

Lemma 3.4. *Let $(\mathcal{D}, \mathbf{q})$ be an n-collection for a knot K such that M_L is irreducible. Let $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$ be an oriented surface with $\partial S = K$ that is taut. For*

$j = 1, \dots, n$, *define* $\mathbf{i}_j := (0, \dots, 0, 1, 0, \dots, 0)$ *where the unique entry 1 appears at the* j*-th place. Then, at least one of* $M_L^+(\mathbf{i}_j)$, $M_L^-(\mathbf{i}_j)$ *is irreducible and* S *remains taut in that 3-manifold.*

Proof: The proof uses a result of [Ga] in the spirit of [ScT]: For $j \in \{1, \dots, n\}$ set $M^+ := M_L^+(\mathbf{i}_j)$ and $M^- := M_L^-(\mathbf{i}_j)$. Also set $L^j := L \setminus L_j$ and $T_j := \partial\eta(L_j)$. We distinguish two cases:

Case 1: Suppose that every embedded torus that is incompressible in M_L and it separates $L^j \cup S$ from L_j, is parallel to T_j. Then, M_L is S_{L_j}-atoroidal (see Definition 1.6 of [Ga]). By Corollary 2.4 of [Ga], there is at most one Dehn filling along T_j that yields a 3-manifold which is either reducible or in which S doesn't remain taut. Thus the desired conclusion follows.

Case 2: There exists an embedded torus $T \subset M_L$ such that i) T is incompressible in M_L; ii) T separates $L^j \cup S$ from L_j; and iii) T is not parallel to T_j. In S^3, T bounds a solid torus V, with $\partial V = T$. Suppose, for a moment, that L_j lies in $\text{int}(V)$ and $L^j \cup S$ lies in $S^3 \setminus V$. If V is knotted in S^3 then, since L_j is unknotted, L_j is homotopically inessential in V. But then T compresses in V and thus in M_L; a contradiction. If V is unknotted in S^3 then the longitude of V bounds a disc E in $S^3 \setminus V$. Since S is disjoint from T, K intersects E at least twice. On the other hand, since T is incompressible in M_L and K intersects D_j twice, L_j is isotopic to the core of V. Hence, T is parallel to T_j in M_L; a contradiction. Hence $L^j \cup S$ lies in $\text{int}(V)$ while L_j lies in $S^3 \setminus V$. We will show that M^+, M^- are irreducible and that S remains taut in both of these 3-manifolds.

Among all tori in M_L that have properties (i)-(iii) stated above, choose T to be one that minimizes $|T \cap D_j|$. Then, that $D_j \cap T$ consists of a single curve which bounds a disc $D^* \subset \text{int}(D_j)$, such that $(K \cap D_j) \subset \text{int}(D^*)$ and D^* is a meridian disc of V. See Figure 2 below. Since T is not parallel to T_j, V must be knotted. For $r \in \mathbf{Z}$, let $M(r)$ denote the 3-manifold obtained from M_L by performing Dehn filling along $\partial\eta(L_j)$ with slope $\frac{1}{r}$. Since the core of V intersects D_j once, the Dehn filling doesn't unknot V and $T = \partial V$ remains incompressible in $M(r) \setminus V$. On the other hand, T is incompressible in $V \setminus (K \cup L^j)$ by definition. Notice that both $M(r) \setminus V$ and $V \setminus (K \cup L^j)$ are irreducible and

$$M(r) = (M(r) \setminus V) \bigcup_T (V \setminus (K \cup L^j)).$$

We conclude that T remains incompressible in $M(r)$ and $M(r)$ is irreducible. In particular M^+ and M^- are both irreducible.

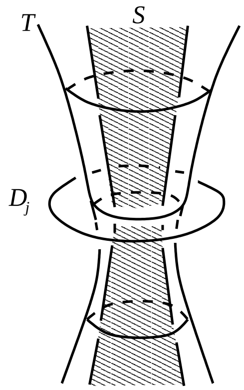

FIGURE 2. The intersection of T and S with D_j.

Next we show that S remains taut in M^+ and M^-. By Lemma 3.3, we must show that S is a minimal genus surface for K in M^+ and in M^-. To that end, let S_1 be a minimal genus surface for K in M^+ or in M^-. We may isotope so that $S_1 \cap T$ is a collection of parallel essential curves on T. Since the linking number of K and L_j is zero, $S_1 \cap T$ is homologically trivial in T. Thus, we may attach annuli along the components of $S_1 \cap T$ and then isotope off T in $\mathrm{int}(V)$, to obtain a Seifert surface S_1' for K that is disjoint from L_j. Thus S_1' is a surface in the complement of L. Since T is incompressible, no component of $S_1 \setminus V$ is a disc. Thus, $\mathrm{genus(S_1')} \leq \mathrm{genus(S_1)}$. On the other hand, by definition of S, $\mathrm{genus(S)} \leq \mathrm{genus(S_1')}$ and thus $\mathrm{genus(S)} \leq \mathrm{genus(S_1)}$. □

Lemma 3.5. *Let $(\mathcal{D}, \mathbf{q})$ be an n-collection for a knot K such that M_L is irreducible. Let $(S,\ \partial S) \subset (M_L,\ \partial\eta(K))$ be an oriented surface with $\partial S = K$ that is taut. There exists at least one sequence $\mathbf{i} := (i_1, \ldots i_n)$, with $i_j \in \{1, 0\}$, such that S remains taut in $M_L(\mathbf{i})$. Thus we have, $g(K(\mathbf{i})) = \mathrm{genus}(S)$.*

Proof: The proof is by induction on n. For $n = 1$, the conclusion follows from Lemma 3.4. Suppose inductively that for every $m < n$ and every m-collection $(\mathcal{D}_1, \mathbf{q}_1)$ of a knot K_1 such that M_{L^1} is irreducible, the conclusion of the lemma is true. Here, L^1 denotes the crossing link associated to $\mathcal{D}_1$ and $M_{L^1} := S^3 \setminus \eta(K_1 \cup L^1)$.

Now let K, $(\mathcal{D}, \mathbf{q})$ and S be as in the statement of the lemma and let $\mathbf{i}_1 := (1, 0, \ldots, 0)$. By Lemma 3.4 at least one of $M_L^{\pm}(\mathbf{i}_1)$, say $M_L^{-}(\mathbf{i}_1)$, is irreducible and S remains taut in that 3-manifold. Let

$$\mathcal{D}_1 := \{D_2, \ldots, D_n\} \text{ and } \mathbf{q}_1 := \{q_2, \ldots, q_n\}.$$

Let $L^1 := L \setminus L_1$ and let K_1 denote the image of K in $M^-_L(\mathbf{i}_1)$. Clearly, $M_{L^1} = M^-_L(\mathbf{i}_1)$ and thus M_{L^1} is irreducible. By the induction hypothesis, applied to K_1 and the $(n-1)$-collection $(\mathcal{D}_1, \mathbf{q}_1)$, it follows that there is at least one sequence $\mathbf{i}_0 := (i_{02}, \dots i_{0n})$, with $i_{0j} \in \{1, 0\}$, such that S remains taut in $M_{L^1}(\mathbf{i}_0)$. Since $M_{L^1}(\mathbf{i}_0) = M_L(\mathbf{i})$, where $\mathbf{i} := (0, i_{02}, \dots i_{0n})$, the desired conclusion follows. □

Proof: [Proof of Theorem 3.1] Let $K \xrightarrow{n} K'$, L and M_L be as in the statement of the theorem. Let S be a Seifert surface for K in the complement of L such that $\text{genus}(S) = g^L_n(K)$. First, assume that M_L is irreducible. By Lemma 3.3, S gives rise to a surface $(S,\ \partial S) \subset (M_L,\ \eta(\partial S))$ that is *taut*. By Lemma 3.5, there exists at least one sequence $\mathbf{i} := (i_1, \dots i_n)$, with $i_j \in \{1, 0\}$, such that S remains taut in $M_L(\mathbf{i})$. There are three cases to consider:

(1) $g(K) > g(K')$,
(2) $g(K) < g(K')$,
(3) $g(K) = g(K')$.

In case (1), for every $\mathbf{i} \neq \mathbf{0}$, we have

$$g(K') = g(K(\mathbf{i})) < g(K) \leq \text{genus}(S).$$

Therefore S doesn't remain taut in $M_L(\mathbf{i}) = S^3 \setminus \eta(K(\mathbf{i}))$. Hence S must remain taut in $M_L(\mathbf{0}) = S^3 \setminus \eta(K)$ and we have $g^L_n(K) = g(K)$. In case (2), notice that we have a n-collection $(\mathcal{D}', \mathbf{q}')$ for K' where $\mathcal{D}' = \mathcal{D}$ and $\mathbf{q}' = -\mathbf{q}$, such that $K'(\mathbf{i}) = K'$ for all $\mathbf{i} \neq \mathbf{1}$ and $K'(\mathbf{1}) = K$. So we may argue similarly as in case (1) that $g^L_n(K) = g(K')$. In fact, in case (2), S must remain taut in $M_L(\mathbf{i})$ for all $\mathbf{i} \neq \mathbf{0}$. Finally in case (3), S remains taut in $M_L(\mathbf{i})$ for all $\mathbf{i}$, and it follows that $g^L_n(K) = g(K') = g(K)$.

Suppose, now, that M_L is reducible. By Lemma 2.3, there is at least one component of L that bounds an embedded disc in the complement of K. Let L^1 denote the union of the components of L that bound disjoint discs in the complement of K and let $L^2 := L \setminus L^1$. We may isotope S so that it is disjoint from the discs bounded by the components of L^1. Now S can be viewed as taut surface in $M_{L^2} := S^3 \setminus \eta(K \cup L^1)$. If $L^2 = \emptyset$, the conclusion is clearly true. Otherwise M_{L^2} is irreducible and the argument described above applies. □

4. Genus reducing n-collections

The purpose of this section is to prove Theorem 1.2 in the cases where either $g(K) > g(K')$ The argument is similar to that in the proof of the main result of [HL].

Proof: [Proof of Theorem 1.3] Let K, K' be as in the statement of the theorem. Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' with associated crossing link L. Let S be a Seifert surface for K that is of minimum genus among all surfaces bounded by K in the complement of L. By Theorem 3.1 we have genus$(S) = g(K)$. Since S is incompressible, after an isotopy, we can arrange so that for $i = 1, \dots, n$, each closed component of $S \cap \mathrm{int}(D_i)$ is essential in $D_i \setminus K$ and thus parallel to $L_i = \partial D_i$ on D_i. Then , after an isotopy of L_i in the complement of K, we may assume that $S \cap \mathrm{int}(D_i)$ consists of a single properly embedded arc $(\alpha_i, \partial\alpha_i) \subset (S, \partial S)$ (see Figure 3). Notice that α_i is essential on S. For, otherwise, D_i would bound a disc in the complement of K and thus the genus of K could not be lowered by surgery on L_i.

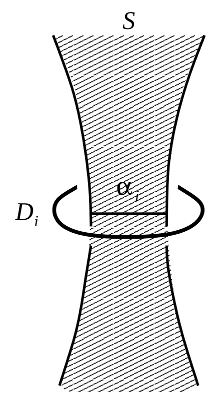

FIGURE 3. The intersection of S with $\mathrm{int}(D_i)$.

