Analysis of Nonlinear Time-Delay Systems using the Sum of Squares Decomposition

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Abstract— The use of the sum of squares decomposition and semidefinite programming have provided an efficient methodology for analysis of nonlinear systems described by ODEs by algorithmically constructing Lyapunov functions. Based on the same methodology we present an algorithmic procedure for constructing Lyapunov-Krasovskii functionals for nonlinear time delay systems described by Functional Differential Equations (FDEs) both for delay-dependent and delay-independent stability analysis. Robust stability analysis of these systems under parametric uncertainty can be treated in a unified way. We illustrate the results with an example from population dynamics.

I. INTRODUCTION

Significant progress has been made in the stability analysis of linear autonomous time-delay systems (TDS) using time-domain (*Lyapunov*) and frequency domain methods [6], [3], [8], [10]. In general there are two types of Lyapunov methods: using Lyapunov-Krasovskii (L-K) functionals and Lyapunov-Razumikhin (L-R) functions. In the linear case these are constructed by solving Linear Matrix Inequalities (LMIs) [4] with a worst-case polynomial time complexity. The L-R LMI criteria are in general more conservative than the L-K ones [6].

This algorithmic procedure is used for both *delay-dependent* and *delay-independent* stability analysis, a classification based on the persistence of stability as the delay is increased. In the linear case, the structures of general quadratic Lyapunov-Krasovskii functionals necessary and sufficient for delay-dependent and strong delay-independent stability analysis are known [6], [1].

On the other hand, the stability analysis of nonlinear time delay systems is far more difficult and L-R functions are usually constructed 'manually' in this case [11]. Even in the case of systems described by ordinary differential equations, the stability analysis of nonlinear systems has always been a challenging task. It is not until recently [16], [15] that a methodology has been proposed to analyze such systems by algorithmically constructing a *Lyapunov function* as a certificate for stability of the zero equilibrium using the Sum of Squares decomposition and SOSTOOLS [18].

In this paper we present an extension of this methodology to the construction of Lyapunov-Krasovskii functionals for time-delay systems. The functionals that we use have structures that are similar to the complete functionals used for delay-independent and delay-dependent stability analysis of linear systems but they have kernels that are *polynomials*. This allows the use of the Sum of Squares decomposition to check the resulting stability conditions through the solution of LMIs. The methodology reduces to the standard LMI conditions when the system under consideration is linear and the functional has quadratic kernels. The same methodology can be used to analyze robust stability of nonlinear time delay systems under parametric uncertainty.

The structure of this paper is as follows. In Section II we present the general setting, the problem formulation and the algorithmic methodology that we propose to use. In Section III we show how it can be applied to solve the problem as it was stated in Section II. In Section IV we present how robust stability analysis can be performed using the same methodology; we apply the theory developed to the stability analysis of a predator-prey system in Section V that concludes the paper.

The notation we will be using is standard, and is the one that is used in [8]. $\mathbb{R}^n$ is an n-dimensional real Euclidean space with norm $| \cdot |$. For $b > a$ denote $C([a,b],\mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For $\phi \in C([a,b],\mathbb{R}^n)$ the norm of ϕ is defined as $\|\phi\| = \sup_{a \le \theta \le b} |\phi(\theta)|$, where $| \cdot |$ is a norm in $\mathbb{R}^n$. Also $C^\gamma = \{\phi \in C : \|\phi\| < \gamma\}$.

II. THE GENERAL SETTING

We will be concerned with autonomous Retarded Functional Differential Equations (RFDEs) given by

$$\dot{x}(t) = f(x_t). \tag{1}$$

where $f : \Omega \to R^n$, $\Omega \subset C$, ' ˙ ' represents the right-hand derivative and $x_t \in \Omega$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. Definitions of stability of the equilibrium x^* of this system satisfying $f(x^*) = 0$ can be found in [8].

Assessing the stability properties of the equilibrium of (1) can be done using *time-domain* methodologies by constructing either a Lyapunov-Razumikhin (L-R) function or a Lyapunov-Krasovskii (L-K) functional. L-R functions attempt to assess stability of an infinite-dimensional system using finite-dimensional tools; the results are conservative even in the linear case [6].

On the other hand, the Lyapunov-Krasovskii theorem can be seen as a generalization of the Lyapunov theorem for the case of systems described by ODEs, in which the existence of a positive definite function $V(x)$ defined in a region of the zero equilibrium with a negative definite derivative proves its asymptotic stability.