We claim that no two of the arcs $\alpha_1, \dots \alpha_n$, can be parallel on S. For, suppose on the contrary, that the arcs $\alpha_i := \mathrm{int}(D_i) \cap S$ and $\alpha_j := \mathrm{int}(D_j) \cap S$ are parallel on S. Then the crossing knots L_i and L_j cobound an embedded annulus that is disjoint from K. Let

$$M := S^3 \setminus \eta(K \cup L_i) \text{ and } M_1 := S^3 \setminus \eta(K \cup L_i \cup L_j).$$

For $r, s \in \mathbf{Z}$ let $M(r)$ (resp. $M_1(r,s)$) denote the 3-manifold obtained from M (resp. M_1) by filling in $\partial\eta(L_i)$ (resp. $\partial\eta(L_i \cup L_j)$) with slope $\frac{1}{r}$ (resp. slopes $\frac{1}{r}, \frac{1}{s}$). By assumption, S doesn't remain taut in any of $M(q_i)$, $M_1(q_i, q_j)$. Since L_i, L_j are coannular we see that $M_1(q_i, q_j) = M(q_i + q_j)$. Notice that $q_i + q_j \neq q_i$ since otherwise we would conclude that a twist of order q_j along L_j cannot reduce the genus of K. Hence we would have two distinct Dehn fillings of M along $\partial\eta(L_i)$ under which S doesn't remain taut, contradicting Corollary 2.4 of [Ga]. Therefore, we conclude that no two of the arcs $\alpha_1, \dots \alpha_n$, can be parallel on S. Now the conclusion follows since a Seifert surface of genus g contains $6g-3$ essential arcs no pair of which is parallel. □

5. Knot adjacency and essential tori

In this section we will complete the proof of Theorem 1.2. For this we need to study the case of n-adjacent knots $K \xrightarrow{n} K'$ in the special case where all the crossing changes from K to K' are supported on a single crossing link of K. Using Theorem 3.1, we will see that the general case is reduced to this special one.

5.1. **Knot adjacency with respect to a crossing circle.** We begin with the following definition that provides a refined version of knot adjacency:

Definition 5.1. *Let K, K' be knots and let D_1 be a crossing disc for K. We will say that K is m-adjacent to K' with respect to the crossing knot $L_1 := \partial D_1$, if there exist non-zero integers $s_1, \ldots, s_m$ such that the following is true: For every $\emptyset \neq J \subset \{1, \ldots, m\}$, the knot obtained from K by a surgery modification of order $s_J := \sum_{j \in J} s_j$ along L_1 is isotopic to K'. We will write $K \xrightarrow{(m,L_1)} K'$.*

Suppose that $K \xrightarrow{(m,L_1)} K'$ and consider the m-collection obtained by taking m parallel copies of D_1 and labeling the i-th copy of L_1 by $\frac{1}{s_i}$. As it follows immediately from the definitions, this m-collection transforms K to K' in the sense of Definition 2.1; thus $K \xrightarrow{m} K'$. The following lemma provides a converse statement that is needed for the proof of Theorem 1.2:

Lemma 5.2. *Let K, K' be knots and set $g := \max\{\, g(K), g(K')\,\}$. Suppose that $K \xrightarrow{n} K'$. If $n > m(6g-3)$ for some $m > 0$, then there exists a crossing link L_1 for K such that $K \xrightarrow{(m+1,L_1)} K'$.*

Proof: Let $(\mathcal{D}, \mathbf{q})$ be an n-collection that transforms K to K' and let L denote the associated crossing link. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L. Isotope so that, for $i = 1, \ldots, n$, the intersection $S \cap \text{int}(D_i)$ is an arc α_i that is properly embedded and essential on S. By Theorem 3.1, we have genus$(S) = g$. Since $n > m(6g-3)$, the set $\{\alpha_i \mid i = 1, \ldots, n\}$ contains at least $m+1$ arcs that are parallel on S. Suppose, without loss of generality, that these are the arcs α_i, $i = 1, \ldots, m+1$. It follows that the components $L_1, \ldots, L_{m+1}$ of L are isotopic in the complement of K; thus any surgery along any of these components can be realized as surgery on L_1. It now follows from Definitions 2.1 and 5.1 that $K \xrightarrow{(m+1,L_1)} K'$. □

The main ingredient needed to complete the proof of Theorem 1.2 is provided by the following theorem:

Theorem 5.3. *Given knots K, K', there exists a constant $b(K,\ K') \in \mathbf{N}$, that depends only on K and K', such that if L_1 is a crossing knot of K and $K \stackrel{(m,L_1)}{\longrightarrow} K'$, then either $m \leq b(K,\ K')$ or L_1 bounds an embedded disc in the complement of K.*

Proof: [Proof of Theorem 1.2 assuming Theorem 5.3] Suppose that K, K' are non-isotopic knots with $K \stackrel{n}{\longrightarrow} K'$. If $g(K) > g(K')$ the conclusion follows from Theorem 1.3 by simply taking $C(K,\ K') := 6g - 3$. In general, let $C(K,\ K') := b(K,\ K')\ (6g-3)$, where $b := b(K,\ K')$ is the constant of Theorem 5.3. We claim that we must have $n \leq C(K,\ K')$. Suppose, on the contrary, that $n > C(K,\ K')$. By Lemma 5.2, there exists a crossing circle L_1 for K, such that $K \stackrel{(b+1,L_1)}{\longrightarrow} K'$. By Theorem 5.3, L_1 bounds an embedded disc in the complement of K. But this implies that K is isotopic to K' contrary to our assumption. □

The remaining of this section will be devoted to the proof of Theorem 5.3. For that we need to study whether the complement of $K \cup L_1$ contains essential tori and how these tori behave under the crossing changes from K to K'. To begin, given K, K' and L_1 such that $K \stackrel{(m,L_1)}{\longrightarrow} K'$, set $N := S^3 \setminus \eta(K \cup L_1)$ and $N' := S^3 \setminus \eta(K')$. We may assume that N is irreducible. For, otherwise, Lemma 2.3 implies that L_1 bounds a disc in the complement of K. By assumption, N' is obtained by Dehn filling along the torus $T_1 := \partial\eta(L_1)$. It turns out that there are three basic cases to consider:

(a) K' is a composite knot (i.e. N' contains an essential torus in which K' has winding number one).

(b) N is atoroidal.

(c) N is toroidal and K' is not a composite knot.

We will consider each of these cases separately. By Thurston ([T]), if N is atoroidal then it is either hyperbolic (it admits a complete hyperbolic metric of finite volume) or it is a *small* Seifert fibred space. To handle the hyperbolic case we will use a result of Cooper and Lackenby ([CoLa]). The small Seifert fibred spaces are known to be very special and this case is handled by a case-by-case analysis. Case (c) is handled by induction on the number of essential tori contained in N. To set up the induction one also needs to study the behavior of these essential tori under the Dehn fillings from N to N'. In particular, we need to examine the circumstances under which these Dehn fillings *create* essential tori in N'. For this step, we employ a result of Gordon ([Go]).

5.2. **Composite knots.** Here we examine the circumstances under which a knot is n-adjacent with respect to a crossing knot to a composite knot. We will need the following theorem.

Theorem 5.4. (Torisu, [To]) *Let $K' := K'_1 \# K'_2$ be a composite knot and K'' a knot that is obtained from K' by a generalized crossing change with corresponding crossing disc D. If K'' is isotopic to K' then either ∂D bounds a disc in the complement of K' or the crossing change occurs within one of K'_1, K'_2.*

Proof: For an ordinary crossing the result is given as Theorem 2.1 in [To]. The proof given in there goes through for generalized crossings as well . □

The next lemma handles possibility (a) above as it reduces Theorem 5.3 to the case that K' is a prime knot.

Lemma 5.5. *Let K, K' be knots such that $K \overset{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing knot for K. Suppose that $K' := K'_1 \# K'_2$ is a composite knot. Then, either L_1 bounds a disc in the complement of K or K is a connect sum $K = K_1 \# K_2$ and there exist $J \in \{K_1, K_2\}$ and $J' \in \{K'_1, K'_2\}$ such that $J \overset{(m,L_1)}{\longrightarrow} J'$.*

Proof: By assumption there is an integer $r \neq 0$ so that the knot K'' obtained from K' by a generalized crossing change of order r is isotopic to K'. By Theorem 5.4, either L_1 bounds a disc in the complement of K' or the crossing change occurs on one of K'_1, K'_2; say on K'_1. Thus, in particular, in the latter case L_1 is a crossing link for K'_1. Since K is obtained from K' by twisting along L_1, K is a, not necessarily non-trivial, connect sum of the form $K_1 \# K'_2$. By the uniqueness of knot decompositions it follows that $K_1 \overset{(m,L_1)}{\longrightarrow} K'_1$. □

5.3. **Dehn surgeries that create essential tori.** Let M be a compact orientable 3-manifold. For a collection $\mathcal{T}$ of disjointedly embedded, pairwise non-parallel, essential tori in M we will use $|\mathcal{T}|$ to denote the number of components of $\mathcal{T}$. By Haken's finiteness theorem ([H], Lemma 13.2), the number

$$\tau(M) = \max \{ |\mathcal{T}| \mid \mathcal{T} \text{ a collection of tori as above} \}$$

is well defined. A collection $\mathcal{T}$ for which $\tau(M) = |\mathcal{T}|$ will be called a *Haken system.*

In this section we will study the behavior of essential tori under the various Dehn fillings from $N := S^3 \setminus \eta(K \cup L_1)$ to $N' := S^3 \setminus \eta(K')$. Since N' is obtained from

N by Dehn filling along $T_1 := \partial\eta(L_1)$, essential tori in N' occur in the following two ways (see [Go] for relating discussion):

Type I: An essential torus $T' \subset N'$ that can be isotoped in $N \subset N'$; thus such a torus is the image of an essential torus $T \subset N$.

Type II: An essential torus $T' \subset N'$ that is the image of an essential punctured torus $(P, \partial P) \subset (N, T_1)$, such that each component of ∂P is parallel on T_1 to the curve along which the Dehn filling from N to N' is done.

We begin with the following lemma that examines circumstances under which twisting a knot that is geometrically essential inside a knotted solid torus V yields a knot that is geometrically inessential inside V. In the notation of Definition 5.1, the lemma implies that an essential torus in N either remains essential in $N(s_J)$, for all $\emptyset \neq J \subset \{1, \ldots, m\}$, or it becomes inessential in all $N(s_J)$.

Lemma 5.6. *Let $V \subset S^3$ be a knotted solid torus and let $K_1 \subset V$ be a knot that is geometrically essential in V. Let $D \subset \mathrm{int}(V)$ be a crossing disc for K_1 and let K_2 be a knot obtained from K_1 by a non-trivial twist along D. Suppose that K_1 is isotopic to K_2 in S^3. Then, K_2 is geometrically essential in V. Furthermore, if K_1 is not the core of V then K_2 is not the core of V.*

Proof: Suppose that K_2 is not geometrically essential in V. Then there is an embedded 3-ball $B \subset \mathrm{int}(V)$ that contains K_2. Since making crossing changes on K_2 doesn't change the homology class it represents in V, the winding number of K_1 in V must be zero. Set $L := \partial D$ and $N := S^3 \setminus \eta(K \cup L)$. Let S be a Seifert surface for K_1 such that among all the surfaces bounded by K_1 in N, S has minimum genus. As usual we isotope S so that $S \cap D$ is an arc α properly embedded on S. As in the proof of Theorem 3.1, S gives rise to Seifert surfaces S_1, S_2 of K_1, K_2, respectively. Now K_1 can be recovered from K_2 by twisting ∂S_2 along α.

Claim: L can be isotoped inside B in the complement of K.

Since K_1 is obtained from K_2 by a generalized crossing change supported on L it follows that K_1 lies in B. Since this contradicts our assumption that K_1 is geometrically essential in V, K_2 must be geometrically essential in V. The last claim follows from the observation that if K_1 is not the core C of V, then C is a companion knot of K_1. If K_2 is the core of V, C and K_1 are isotopic in S^3 which by Schubert ([S]) is impossible.

Proof of Claim: Since K_1, K_2 are isotopic in S^3 using Corollary 2.4 of [Ga], as used in the proof of Theorem 3.1, we see that S_1 (resp. S_2) is a minimum genus surface for K_1 (resp. K_2) in S^3. By assumption ∂V is a non-trivial companion torus of K_1. Since the winding number of K_1 in V is zero, the intersection $S_1 \cap \partial V$ (resp. $S_1 \cap \partial V$) is homologically trivial in ∂V. Thus we may replace the components of $S_1 \cap \overline{S^3 \setminus V}$, (resp. $S_2 \cap \overline{S^3 \setminus V}$) with boundary parallel annuli in $\mathrm{int}(V)$ to obtain a Seifert surface S_1' (resp. S_2') inside V. It follows, that $S_1 \cap \overline{S^3 \setminus V}$, (resp. $S_2 \cap \overline{S^3 \setminus V}$) is a collection of annuli and S_1' (resp. S_2') is a minimum genus Seifert surface for K_1 (resp. K_2). Now S_2' is a minimum genus Seifert surface for K_2 such that $\alpha \subset S_2'$. By assumption, K_2 lies inside B. Since S_2' is incompressible and V is irreducible, S_2' can be isotoped in B by a sequence of disc trading isotopies in $\mathrm{int}(V)$. But this isotopy will also bring α inside B and thus L. □

Next we focus on the case that N' is toroidal and examine the circumstances under which N' contains type II tori. We have the following:

Proposition 5.7. *Let K, K' be knots such that K' is a non-trivial satellite. Suppose that $K \stackrel{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing knot for K and let the notation be as in Definition 5.1. Then, at least one of the following is true:*

a) L_1 bounds an embedded disc in the complement of K.

b) For every $\emptyset \neq J \subset \{1, \dots, m\}$, $N(s_J)$ has a Haken system that doesn't contain tori of type II.

c) We have $m \leq 6$.

Proof: For $s \in \mathbf{Z}$, let $N(s)$ denote the 3-manifold obtained from N by Dehn filling along T_1 with slope $\frac{1}{s}$. Assume that L_1 doesn't bound an embedded disc in the complement of K and that, for some $\emptyset \neq J_1 \subset \{1, \dots, m\}$, $N(s_{J_1})$ admits a Haken system that contains tori of type II. We claim that, for every $\emptyset \neq J \subset \{1, \dots, m\}$, $N(s_J)$ has such a Haken system. To see this, first assume that N doesn't contain essential embedded tori. Then, since $N' = N(s_J)$ and K' is a non-trivial satellite the conclusion follows. Suppose that N contains essential embedded tori. By Lemma 5.6 it follows that an essential torus in N either remains essential in $N(s_J)$, for all $\emptyset \neq J \subset \{1, \dots, m\}$, or it becomes inessential in all $N(s_J)$ as above. Thus the number of type I tori in a Haken system of $N(s_J)$ is the same for all J as above. Thus, since we assume that $N(s_{J_1})$ has a Haken system containing tori of type II, a Haken system of $N(s_J)$ must contain tori of type II, for every $\emptyset \neq J \subset \{1, \dots, m\}$. We distinguish two cases:

Case 1: Suppose that $s_1, \dots, s_m > 0$ or $s_1, \dots, s_m < 0$. Let $s := \sum_{j=1}^{m} s_j$ and recall that we assumed that N is irreducible. By our discussion above, both of

$N(s_1), N(s)$ contain essential embedded tori of type II. By Theorem 1.1 of [Go], we must have

$$\Delta(s, s_1) \leq 5, \tag{5.1}$$

where $\Delta(s, s_1)$ denotes the geometric intersection on T_1 of the slopes represented by $\frac{1}{s_1}$, and $\frac{1}{s}$. Since $\Delta(s, s_1) = |\sum_{j=2}^{m} s_j|$, and $|s_j| \geq 1$, in order for (5.1) to be true we must have $m - 1 \leq 5$ or $m \leq 6$.

Case 2: Suppose that not all of $s_1, \ldots, s_m$ have the same sign. Suppose, without loss of generality, that $s_1, \ldots, s_k > 0$ and $s_{k+1}, \ldots, s_m < 0$. Let $s := \sum_{j=1}^{k} s_j$ and $t := \sum_{j=k+1}^{m} s_j$. Since both of $N(s), N(t)$ contain essential embedded tori of type II, by Theorem 1.1 of [Go]

$$\Delta(s, t) \leq 5. \tag{5.2}$$

But $\Delta(t, s) = s - t = \sum_{j=1}^{m} |s_j|$. Thus, in order for (5.2) to be true we must have $m \leq 5$ and the result follows. □

Proposition 5.7 yields immediately the following corollary:

Corollary 5.8. *Let K, K' be knots and let L_1 be a crossing knot for K. Suppose that the 3-manifold N contains no essential embedded torus and that $K \overset{(m,L_1)}{\longrightarrow} K'$. If K' is a non-trivial satellite, then either $m \leq 6$ or L_1 bounds an embedded disc in the complement of K.*

5.4. **Hyperbolic and Seifert fibred manifolds.** In this section we will deal with the case that the manifold N is atoroidal and thus either hyperbolic or a Seifert fibred manifold.

First we recall some terminology about hyperbolic 3-manifolds. Let N be a hyperbolic 3-manifold with boundary and let T_1 a component of ∂N. In $\mathrm{int}(N)$ there is a cusp, which is homeomorphic to $T_1 \times [1, \infty)$, associated with the torus T_1. The cusp lifts to an infinite set, say $\mathcal{H}$, of disjoint horoballs in the hyperbolic space $\mathbf{H}^3$ which can be expanded so that each horoball in $\mathcal{H}$ has a point of tangency with some other. The image of these horoballs under the projection $\mathbf{H}^3 \longrightarrow \mathrm{int}(N)$, is the *maximal horoball neighborhood* of T_1; note, it is unique. The boundary $\mathbf{R^2}$ of each horoball in $\mathcal{H}$ inherits a Euclidean metric from $\mathbf{H}^3$ which in turn induces a Euclidean metric on T_1. A slope $\mathbf{s}$ on T_1 defines a primitive element in $\pi_1(T_1)$ corresponding to a Euclidean translation in $\mathbf{R^2}$. The length of $\mathbf{s}$, denoted by $l(\mathbf{s})$, is given by the length of corresponding translation vector.

Given a slope $\mathbf{s}$ on T_1, let us use $N[\mathbf{s}]$ to denote the manifold obtained from N by Dehn filling along T_1 with slope $\mathbf{s}$. We remind the reader that in the case that the slope $\mathbf{s}$ is represented by $\frac{1}{s}$, for some $s \in \mathbf{Z}$, we use the notation $N(s)$ instead. Next we recall a result of Cooper and Lackenby that we will need; we only state it in the special case needed here:

Theorem 5.9. (Cooper-Lackenby, [CoLa]) *Let N' be a compact orientable manifold, with $\partial N'$ a collection of tori. Let N be a hyperbolic manifold and let $\mathbf{s}$ be a slope on a toral component T_1 of ∂N such that $N[\mathbf{s}]$ is homeomorphic to N'. Suppose that the length of $\mathbf{s}$ on the maximal horoball of T_1 in $\mathrm{int}(N)$ is at least $2\pi + \epsilon$, for some $\epsilon > 0$. Then, for any given N' and $\epsilon > 0$, there is only a finite number of possibilities (up to isometry) for N and $\mathbf{s}$.*

Remark 5.10. With the notation of Theorem 5.9, let E denote the set of all slopes $\mathbf{s}$ on T_1, such that $l(\mathbf{s}) \leq 2\pi$. It is a consequence of the Gromov-Thurston "2π " theorem that E is finite. More specifically, the Gromov-Thurston theorem (a proof of which is found in [BHo]) states that if $l(\mathbf{s}) > 2\pi$, then $N[\mathbf{s}]$ admits a negatively curved metric. But in Theorem 11 of [BHo], Bleiler and Hodgson show that there can be at most 48 slopes on T_1 for which $N[\mathbf{s}]$ admits no negatively curved metric. Thus, there can be at most 48 slopes on T_1 with length $\leq 2\pi$.

Using Theorem 5.9 we will prove the following proposition which is a special case of Theorem 5.3 (compare, possibility (b) of §5.1):

Proposition 5.11. *Let K, K' be knots such that $K \overset{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing knot for K and $m > 0$. Suppose that $N := S^3 \setminus \eta(K \cup L_1)$ is a hyperbolic manifold. Then, there is a constant $b(K,\ K')$, that depends only on K, K', such that $m \leq b(K,\ K')$.*

Proof: We will apply Theorem 5.9 for the manifolds $N := S^3 \setminus \eta(K \cup L_1)$, $N' := S^3 \setminus \eta(K')$ and the component $T_1 := \partial\eta(L_1)$ of ∂N. Let $s_1, \ldots, s_m$ be integers that satisfy Definition 5.1. That is, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, $N(s_J)$ is homeomorphic to N'. By abusing the notation, for $r \in \mathbf{Z}$ we will use $l(r)$ to denote the length on T_1 of the slope represented by $\frac{1}{r}$. Also, as in the proof of Proposition 5.7, we will use $\Delta(r, t)$ to denote the geometric intersection on T_1 of the slopes represented by $\frac{1}{r}$, $\frac{1}{t}$. Let $A(r, t)$ denote the area of the parallelogram in $\mathbf{R}^2$ spanned by the lifts of these slopes and let $A(T_1)$ denote the area of a fundamental domain of the torus T_1. It is known that $A(T_1) \geq \frac{\sqrt{3}}{2}$ (see, [BHo]) and that $\Delta(r, t)$ is the quotient of $A(r, t)$ by $A(T_1)$. Thus, for every $r, t \in \mathbf{Z}$, we have

$$l(r)l(t) \geq \Delta(r, t)\frac{\sqrt{3}}{2}. \tag{5.3}$$

Let $\lambda > 0$ denote the length of a meridian of T_1; in fact it is known that $\lambda \geq 1$. Assume on the contrary that no constant $b(K,\ K')$ as in the statement of the proposition exists. Then, there exist infinitely many integers s such that $N(s)$ is homeomorphic to N'. Applying (5.3) for $l(s)$ and λ we obtain

$$l(s) \geq |s| \frac{\sqrt{3}}{2\lambda}.$$

Thus, for $|s| \geq \frac{4\pi\lambda+2\lambda}{\sqrt{3}}$ we have $l(s) \geq 2\pi+1$. But then for $\epsilon = 1$, we have infinitely many integers such that $l(s) \geq 2\pi + \epsilon$ and $N(s)$ is homeomorphic to N'. Since this contradicts Theorem 5.9 the proof of the Proposition is finished. □

Next we turn our attention to the case where $N := S^3 \setminus \eta(K \cup L_1)$ is a Seifert fibred space. Since ∂N has two components and is contained in a knot complement in S^3, N can either be a *cable space* or a torus bundle $T^2 \times I$. Let us recall how a cable space is formed: Let $V'' \subset V' \subset S^3$ be concentic solid tori. Let J be a simple closed curve on $\partial V''$ having slope $\frac{a}{b}$, for some $a, b \in \mathbf{Z}$ with $|b| \geq 2$. The complement $X := V' \setminus \mathrm{int}(\eta(J))$ is a $\frac{a}{b}$-cable space. Topologically, X is a Seifert fibred space over the annulus with one exceptional fiber of multiplicity $|b|$. We show the following:

Lemma 5.12. *Let K, K' be knots such that $K \overset{(m,L_1)}{\longrightarrow} K'$, where L_1 is a crossing knot for K and $m > 0$. Suppose that $N := S^3 \setminus \eta(K \cup L_1)$ is an atoroidal Seifert fibred manifold. Then, there is a constant $b(K,\ K')$ such that $m \leq b(K,\ K')$.*

Proof: As discussed above, N is either a cable space or a torus bundle $T^2 \times I$. Note, however, that in a cable space the cores of the solid tori bounded in S^3 by the two components of ∂N have non-zero linking number. Thus, since the linking number of K and L_1 is zero, N cannot be a cable space. Hence, we only have to consider the case where $N \cong T^2 \times I$. Suppose $T_1 = T^2 \times \{1\}$ and $T_2 := \partial\eta(K) = T^2 \times \{0\}$. By assumption there is a slope $\mathbf{s}$ on T_1 such that the Dehn filling of T_1 along $\mathbf{s}$ produces N'. Now $\mathbf{s}$ corresponds to a simple closed curve on T_2 that must compress in N'. By Dehn's Lemma, K' must be the unknot. It follows that either $g(K) > g(K')$ or K is the unknot. In the later case, we obtain that L_1 bounds a disc disjoint from K contrary to our assumption that N is irreducible. Thus, $g(K) > g(K')$ and the conclusion follows from Theorem 1.3. □.

5.5. **The proof of Theorem 5.3.** Before we embark on the proof of Theorem 5.3 we recall that for a compact orientable 3-manifold M, $\tau(M)$ denotes the cardinality of a Haken system of tori (see subsection §5.3). In particular, M is atoroidal if and only if $\tau(M) = 0$.

Proof: [Proof of Theorem 5.3] Let K, K' be knots and let L_1 be a crossing knot of K such that $K \overset{(m,L_1)}{\longrightarrow} K'$. As before set $N := S^3 \setminus \eta(K \cup L_1)$ and $N' := S^3 \setminus \eta(K')$ and consider the complexity

$$\rho = \rho(K,\, K',\, L_1) := \tau(N).$$

If $g(K) > g(K')$, by Theorem 1.3, we have $m \leq 3g(K) - 1$. Thus, in this case, we can take $b(K,\, K') := 3g(K) - 1$ and we only have to consider that case that $g(K) \leq g(K')$.

First, suppose that $\rho = 0$, that is N is atoroidal. Then, N is either hyperbolic or a Seifert fibred manifold. In the former case, the conclusion follows from Proposition 5.11; in the later case it follows from Lemma 5.12.

Assume now that $\tau(N) > 0$; that is N is toroidal. Suppose, inductively, that for every triple K_1, K_1', L_1', with $\rho(K_1,\, K_1',\, L_1') < r$, there is a constant $d = d(K_1,\, K_1')$ such that: If $K_1 \overset{(m,L_1')}{\longrightarrow} K_1'$, then either $m \leq d$ or L_1' bounds an embedded disc in the complement of K_1. Let K, K', L_1 be knots and a crossing link for K, such that $K \overset{(m,L_1)}{\longrightarrow} K'$ and $\rho(K,\, K',\, L_1) = r$. Let $s_1, \ldots, s_m$ be integers satisfying Definition 5.1 for K, K' and L_1. For every $\emptyset \neq J \subset \{1, \ldots, m\}$, let $N(s_J)$ be the 3-manifold obtained from N by Dehn filling of $\partial\eta(L_1)$ with slope $\frac{1}{s_J}$. By assumption, $N' = N(s_J)$. Assume, for a moment, that for some $\emptyset \neq J_1 \subset \{1, \ldots, m\}$, $N(s_{J_1})$ contains essential embedded tori of type II. Then Proposition 5.7 implies that either $m \leq 6$ or L_1 bounds an embedded disc in the complement of K. Hence, in this case, the conclusion of the theorem is true for K, K', L_1, with $b(K,\ K') := 6$. Thus we may assume that, for every $\emptyset \neq J \subset \{1, \ldots, m\}$, $N(s_J)$ doesn't contain essential embedded tori of type II. We will show the following:

Claim 1: There exist knots K_1, K_1' and a crossing link L_1' for K_1 such that:
(1) $K_1 \overset{(m,L_1')}{\longrightarrow} K_1'$ and $\rho(K_1,\, K_1',\, L_1') < \rho(K,\, K',\, L_1) = r$.
(2) If L_1' bounds an embedded disc in the complement of K_1 then L_1 bounds an embedded disc in the complement of K.

The proof of the theorem assuming Claim 1: By induction, there is $d = d(K_1,\ K_1')$ such that either $m \leq d$ or L_1' bounds a disc in the complement of K_1. Let $\mathcal{K}_m$ denote the set of all pairs of knots K_1, K_1' such that there exists a crossing link L_1' for K_1 satisfying properties (1) and (2) of Claim 1. Define

$$b = b(K,\, K') := \min\{\, d(K_1,\, K_1') \,|\, K_1, K_1' \in \mathcal{K}_m \,\}.$$

Clearly b satisfies the conclusion of the statement of the theorem.

Proof of Claim 1: Let T be an essential embedded torus in N. Since T is essential in N, T has to be knotted (otherwise T is either boundary parallel or compressible in N and thus non-essential). Let V denote the solid torus component of $S^3 \setminus T$. Note that K must lie inside V. For, otherwise L_1 must be geometrically essential in V and thus it can't be the unknot. There are various cases to consider according to whether L_1 lies outside or inside V.

Case 1: Suppose that L_1 lies outside V and it cannot be isotoped to lie inside V. Now K is a non-trivial satellite with companion torus T. Let D_1 be a crossing disc bounded by L_1. Notice that if all the components of $D_1 \cap T$ were either homotopically trivial in $D_1 \setminus (D_1 \cap K)$ or parallel to ∂D_1, then we would be able to isotope L_1 inside V contrary to our assumption. Thus $D_1 \cap T$ contains a component that encircles a single point of the intersection $K \cap D_1$. This implies that the winding number of K in V is one. Since T is essential in N we conclude that K is composite, say $K := K_1 \# K_2$, and T is the follow-swallow torus. Moreover, the generalized crossings realized by the surgeries on L_1 occur along a summand of K, say along K_1. By the uniqueness of prime decompositions of knots, it follows that there exists a (not necessarily non-trivial) knot K_1', such that $K' = {K'}_1 \# K_2$ and $K_1 \overset{(m,L_1)}{\longrightarrow} K_1'$. Set $N_1 := S^3 \setminus \eta(K_1 \cup L_1)$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Thus, $\rho(K_1,\, K_1',\, L_1) < \rho(K,\, K',\, L_1)$ and part (1) of the claim has been proven in this case. To see part (2) notice that if L_1 bounds a disc D in the complement of K_1, we may assume $D \cap K = \emptyset$.

Case 2: Suppose that L_1 can be isotoped to lie inside V. Now the link $K \cup L_1$ is a non-trivial satellite with companion torus T. We can find a standardly embedded solid torus $V_1 \subset S^3$, and a 2-component link $(K_1 \cup L_1') \subset V_1$ such that: i) $K_1 \cup L_1'$ is geometrically essential in V_1; ii) L_1' is a crossing disc for K_1; and iii) there is a homeomorphism $f : V_1 \longrightarrow V$ such that $f(K_1) = K$ and $f(L_1') = L_1$ and f preserves the longitudes of V_1 and V. In other words, $K_1 \cup L_1'$ is the model link for the satellite. Let $\mathcal{T}$ be a Haken system for N containing T. We will assume that the torus T is innermost; i.e. the boundary of the component of $N \setminus \mathcal{T}$ that contains T also contains $\partial\eta(K)$. By twisting along L_1 if necessary, we may without loss of generality assume that $\bar{V} := \overline{V \setminus (K \cup L_1)}$ is atoroidal. Then, $\bar{V}_1 := \overline{V_1 \setminus (K_1 \cup L_1')}$ is also atoroidal. For every $\emptyset \neq J \subset \{1, \ldots, m\}$, let $K(s_J)$

denote the knot obtained from K_1 by performing $\frac{1}{s_J}$-surgery on L_1'. By assumption the knots $f(K(s_J))$ are all isotopic to K'.

Subcase 1: There is $\emptyset \neq J_1 \subset \{1,\ldots,m\}$, such that ∂V is compressible in $V \setminus f(K(s_{J_1}))$. By Lemma 5.6, for every $\emptyset \neq J \subset \{1,\ldots,m\}$, ∂V is compressible in $V \setminus f(K(s_J))$. It follows that there is an embedded 3-ball $B \subset \mathrm{int}(V)$ such that: i) $f(K(s_J)) \subset \mathrm{int}(B)$, for every $\emptyset \neq J \subset \{1,\ldots,m\}$; and ii) the isotopy from $f(K(s_{J_1}))$ to $f(K(s_{J_2}))$ can be realized inside B, for every $J_1 \neq J_2$ as above. From this observation it follows that there is a knot $K_1' \subset \mathrm{int}(V_1)$ such that $f(K_1') = K'$ and $K_1 \overset{(m,L_1')}{\longrightarrow} K_1'$ in V_1. Let $N_1 := S^3 \setminus \eta(K_1 \cup L_1')$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Hence, $\rho(K_1, K_1', L_1) < \rho(K, K', L_1)$ and the part (1) of Claim 1 has been proven.

We will prove part (2) of Claim 1 for this subcase together with the next subcase.

Subcase 2: For every $\emptyset \neq J \subset \{1,\ldots,m\}$, $f(K(s_J))$ is geometrically essential in V. By Lemma 5.5, the conclusion of the claim is true if K' is composite. Thus, we may assume that K' is a prime knot. In this case, we claim that, for every $\emptyset \neq J \subset \{1,\ldots,m\}$, there is an orientation preserving homeomorphism $\phi : S^3 \longrightarrow S^3$ such that $\phi(V) = V$ and $\phi(f(K(s_{J_1}))) = f(K(s_{J_2}))$. We may assume that T is a non-trivial companion torus for $f(K(s_J))$; since otherwise the conclusion is clearly true. Since we assumed that $N(s_{J_1})$, $N(s_{J_1})$ do not contain essential tori of type II, T remains innermost in the complement of $f(K(s_{J_1}))$, $f(K(s_{J_2}))$. By the uniqueness of the torus decomposition of knot complements [JS] or the uniqueness of satellite structures of knots [S], there is an orientation preserving homeomorphism $\phi : S^3 \longrightarrow S^3$ such that $\phi(V) \cap V = \emptyset$ and $\bar{K} := \phi(f(K(s_{J_1}))) = f(K(s_{J_2}))$ (compare, Lemma 2.3 of [Mo]). Since T is innermost in $\bar{V}$, we have $S^3 \setminus \mathrm{int}(V) \subset \mathrm{int}(\phi(S^3 \setminus \mathrm{int}(V)))$ or $\phi(S^3 \setminus \mathrm{int}(V)) \subset \mathrm{int}(S^3 \setminus \mathrm{int}(V))$. In both cases, by Haken's finiteness theorem, it follows that T and $\phi(T)$ are parallel in the complement of $\bar{K}$. Thus after an ambient isotopy, leaving $\bar{K}$ fixed, we have $\phi(V) = V$. Let $h = f \circ \phi \circ f^{-1} : V_1 \longrightarrow V_1$. Then h preserves the longitude of V_1 up to a sign and $h(K(s_{J_1})) = K(s_{J_2})$. So, in particular, the knots $K(s_{J_1})$ and $K(s_{J_2})$ are isotopic in S^3. Let K_1' denote the knot type in S^3 of $\{K(s_J)\}_{J \subset \{1,\ldots,m\}}$. By our earlier assumptions, $K_1 \overset{(m,L_1')}{\longrightarrow} K_1'$. Let $N_1 := S^3 \setminus \eta(K_1 \cup L_1')$ and $N_1' := S^3 \setminus \eta(K_1')$. Clearly, $\tau(N_1) < \tau(N)$. Thus part (1) of Claim 1 has been proven also in this subcase.

We now prove the part (2) of Claim 1 for both subcases. Note that it is enough to show that if L'_1 bounds an embedded disc, say D', in the complement of K_1 in S^3, then it bounds one inside V_1.

Let $D'_1 \subset V_1$ be a crossing disc bounded by L'_1 and such that $\mathrm{int}(D') \cap \mathrm{int}(D'_1) = \emptyset$. Since ∂V_1 is incompressible in $V_1 \setminus K_1$, after a cut and paste argument, we may assume that $E = D'_1 \cup (D \cap V_1)$ is a proper annulus whose boundary are longitudes of V_1.

By assumption, in both subcases, there exist non-zero integers s, r, such that $K(s)$ and $K(s+r)$ are isotopic in S^3. Here, $K(s)$ and $K(s+r)$ denotes the knots obtained from K_1 by a twist along L'_1 of order s and $s+r$ respectively. Let the extension of $h : V_1 \longrightarrow V_1$ to S^3 be $\hat{h} : S^3 \longrightarrow S^3$. We assume that $\hat{h}$ fixes the core circle C of the complementary solid torus of V_1. Since the 2-sphere $D \cup D'_1$ gives the same (possible trivial) connected sum decomposition of $K'_1 = K(s) = K(s+r)$ in S^3, we may assume that $\hat{h}(D) = D$ and $\hat{h}(D'_1) = D'_1$ up to an isotopy. During this isotopy of $\hat{h}$, $\hat{h}(C)$ and $\hat{h}(V_1)$ remain disjoint. So we may assume that at the end of the isotopy, we still have $\hat{h}(V_1) = V_1$. Thus, we can assume that $h(E) = E$.

The annulus E cuts V_1 into two solid tori V'_1 and V''_1. See Figure 4, where the solid torus above E is V'_1 and below E is V''_1. We have either $h(V'_1) = V'_1$ and $h(V''_1) = V''_1$ or $h(V'_1) = V''_1$ and $h(V''_1) = V'_1$. In the case when $h(V'_1) = V'_1$ and $h(V''_1) = V''_1$, we may assume that $h|\partial V_1 = \mathrm{id}$ and $h|E = \mathrm{id}$. Thus $K(s+r) \cap V'_1 = K(s) \cap V'_1$ and $K(s+r) \cap V''_1$ is equal to $K(s) \cap V'_1$ twisted by a twist of order r along L'_1. Let M denote the 3-manifold obtained from $V''_1 \setminus (V''_1 \cap K(s))$ by attaching a 2-handle to $\partial V''_1 \cap E$ along $K(s) \cap V''_1$. Now $h|\partial M$ can be realized by a Dehn twist of order r along L'_1. By Theorem A.1, L'_1 must bounds a disk in that M. In order words, L'_1 bounds a disk $V_1 \setminus K(s)$. This implies that L'_1 bounds a disk in $V_1 \setminus K_1$

In the case when $h(V'_1) = V''_1$ and $h(V''_1) = V'_1$, we may assume that $h|\partial V_1$ and $h|E$ are rotations of $180°$ with an axis on E passing through the intersection points of D'_1 with $K(s)$ and $K(s+r)$. Thus $K(s+r) \cap V'_1$ and $K(s) \cap V''_1$ differ by a rotation, and $K(s+r) \cap V''_1$ is equal to $K(s) \cap V'_1$ twisted by a twist of order r along L'_1 followed by a rotation. Now we consider the 3-manifold N obtained from $V'_1 \setminus (V'_1 \cap K(s))$ by attaching a 2-handle to $\partial V'_1 \cap E$ along $K(s) \cap V'_1$. As above we conclude that a Dehn twist of order r along L'_1 extends to N and we complete the argument by applying Theorem A.1. $\square$

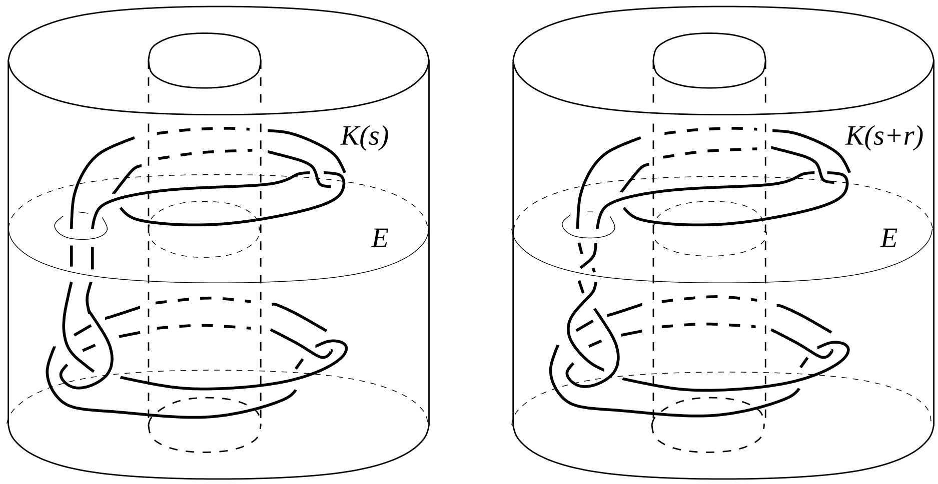

FIGURE 4. The annulus E and knots $K(s)$, $K(s+r)$.

6. Applications and examples

In this section we discuss some corollaries of 5.3. Also, for every $n \in \mathbf{N}$, we construct examples of knots K, K' with $K \xrightarrow{n} K'$.

6.1. **Applications to nugatory crossings.** Recall that a crossing of a knot K with crossing disc D is called *nugatory* if ∂D bounds a disc disjoint from K. This disc and D bound a 2-sphere that decomposes K into a connected sum, where some of the summands may be trivial. Clearly, changing a nugatory crossing doesn't change the isotopy class of a knot. An outstanding open question is whether the converse is true (see Problem 1.58 of Kirby's Problem List ([GT]):

Question 6.1. (Problem 1.58, [GT]) *If a crossing change in a knot K yields a knot isotopic to K is the crossing nugatory?*

The answer is know to be *yes* in the case when K is the unknot ([ScT]) and when K is a 2-bridge knot ([To]). In [To], I. Torisu conjectures that the answer is always *yes*. Our results in Section 5 yield the following corollary that shows that an *essential* crossing circle of a knot K can admit at most finitely many twists that do not change the isotopy type of K:

Corollary 6.2. *For a crossing of a knot K, with crossing disc D, let $K(r)$ denote the knot obtained by a twist of order r along D. The crossing is nugatory if and only if $K(r)$ is isotopic to K for all $r \in \mathbf{Z}$.*

Proof: One direction of the corollary is clear. To obtain the other direction apply Theorem 5.3 for $K = K'$. □

In the view of Corollary 6.2, Question 6.1 is reduced to the following: With the same setting as in Corollary 6.2, let $K_+ := K$ and $K_- := K(1)$. If K_- is isotopic to K_+ is it true that $K(r)$ is isotopic to K, for all $r \in \mathbf{Z}$? In [KLi], we will study knots that admit finitely many minimum genus surfaces up to isotopy in S^3 that leaves the knot fixed. For such knots, we will give explicit bounds on the number of twists along a crossing disc that do not change the isotopy class of the knot.

In the next section we prove the following result which answers Question 6.1. for fibred knots.

Corollary 6.3. *Let K be a fibred knot. A crossing change in K yields a knot isotopic to K if and only if the crossing is nugatory.*

Now combining the results of [To] with Corollary 6.3 we obtain the following:

Corollary 6.4. *Let $\mathcal{U}$ denote the set of knots K' that are 2-bridge knots or fibred and their connect sums. A crossing change in a knot $K \in \mathcal{U}$ yields a knot isotopic to K if and only if the crossing is nugatory.*

6.2. **Examples.** In this subsection, we outline some methods that for every $n > 0$ construct knots K, K' with $K \xrightarrow{n} K'$. It is known that given $n \in \mathbf{N}$ there exists a plethora of knots that are n-adjacent to the unknot. In fact, [AK] provides a method for constructing all such knots. It is easy to see that given knots K, K' such that K_1 is n-adjacent to the unknot, the connected sum $K := K_1 \# K'$ is n-adjacent to K'. Clearly, if K_1 is non-trivial then $g(K) > g(K')$. To construct examples K, K' in which K is not composite, at least in an obvious way, one can proceed as follows: For $n > 0$ let K_1 be a knot that is n-adjacent to the unknot and let $V_1 \subset S^3$ be a standard solid torus. We can embed K_1 in V_1 so that i) it has non-zero winding number; and ii) it is n-adjacent to the core of V_1 inside V_1. Note that there might be many different ways of doing so. Now let $f : V_1 \longrightarrow S^3$ be any embedding that knots V_1. Set $V := f(V_1)$, $K := f(K_1)$ and let K' denote the core of V. By construction, $K \xrightarrow{n} K'$. Since K_1 has non-zero winding number in V_1 we have $g(K) > g(K')$ (see [BZ]).

We will say that two ordered pairs of knots $(K_1,\ K_1')$, $(K_2,\ K_2')$ are isotopic iff K_1 is isotopic to K_2 and K_1' is isotopic to K_2'. From our discussion above we obtain the following:

Proposition 6.5. *For every $n \in \mathbf{N}$ there exist infinitely many non-isotopic pairs of knots $(K,\ K')$ such that $K \xrightarrow{n} K'$ and $g(K) > g(K')$.*

In [AK] Brunnian n-graphs were introduced and were used to describe the knots that are n-adjacent to the unknot. These graphs can be generalized and the constructions of [AK] can be adjusted to provide further examples of knots K, K' with $K \xrightarrow{n} K'$. Unfortunately for these constructions it is hard to understand the relation between $g(K)$ and $g(K')$.

7. Adjacency to fibred knots

This section is concerned with the question of the degree to which a knot can be adjacent to a fibred knot.The main result here is the following theorem.

Theorem 7.1. *Suppose that K' is a fibered knot and that K is a knot such that $K \xrightarrow{n} K'$, for some $n > 1$. Then, either $g(K) > g(K')$ or K is isotopic to K'.*

7.1. **Winding numbers of isotopies and rolling of Seifert surfaces.** A 3-ball B is called a *base 3-ball* of a knot K' if $K' \cap B$ is a proper unknotted arc in B. If S is a Seifert surface of K', then B is called a base 3-ball of S if B is a base 3-ball of K' and $S \cap B$ is a disc. Two minimum genus Seifert surfaces S, S_1 of a knot K' are called *strongly equivalent* iff they are ambiently isotopic by an isotopy of S^3 that leaves the boundary $\partial S = \partial S_1$ fixed point wise.

To continue suppose that S, S_1 are strongly equivalent surfaces of a knot K' and that there is a base 3-ball such that $B \cap S_1 = B \cap S = D$. We need to show that if K' is fibred S can be isotoped to S_1 relatively $K' \cup B$. The strong equivalence guarantees an ambient isotopy G that fixes K' point wise and brings S to S_1. Now G moves D to itself fixing an arc on ∂D point wise. Pick a point $y \in \mathrm{int}(D)$. The image of y under this isotopy is a loop $\gamma \subset S^3 \setminus K'$. The homology class of γ in $H_1(S^3 \setminus K')$ is represented by an integer. We define this integer to be the winding number $w(G, D)$, of the self isotopy of D.

Lemma 7.2. *Suppose that the Seifert surfaces S, S_1 are strongly equivalent by an ambient isotopy G for which $w(G, D) = 0$. Then, by reducing the radius of B if necessary, we can find an ambient isotopy that sends S to S_1 and leaves $K' \cup B$ point wise fixed.*

Proof: Let $\mathcal{R}(B)$ denote the space of rotations of the 3-ball B with axis $K' \cap B$. The isotopy G gives a homotopically trivial loop γ in $\mathcal{R}(B)$. By reducing the radius of B if necessary, we can find an ambient isotopy of S^3, which fixes the complement of B point wise and such that the loop in $\mathcal{R}(B)$ defined by its restriction of B is γ. Now the concatenation of the inverse of this last isotopy with G will give us the desired isotopy. □

Given a knot K' with a tubular neighborhood $\eta(K')$ consider the torus $\partial\eta(K')$ in the form $S^1 \times S^1$ parameterized by the meridian and longitude of K'. Assume that the orientation of the meridian is consistent with the orientations of S^3 and K'. The *meridian roll* R of $\partial\eta(K')$ is the isotopic deformation of the torus given by

$$R_t(\theta_1, \theta_2) = (\theta_1 + 2\pi t, \theta_2), \quad 0 \le t \le 1.$$

Since $R_1 = \mathrm{id}$, it extends to an orientation preserving homeomorphism of S^3 that leaves K' fixed and preserves its orientation. If S is a minimum genus surface of K', isotoped so that it intersects $\eta(K')$ in an annulus, then R_1 "rolls" S around K' as illustrated in Figure 5. We will denote the n-th rolling of S by $R^n(S)$. In the presence of a base 3-ball B, we will assume that $S \cap B = R^n(S) \cap B = D$. Clearly, the meridian roll R extends to an isotopic deformation of the solid torus $\eta(K')$ that leaves K' fixed. Furthermore, we can use a collar of $\partial\eta(K')$ in the complement $\overline{S^3 \setminus \eta(K')}$ to extend R to an isotopic deformation of $\overline{S^3 \setminus \eta(K')}$. Thus, we have an ambient isotopy from $R^n(S)$ and S that leaves K' fixed (i.e. $R^n(S)$ is strongly equivalent to S). The winding number of the induced isotopy on D is n.

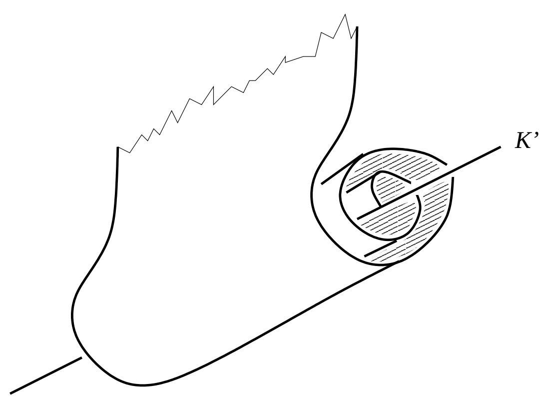

FIGURE 5. Rolling of a Seifert surface around its boundary.

Lemma 7.3. *Suppose that S is strongly equivalent to S_1 by an ambient isotopy G with $w(G, D) = n$. Then there is an ambient isotopy that sends S to $R^n(S_1)$ and leaves $K' \cup B$ point wise fixed.*

Proof: As observed above, S_1 is strongly equivalent to $R^n(S_1)$ by an ambient whose induced isotopy on D has winding number $-n$. The concatenation of G

with this last isotopy is an ambient isotopy Φ from S to $R^n(S_1)$ with $w(\Phi, D) = 0$. Now the conclusion follows from Lemma 7.2. $\square$

Let Q, Q', Q'' be k-complexes in S^3, where $1 \leq k \leq 2$. Suppose that there exist ambient isotopies H and G from Q to Q' and from Q' to Q'', respectively. Recall that the concatenation of H and G, denoted by $H * G$, is the ambient isotopy $\Phi := \{\Phi_t\}_{t \in I}$ defined as follows: For $x \in S^3$, $\Phi_t(x) := H_{2t}(x)$ if $0 \leq t < \frac{1}{2}$ and $\Phi_t(x) := G_{2t-1}(H_1(x))$, if $\frac{1}{2} \leq t \leq 1$. Clearly we have $\Phi_0 = \mathrm{id}$, $\Phi_1(Q) = Q''$.

Lemma 7.4. *Let S, S_1 be minimum genus Seifert surfaces of a fibred knot K'. Suppose that there is a base 3-ball B for K' with $B \cap S = B \cap S_1$. Then, there exists an ambient isotopy $G := \{G_t\}_{0 \leq t \leq 1}$ such that $G_1(S) = S_1$ and $G|(B \cup K') = \mathrm{id}$.*

Proof: It is known that any two minimum genus Seifert surfaces of a fibred knot are strongly equivalent; the proof is given implicitly in Proposition 5.10 of [BZ]. Thus there exists an ambient isotopy that brings S to S_1 and leaves K' point wise fixed. Let n denote the winding number of the induced isotopy on D. If $n = 0$, then the claim is true by Lemma 7.2. Assume that $n \neq 0$. By Lemma 7.3, there exists an ambient isotopy $G' := \{G'_t\}_{0 \leq t \leq 1}$ such that $G'_1(S) = R^n(S_1)$ and $G'|(K' \cup B) = \mathrm{id}$. By Proposition 2.1 of [E], there exists an ambient isotopy that "unrolls" $R^n(S_1)$ back to S_1 through the fibration of $S^3 \setminus \eta(K')$. We will denote this unrolling isotopy by U^n. Since the restriction $U^n | \partial\eta(K') \times I$ is the n-th power of the meridian rolling of $\partial\eta(K')$, by reducing the radius of B if necessary, we can assume that B is left point wise fixed. See Figure 5 of [E]. Hence, if K' is fibered, $R^n(S_1)$ is strongly equivalent to S_1, relatively $K' \cup B$. Now the concatenation $G' * U^n$ gives the desired isotopy G. $\square$

7.2. **An extension lemma.** Let C be a set of disjoint generalized crossings on a knot K_+. Let K_- denote the knot obtained from K_+ by changing all the generalized crossings in C simultaneously. Let $\mathcal{D}$ be a collection of crossing discs for the crossing changes in C and let L be the corresponding crossing link. Let S be a Seifert surface that is of minimum genus among all surfaces bounded by K_+ in the complement of L. As explained earlier, we may isotope S so that its intersection with every disc in $\mathcal{D}$ is a properly embedded arc. The surface $S = S_+$ gives rise to a Seifert surface S_- of K_- in a natural way. Choose a base 3-ball B for S_+ and S_- with $S_+ \cap B = S_- \cap B = D$ and such that B is disjoint from $\mathcal{D}$. Since S_- is obtained from S_+ by twisting along a collection of arcs we can find a spine Γ, based at a point in $p \in B$, that belongs on both S_+ and S_-. Let N be a regular neighborhood of S_+ that is also a regular neighborhood of S_-. Each of

$\pi_1(S_+, p)$, $\pi_1(S_-, p)$ and $\pi_1(N, p)$ can be identified with $\pi_1(\Gamma, p)$. Since S_+, S_- are incompressible, $\pi_1(\Gamma, p)$ can be thought of a subgroup of both of $\pi_1(S^3 \setminus K_+, p)$ and $\pi_1(S^3 \setminus K_-, p)$. We may assume that the components of L are disjoint simple closed curves on ∂N and the corresponding crossing discs are meridian discs of N. Let $h : N \longrightarrow N$ be a composition of Dehn twists about the discs in $\mathcal{D}$ such that $h(S_+) = S_-$. The restriction $h_L := h|\partial N$ is a composition of non-trivial Dehn twists about the curves in L.

Lemma 7.5. *Suppose that S_+, S_- are minimum genus Seifert surfaces for K_+, K_- respectively. Suppose, moreover, that K_+, K_- are fibred isotopic knots. Then, there exists an orientation preserving homeomorphism $\phi : S^3 \longrightarrow S^3$ such that $\phi(N) = N$ and whose restriction on ∂N is isotopic to h_L.*

Proof: Let Γ be a common spine for S_+, S_-, based at a point $p \in D$ as above. As explained, $\pi_1(\Gamma, p)$ can be thought of a subgroup of both of $\pi_1(S^3 \setminus K_+, p)$ and $\pi_1(S^3 \setminus K_-, p)$. Choose an isotopy $H := \{H_t\}_{0 \leq t \leq 1}$ from K_+ to K_- such that $H|B = \text{id}$ and set $S'_+ := H_1(S_+)$. By the proof of Proposition 5.10 in [BZ], we can choose H so that $\pi_1(\Gamma, p) = \pi_1(S'_+, p)$ as subgroups of $\pi_1(M, p)$. By Lemma 7.4, there exists an ambient isotopy $G := \{G_t\}_{0 \leq t \leq 1}$ such that $G_1(S'_+) = S_-$ and $G|(B \cup K_-) = \text{id}$. Clearly, the induced map $(G_1)_* : \pi_1(M, p) \longrightarrow \pi_1(M, p)$, is the identity. Consider the concatenation $\Phi := H * G$. We have $\Phi_0 = \text{id}$, $\Phi_1(S_+) = S_-$ and $\Phi|B = \text{id}$. Since $\Phi_1(N)$ is a regular neighborhood of S_-, by the uniqueness property of regular neighborhoods we can assume that $\Phi_1(N) = N$. One can see that the map $\phi := \Phi_1$ induces the identity on $\pi_1(N, p)$.

Since $h := h_L$ is a product of Dehn twists on meridian discs of N it induces the identity map on $\pi_1(N, p)$. Let $f : S_+ \longrightarrow S_+$ denote the restriction of the map $h^{-1} \circ \phi$ on S_+ and consider the following commutative diagram where the vertical arrows represent isomorphisms induced by the inclusion $S_+ \hookrightarrow N$:

$$\begin{array}{ccc} \pi_1(S_+, p) & \xrightarrow{f_* = \text{id}} & \pi_1(S_+, p) \\ \cong \downarrow & & \downarrow \cong \\ \pi_1(N, p) & \xrightarrow[h_*^{-1} \circ \phi_* = \text{id}]{} & \pi_1(N, p) \end{array}$$

Since $h_*^{-1} \circ \phi_* = \text{id}$, we have $f_* = \text{id}$. Now let us think of S_+ as an abstract surface of genus $g := \text{genus}(S_+)$ with one boundary component and let $\mathcal{M}_{g,1}$ denote the mapping class group of S_+. Also let S_g denote the closed surface of genus g (abstractly obtained from S_+ by capping off ∂S_+) and let $\mathcal{M}_g$ denote the mapping class group of S_g. There exist obvious homeomorphisms $\mathcal{M}_g \xrightarrow{k_1} \mathrm{Aut}(\pi_1(S_g))$ and $\mathcal{M}_{g,1} \xrightarrow{k_2} \mathrm{Aut}(\pi_1(S_g))$, sending a surface homeomorphism to the induced map on π_1. A well known result of Nielsen ([Ni]) states that k_1 gives an isomorphism between $\mathcal{M}_g$ and $\mathrm{Out}(\pi_1(S_g))$. From this and well known results about the relation of $\mathcal{M}_{g,1}$, $\mathcal{M}_g$ stated in [Bi], it follows that there exists a short exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{M}_{g,1} \xrightarrow{k_2} \mathrm{Aut}(\pi_1(S_g)) \longrightarrow 1,$$

where $\mathbf{Z}$ is a normal subgroup of $\mathcal{M}_{g,1}$ generated by a Dehn twist on ∂S_+. (The precise argument is, for example, given in Lemmas 3.2, 3.4 of [BHT]). From the construction of f it follows that it cannot be a non-trivial power of a Dehn twist on ∂S_+. Since $f_* = \mathrm{id}$ we conclude that f is isotopic to the identity on S_+. It follows that $h^{-1} \circ \phi : N \longrightarrow N$ is isotopic to the identity and hence $\phi, h : N \longrightarrow N$ are isotopic. Thus, in particular, $\phi|\partial N$ and h_L are isotopic. □

7.3. **The proofs of the main results.** We are now ready to prove Theorem 7.1 and Corollary 6.3. We begin with the proof of Theorem 7.1.

Proof: [Proof of Theorem 7.1] Let K' be a fibred knot. Suppose that K is a knot with $K \xrightarrow{n} K'$, for some $n > 1$. Suppose, moreover, that $g(K) \leq g(K')$ and let $(\mathcal{D}, \mathbf{q})$ be an n-collection transforming K to K'. Since $n > 1$, L has at least two components. We will show that a component of L bounds a disc in the complement of K and thus K is isotopic to K'. Let S be a Seifert surface for K that is of minimal genus among all surfaces bounded by K in the complement of L such that, for $i = 1, \ldots, n$, $S \cap \mathrm{int}(D_i)$ is an arc. For $\mathbf{i} \neq \mathbf{0}$, let $K(\mathbf{i})$ (resp. $S(\mathbf{i})$) denote the image of K (resp. S) in $M_L(\mathbf{i})$. By assumption, $K(\mathbf{i})$ is isotopic to K' and $S(\mathbf{i})$ is a Seifert surface for $K(\mathbf{i})$. Since $g(K) \leq g(K')$, Theorem 3.1 implies that $S(\mathbf{i})$ is a minimum genus surface for $K(\mathbf{i})$. Let N denote a regular neighborhood of S that is also a regular neighborhood of $S(\mathbf{i})$, for every $\mathbf{i} \neq \mathbf{0}$.

We can find $\mathbf{i}_1, \mathbf{i}_2 \neq \mathbf{0}$, with $\mathbf{i}_1 \neq \mathbf{i}_2$ and such that $S(\mathbf{i}_2)$ is obtained from $S(\mathbf{i}_1)$ by twisting along a properly embedded, essential arc $\alpha \subset S(\mathbf{i}_1)$. Choose a base 3-ball B such that $S(\mathbf{i}_1) \cap B = S(\mathbf{i}_2) \cap B = D$. Let d the meridian disc of N corresponding to α. By construction, the boundary $C := \partial d$ is a component of L. Let h_C denote the positive Dehn twist of ∂N along C and set $M := \overline{S^3 \setminus N}$. Clearly $\partial M = \partial N$. We apply Lemma 7.5 to $K_+ := K(\mathbf{i}_1)$, $K_- = K(\mathbf{i}_2)$, $S_+ := S(\mathbf{i}_1)$ and $S_- := S(\mathbf{i}_2)$ to conclude that there is a homeomorphism $\phi : M \longrightarrow M$ such that $\phi|\partial M$ is isotopic to a power of h_C. By Theorem A.1, C bounds a disc properly embedded in M. This disc can easily be chosen to be disjoint from K. □

We finish with the following theorem which immediately yields Corollary 6.3.

Theorem 7.6. *Let $K+$ be a fibred knot and let K_- a knot obtained from K_+ by a crossing change supported on a crossing circle L. If K_- is isotopic to K_+ then L bounds disc in the complement of K_+.*

Proof: Let D be a crossing disc with $L := \partial D$. Let S_+ be a Seifert surface of K_+ that is of minimum genus in the complement of L isotoped so that $S_+ \cap D$ is a simple arc. Let N be a regular neighborhood of S_+ such that L is a simple curve on ∂N and D is a meridian disc of N. Now S_+ gives rise to a Seifert surface S_- for K_- and N is also a regular neighborhood of S_-. There is a Dehn twist about D, say $h : N \longrightarrow N$, such that $h(S_+) = S_-$ and $h|\partial N$ is a Dehn twist about L. Since K_+ and K_- are isotopic, by Corollary 2.4 of [Ga], S_+ and S_- are Seifert surfaces of minimum genus. Choose a base 3-ball B for S_+, S_- and set $M := \overline{S^3 \setminus N}$. By Lemma 7.5 there is an orientation preserving homeomorphism $\phi : M \longrightarrow M$ such that $\phi|\partial M$ is isotopic to $h|\partial N$. By Theorem A.1, L bounds a properly embedded disc in M, which is disjoint from K_+. □

Appendix A. Homeomorphisms which are Dehn twists on the boundary, by D. McCullough

In this section we will prove the following result about 3-manifold homeomorphisms which restrict on the boundary to a composition of Dehn twists about disjoint loops.

Theorem A.1. *Let M be a compact orientable 3-manifold, and let $C_1, \ldots, C_n$ be disjoint non isotopic simple loops in ∂M. Suppose that $h\colon M \to M$ is a homeomorphism whose restriction to ∂M is isotopic to a composition of nontrivial Dehn twists about the C_i. Then for each i, either C_i bounds a disc in M, or for some $j \neq i$, C_i and C_j cobound an incompressible annulus in M.*

In theorem A.1 and throughout, a collection of disjoint objects means a collection any two of whose objects are disjoint. Also, a Dehn twist about a loop in a surface means cutting along the loop and performing any number of twists before regluing, not necessarily a "single" Dehn twist.

The proof of theorem A.1 will use Dehn twists of 3-manifolds. To define these, consider a properly embedded and 2-sided disc or annulus F in a 3-manifold M. Embed $F\times[0,1]$ in M so that $(F\times[0,1])\cap\partial M = \partial F\times[0,1]$. Let r_θ rotate F through an angle θ (that is, if F is a disc, rotate about the origin, and if it is an annulus

$S^1 \times [0,1]$, rotate in the S^1-factor). Fixing some integer n, define $t\colon M \to M$ by $t(x) = x$ for $x \notin F \times [0,1]$ and $t(z,s) = (r_{2\pi ns}(z), s)$ if $(z,s) \in F \times [0,1]$. The restriction of t to ∂M is a Dehn twist about each circle of ∂F. Dehn twists are defined similarly when F is a 2-sphere, a torus, or a two-sided projective plane, Möbius band, or Klein bottle (for the case of tori, there are infinitely many non isotopic choices of an S^1-factor to define r_θ). Since a properly embedded closed surface in M is disjoint from the boundary, Dehn twists about a torus or Klein bottle are the identity on ∂M.

We will give some applications of theorem A.1. The first provides some structural information about these homeomorphisms.

Theorem A.2. *Let M be a compact orientable 3-manifold, and let $C_1, \ldots, C_n$ be disjoint simple loops in ∂M. Suppose that $h\colon M \to M$ is a homeomorphism whose restriction to ∂M is isotopic to a composition of Dehn twists about the C_i. Then there exists a collection of disjoint embedded discs and annuli in M, each of whose boundary circles is isotopic to one of the C_i, for which some composition of Dehn twists about these discs and annuli is isotopic to h on ∂M.*

We will discuss the other applications after proving theorems A.1 and A.2.

The proof of theorem A.1 will use the following result on Dehn twists about annuli in orientable 3-manifolds.

Lemma A.3. *Let A_1 and A_2 be properly embedded annuli in an orientable 3-manifold M, with common boundary consisting of the loops C' and C''. Let N' and N'' be disjoint closed regular neighborhoods in ∂M of C' and C'' respectively, and let t_i be Dehn twists about the A_i whose restrictions to ∂M are supported on $N' \cup N''$. If the restrictions of t_1 and t_2 to N' are isotopic relative to $\partial N'$, then their restrictions to N'' are isotopic relative to $\partial N''$. Consequently, if A is a properly embedded annulus whose boundary circles are isotopic in ∂M (in particular, if they are contained in a torus boundary component of M), then any Dehn twist about A is isotopic to the identity on ∂M.*

Proof: The result is clear if the A_i have orientations so that their induced orientations on $C' \cup C''$ are equal, since then the embeddings of $S^1 \times I \times [0,1]$ into M used to define the Dehn twists can be chosen to agree on $S^1 \times \partial I \times [0,1]$. So we assume that the oriented boundary of A_1 is $C' \cup C''$ and the oriented boundary of A_2 is $C' \cup (-C'')$.

By assumption, t_1 and t_2 restrict to the same Dehn twist near C'. Their effects near C'' differ in that after cutting along C'', the twisting of C'' occurs in opposite directions, but they also differ in that this twisting is extended to collar neighborhoods on opposite sides of C'' (that is, the embeddings of $S^1 \times \partial I \times [0,1]$ used to define the Dehn twists fall on the same sides of C' but opposite sides of C''). Each of these differences changes a Dehn twist about C'' to its inverse, so their combined effect is to give the same Dehn twist near C''.

The last remark of the lemma follows by taking $A_1 = A$ and A_2 to be an annulus with $\partial A_2 = \partial A_1$, with A_2 parallel into ∂M. All Dehn twists about A_2 are isotopic to the identity on ∂M, so the same is true for all Dehn twists about A_1. $\square$

We will also use a fact about homeomorphisms of reducible 3-manifolds, even in many of the cases when M itself is irreducible.

Lemma A.4. *Let $W = P\#Q$ be a connected sum of compact orientable 3-manifolds, with P irreducible. Let S be the sum 2-sphere. Suppose that ∂P is nonempty, and that $g\colon W \to W$ is a homeomorphism which preserves a component of ∂P. Then there is a homeomorphism $j\colon W \to W$, which is the identity on ∂W, such that $jg(S) = S$.*

Proof: Consider a prime factorization $P_1\#\cdots\#P_r\#N_1\#\cdots\#N_s$ of W, where each P_i is irreducible and each N_j is a 2-sphere bundle over S^1. Let Σ be the result of removing from a 3-sphere the interiors of $r+2s$ disjoint 3-balls $B_1, \ldots, B_r$, D_1, E_1, $D_2, \ldots, E_s$. For $1 \le i \le r$, let P_i' be the result of removing the interior of a small open 3-ball B_i' from P_i, and regard W as obtained from Σ and the union of the P_i' by identifying each ∂B_i with $\partial B_i'$ and each ∂D_j with ∂E_j.

In [M], certain *slide homeomorphisms* of W are constructed. These can be informally described as cutting W apart along a ∂B_i or ∂D_j, filling in one of the removed 3-balls to obtain a manifold Y, performing an isotopy that slides that ball around a loop in the interior of Y, removing the 3-ball, and gluing back together to obtain a homeomorphism on the original W. Slide homeomorphisms are assumed to be the identity on ∂W (this is ensured by requiring that the isotopy that slides the 3-ball around the loop in Y be the identity on ∂Y at all times). Lemma 1.4 of [M], essentially due to M. Scharlemann, says that if W is orientable and T is a collection of disjoint embedded 2-spheres in W, then there is a composition j of slide homeomorphisms such that $j(T) \subset \Sigma$.

In our context, we may choose notation so that $P = P_1$ and $S = \partial B_1$. Applying Lemma 1.4 of [M] with $T = g(S)$, we obtain j so that $jg(S) \subset \Sigma$. In particular, there is a component Z of $W - jg(S)$ whose closure contains P_1'. Since g preserves a component of the boundary of P_1, the closure of Z must be $jg(P_1')$. Since P_1 is irreducible, $jg(S)$ must be isotopic to S in W, so changing j by isotopy we obtain $jg(S) = S$. □

Proof: [Proof of Theorem A.1] Let N_j be disjoint closed regular neighborhoods of the C_j in ∂M, and let F be the closure of $\partial M - \cup_j N_j$. By hypothesis, we may assume that h is the identity on F. Let M' be another copy of M, and identify F with its copy F' to form a manifold W with boundary a union of tori, one containing each C_j. Denote by T_j the one containing C_j. Let $g\colon W \to W$ be h on M and the identity map on M', so that on each T_j, g restricts to a nontrivial Dehn twist about C_j.

Fix any C_i, and for notational convenience call it C_1. If W is irreducible, put $W_1 = W$. Otherwise, write W as $W_1 \# W_2$ where W_1 is irreducible and $C_1 \in \partial W_1$, and let S be the sum sphere. By lemma A.4, there is a homeomorphism j of W that is the identity on ∂W, such that $jg(S) = S$. Split W along S, fill in one of the resulting 2-sphere boundary components to obtain W_1, and extend jg to that ball. This produces a homeomorphism of W_1, which we again call g, that restricts to a nontrivial Dehn twist about one of the C_j on each boundary torus of W_1. Since W_1 is irreducible with nonempty boundary, it is Haken.

Assume first that W_1 has compressible boundary. Since W_1 is irreducible, it is a solid torus. The only nontrivial Dehn twists on T_1 that extend to W_1 are Dehn twists about a meridian circle, showing that C_1 bounds a disc in W_1, and hence a disc E in W. Since C_1 does not meet F, we may assume that E meets F transversely in a collection of disjoint circles. Consider a component X of $E - (E \cap F)$ that does not meet ∂E. Since W is the double of M along F, X is contained in one copy of M and there is a corresponding mirror image X' in the other copy. Change E by replacing X with X'. The resulting disc might no longer be embedded in W, but its intersection with F is the same as the original E. By a slight push of the new disc, we may eliminate the intersection circles $\overline{X} \cap F$. Continuing this process, we eventually obtain a singular disc in M bounded by C_1. By the Loop Theorem, C_1 bounds an embedded disc in M.

We call the argument in the previous paragraph that started with E in W and obtained a singular version of E in M, having the same boundary as E, a *swapping* argument (since we are swapping pieces of the surface on one side of F for pieces on the other side).

Suppose now that W_1 has incompressible boundary. Let Σ be Johannson's characteristic submanifold of W_1 ([Jo], also see Chapter 2 of [CM] for an exposition of Johannson's theory). Since ∂W_1 consists of tori, Σ can be Seifert-fibered and contains all of ∂W_1 (in Johannson's definition, a component of Σ can be just a collar neighborhood of a torus boundary component). Note that each C_j in W_1 is noncontractible in T_j, and T_j is incompressible in W_1, so C_j is noncontracible in W_1. This implies that C_j is noncontractible in W, hence also in M.

It suffices to prove that C_1 and some other C_i cobound an embedded annulus in W_1 and hence in W. For then, a swapping argument, similar to the argument used to produce the singular disc in M from the disc E above, produces a singular annulus in M cobounded by C_1 and C_i. Since C_1 and C_i are noncontractible, a direct application of the Generalized Loop Theorem [W1] (see p. 55 of [H]) produces an embedded annulus in M cobounded by C_1 and C_i.

By Corollary 27.6 of [Jo], the mapping class group of W_1 contains a subgroup of finite index generated by Dehn twists about essential annuli and tori. So by raising g to a power, we may assume that it is a composition of such Dehn twists. The Dehn twists about tori do not affect ∂W_1, so we may discard them to assume that g is a composition $t_1 \cdots t_m$, where each t_k is a Dehn twist about an essential annulus A_k. By Corollary 10.10 of [Jo], each A_k is isotopic into Σ. By Proposition 5.6 of [Jo], we may further change each A_k by isotopy to be either horizontal or vertical with respect to the Seifert fibering of Σ.

Suppose first that some A_k is horizontal. Then Σ is either $S^1 \times S^1 \times I$ or the twisted I-bundle over the Klein bottle (for a horizontal annulus projects by a covering map to the base orbifold, and an orbifold Euler characteristic argument shows that the base orbifold is either an annulus, a Möbius band, or the disc with two order-2 cone points, the latter two possibilities yielding the two Seifert fiberings of the twisted I-bundle over the Klein bottle). In the latter case, $\partial\Sigma = T_1$, so $W_1 = \Sigma$ and therefore $\partial W_1 = T_1$. By lemma A.3, each t_i is isotopic to the identity on T_1, hence so is g, a contradiction. So $\Sigma = S^1 \times S^1 \times I$.

Since A_k is horizontal, it must meet both components of $\partial\Sigma$, and we have $\Sigma = W_1$ and $\partial W_1 = T_1 \cup T_i$ for some i. Let $A = C_1 \times I \subset S^1 \times S^1 \times I$. For an appropriate Dehn twist t about A_0, $t^{-1}g$ is isotopic to the identity on T_1. Using Lemma 3.5 of [W], $t^{-1}g$ is isotopic to a level-preserving homeomorphism of W_1, and hence to the identity. We conclude that g is isotopic to t, and consequently C_1 and C_i cobound an annulus A in W_1.

It remains to consider the case when all A_k are vertical. In this case, each t_k restricts on ∂W_1 to Dehn twists about loops isotopic to fibers, so each C_j in ∂W_1 is isotopic to a fiber of the Seifert fibering on Σ.

Let Σ_0 be the component of Σ that contains C_1. Suppose first that $\Sigma_0 \cap \partial W_1 = T_1$. Then each A_k has both boundary circles in T_1, so lemma A.3 implies that g is isotopic to the identity on T_1, a contradiction. So Σ_0 contains another T_i. Since C_1 and C_i are isotopic to fibers, we have the annulus A in Σ_0 with boundary $C_1 \cup C_i$. □

Proof: [Proof of theorem A.2] We may assume that no two C_i are isotopic in ∂M, and that all the Dehn twists about the C_i are nontrivial. Theorem A.1 provides a properly embedded surface S which is either an embedded disc with boundary C_n or an incompressible annulus with boundary C_n and some other C_i. For some Dehn twist t_n about S, t_n and h are isotopic near C_n. The composition $t_n^{-1}h$ is isotopic on ∂M to a composition of Dehn twists about $C_1, \ldots, C_{n-1}$ (some of them possibly trivial). Induction on n produces a composition t as in the theorem, except for the assertion that the collection of discs and annuli may be selected to be disjoint.

Let $D_1, \ldots, D_r$ and $A_1, \ldots, A_s$ be the discs and annuli needed for the Dehn twists in t. We first work on the annuli.

We will say that a union $\mathcal{A}$ of disjoint incompressible embedded annuli in M is *sufficient* for $A_1, \ldots, A_k$ if each boundary circle of $\mathcal{A}$ is isotopic in ∂M to a boundary circle of one of the A_i, and if for any composition of Dehn twists about the A_j, there is a composition of Dehn twists about the annuli of $\mathcal{A}$ which has the same effect, up to isotopy, on ∂M. Notice that $\mathcal{A} = A_1$ is sufficient for the collection consisting of A_1 alone. Inductively, suppose that there is a collection $\mathcal{A}$ sufficient for $A_1, \ldots, A_{k-1}$. By a routine surgery process, we may change A_k to make A_k and $\mathcal{A}$ intersect only in circles essential in both A_k and $\mathcal{A}$. (First, make A_k transverse to $\mathcal{A}$. An intersection circle which is contractible in $\mathcal{A}$ must also be contractible in A_k, since both $\mathcal{A}$ and A_k are incompressible. If there is a contractible intersection circle, then there is a disc E in $\mathcal{A}$ with ∂E a component of $A_k \cap \mathcal{A}$ and the interior of E disjoint from A_k. Replace the disc in A_k bounded by ∂E with E, and push off by isotopy to achieve a reduction of $A_k \cap \mathcal{A}$.)

Now let Z be a closed regular neighborhood of $A_k \cup \mathcal{A}$. Since all intersection circles of A_k with $\mathcal{A}$ are essential in both intersecting annuli, each component of Z has a structure of an S^1-bundle, in which the boundary circles of $\mathcal{A}$ and A_k are fibers.

Inducting on k, we will show that Z contains a collection sufficient for $A_k \cup \mathcal{A}$ and hence for $A_1, \ldots, A_k$. We may assume that Z is connected. For notational simplicity, there is no harm in writing $C_1, \ldots, C_m$ for the boundary circles of $\mathcal{A}$ and A_k, since they are isotopic in ∂M to some of the original C_i.

Fix a small annular neighborhood N of C_1 in $Z \cap \partial M$. Using the S^1-bundle structure of Z, we can choose a disjoint collection $B_2, \ldots, B_m$ of annuli, with B_i running from C_i to a loop in N parallel to C_1.

Consider one of the annuli A of $A_k \cup \mathcal{A}$, say with boundary circles C_i and C_j. If either i or j is 1, say $j = 1$, then by lemma A.3, Dehn twists about A have the same effect on ∂M as Dehn twists about B_i. So we assume that neither is 1.

Form an annulus B connecting C_i to C_j by taking the union of B_i, B_j, and the annulus in N connecting $B_i \cap N$ to $B_j \cap N$, then pushing off of N to obtain a properly embedded annulus. Observe that any Dehn twist about B is isotopic on M to a composition of Dehn twists about B_i and B_j. By lemma A.3, there is a Dehn twist about B whose effect on ∂M is the same as the twist about A. This shows that the collection $B_2, \ldots, B_m$ is sufficient for $A_1, \ldots, A_k$ and completes the induction. So there is a collection $\mathcal{A}$ sufficient for $A_1, \ldots, A_s$.

By additional routine surgery, we may assume that each D_i is disjoint from $\mathcal{A}$. Then, surge D_2 to make D_2 disjoint from D_1, surge D_3 to make it disjoint from $D_1 \cup D_2$, and so on, eventually achieving the desired disjoint collection of discs and annuli. □

We now give two applications. The first concerns compression bodies. These were developed by F. Bonahon [Bo], in a study of cobordism of surface homeomorphisms. They were used in work on mapping class groups of 3-manifolds in [MM], [M2], and [M1], and on deformations of hyperbolic structures on 3-manifolds in [CM]. The homeomorphisms of compression bodies were examined in [O], which develops an analogue for compression bodies of the Nielsen-Thurston theory of surface homeomorphisms. A *compression body* is a 3-manifold constructed by starting with a compact surface G with no components that are 2-spheres, forming $G \times [0, 1]$, and then attaching 1-handles to $G \times \{1\}$. Compression bodies are irreducible. They can be handlebodies (when no component of G is closed) or product I-bundles over surfaces with no S^2 components (when there are no 1-handles). The *exterior boundary* of V is the closure of $\partial V - G \times \{0\}$. Note that if F is the exterior boundary of V, and N is a regular neighborhood in V of the union of F with a collection of cocore 2-discs for the 1-handles of V, then each component of the closure of $V - N$ is a product $X \times I$, where $X \times \{0\}$ is a component of $\overline{\partial N - F}$ and $X \times \{1\}$ is a component of $G \times \{0\}$.

Corollary A.5. *Let V be a compact orientable compression body, and let $h\colon V \to V$ be a homeomorphism which restricts on ∂V to a collection of Dehn twists about disjoint simple loops $C_1, \ldots, C_n$. Then h is isotopic to a composition of Dehn twists about a collection of disjoint discs and incompressible annuli in V, each of whose boundary circles is isotopic in ∂V to one of the C_i.*

Corollary A.5 is proven for the case of V a handlebody in [O].

By theorem A.2, there is a composition t of Dehn twists about a collection of disjoint discs and annuli in V, such that t and h are isotopic on ∂V. Changing h by isotopy, we may assume that $t^{-1}h$ is the identity on ∂V. Corollary A.5 is then immediate from the following lemma.

Lemma A.6. *Let V be a compression body with exterior boundary F, and let $g\colon V \to V$ be a homeomorphism which is the identity on F. Then g is isotopic relative to F to the identity.*

Proof: We have noted that there is a collection of disjoint properly embedded discs $D_1, \ldots, D_n$, with boundaries in F, such that if N is a regular neighborhood of $F \cup (\cup_i D_i)$, then each component of $\overline{V - N}$ is a product $X \times I$, where $X \times \{0\}$ is a component of $\overline{\partial N - F}$. Now ∂D_1 is fixed by g, so we may assume that $g(D_1) \cap D_1$ consists of ∂D_1 and a collection of transverse intersection circles. Since V is irreducible, we may change g by isotopy relative to F to eliminate these other intersection circles, and finally to make g fix D_1 as well as F. Inductively, we may assume that g is the identity on $F \cup (\cup_i D_i)$, and then on N. Finally, for each component $X \times I$ of $\overline{V - N}$, g is the identity on $X \times \{0\}$ and $X \times \{1\}$ equals a component of $G \times \{0\}$. Using Lemma 3.5 of [W], g may be assumed to preserve the levels $X \times \{s\}$ of $X \times I$, and then there is an obvious isotopy from g to the identity on $X \times I$, relative to $X \times \{0\} \cup ((X \times I) \cap F)$. Applying these isotopies on the complementary components of N, we make g the identity on V. □

Corollary A.7. *Let M be a compact orientable irreducible 3-manifold. Suppose there is a homeomorphism $h\colon M \to M$ which restricts on ∂M to a nontrivial Dehn twist about a loop C that is essential in a torus component $T \subset \partial M$. Then M is a solid torus and h is isotopic to a Dehn twist about a meridian disc bounded by C.*

Proof: Let h be a Dehn twist about the essential loop C in T. By theorem A.1, C bounds a meridian disc in M. Since M is irreducible, this implies that M is a solid torus, and corollary A.5 implies that h is isotopic to a Dehn twist about the meridian disc. □

It appears that most of our results can be extended to the non-orientable case, adding the possibilities of Dehn twists about Möbius bands in theorems A.1 and A.2, and in corollary A.5, but the proofs require the more elaborate machinery of uniform homeomorphisms, found in [M] or chapter 12 of [CM] (in particular, lemma 12.1.2 of [CM] is a version of lemma 1.4 of [M] that applies to non-orientable 3-manifolds). Corollary A.7 fails in the non-orientable case, however. Not only are there many possible manifolds with torus boundary admitting Dehn twists about Möbius bands, but an annulus can meet the torus boundary in such a way that a Dehn twist about the annulus will be isotopic on the boundary torus to an even power of a simple Dehn twist about one of its boundary circles.

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