Let $\Omega \subset C^\gamma$, define $V : \Omega \to \mathbb{R}$ a continuous function and let $\dot{V}$ denote the *Upper Right Dini Derivative*. Then we have the following theorem [10]:

Theorem 1: (Lyapunov-Krasovskii) Let $\Omega \subset C^\gamma$. Suppose $V : \Omega \to \mathbb{R}$ is continuous and there exist nonnegative functions $a(s)$ and $b(s)$ such that $a(s) \to \infty$ as $s \to \infty$, and $a(0) = b(0) = 0$ such that

$$a(|\phi(0)|) \leq V(\phi), \quad \dot{V}(\phi) \leq -b(|\phi(0)|) \ \forall \ \phi \in \Omega. \quad (2)$$

Then the solution $x = 0$ of (1) is uniformly stable. If, in addition, $b(s)$ is positive definite, then the solution $x = 0$ of (1) is uniformly asymptotically stable.

Just as in the case of ODEs, this is a powerful theorem as it answers questions about stability without requiring a solution to (1). At the same time, however, no methodology exists to construct these functions. In particular, there are two questions that arise. Firstly, what should be the structure of V, and secondly, how one can check the two Lyapunov conditions (2) *algorithmically*.

To answer the first question we can use the structure of *complete* Lyapunov functionals in the case in which the RFDEs are linear (which is known) to guide the choice of V in the case in which f is non-linear. Just as in the case of ODEs with polynomial vector fields where a good choice would be a *polynomial V*, in the case of polynomial FDEs, we will be constructing Lyapunov functionals that have *polynomial kernels*, sacrificing non-negativity of V with the non-negativity of its kernel. Non-polynomial FDEs can be handled in a way similar to non-polynomial ODEs [15].

To answer the second question, we propose a methodology based on the Sum of Squares decomposition [16].

Definition 2: A multivariate polynomial $p(x)$, $x \in \mathbb{R}^n$ is a *Sum of Squares* (SOS) if there exist polynomials $f_i(x)$, $i = 1, \ldots, M$ such that $p(x) = \sum_{i=1}^{M} f_i^2(x)$.

An equivalent characterization of SOS polynomials is given in the following proposition.

Proposition 3: [16] A polynomial $p(x)$ of degree $2d$ is SOS if and only if there exists a positive semidefinite matrix Q and a vector $Z(x)$ containing monomials in x of degree $\leq d$ so that $p = Z(x)^T Q Z(x)$.

In general, the monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T Q Z(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of Q. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a Q which proves that $p(x)$ is an SOS can be cast as a semidefinite program (SDP) [16]. If the monomials in the polynomial $p(x)$ have *unknown* coefficients then the search for feasible values of those

coefficients such that $p(x)$ is nonnegative is also an SDP, a fact that is important for the construction of Lyapunov functions and other S-procedure type multipliers. Constructing the equivalent SDP can be quite involved when the degree of the polynomials is high, but this has been automated in SOSTOOLS [17], [18].

In the next sections we will deliberately be using the SOS decomposition to test nonnegativity and we will be formulating Lyapunov and S-procedure type SOS conditions for analysis of time-delay systems. We will then be searching for polynomials that satisfy those conditions using semidefinite programming and SOSTOOLS.

III. STABILITY OF RFDEs USING SOSTOOLS

In this section we present the methodology for assessing the stability properties of RFDEs through the construction of Lyapunov-Krasovskii functionals using the Sum of Squares decomposition, for delay-independent and delay-dependent stability analysis. In all that will follow, we assume without any loss of generality that the equilibrium of interest is at the origin.

A. Delay-independent stability

Delay-independent stability for nonlinear systems has been investigated deeply under Lyapunov-Razumikhin conditions [12]. In [20] connections between appropriate Lyapunov-Razumikhin conditions and Input-to-State Stability small-gain are made, and relaxed Razumikhin-type conditions guaranteeing global asymptotic stability are derived. In [2] a relationship between a criterion obtained using a Lyapunov-Krasovskii functional and the delay-independent small gain theorem is established for a special class of nonlinear time-delay systems.

In this paper we concentrate on Lyapunov-Krasovskii functionals. Finding the proper structure involves some guessing, particularly for nonlinear systems. Here we consider the following Lyapunov functional for the nonlinear system given by (1):

$$V(x_t) = V_0(x(t)) + \int_{-r}^{0} V_1(x(t+\theta)) d\theta \quad (3)$$

where by $V_0(x(t))$ we denote a polynomial in $x(t)$ of bounded degree, instead of just quadratic. Stability conditions can then be summarized as follows:

Proposition 4: Let 0 be an equilibrium of (1), and let there exist polynomials $V_0(x(t))$, $V_1(x(t+\theta))$ and a positive definite function $\varphi(x(t))$.

1) $V_0(x(t)) - \varphi(x(t)) \geq 0$,
2) $V_1(x(t+\theta)) \geq 0$,
3) $\frac{dV}{dt} = \frac{\partial V_0}{\partial x(t)} f + V_1(x(t)) - V_1(x(t-r)) \leq 0$.

then the equilibrium is globally delay-independent stable.

Proof: The first two conditions guarantee that $V(x_t)$ is positive definite, as $V(x_t) \geq \varphi(x(t)) > 0$. By the third condition, the derivative of V is also non-positive; V is a Lyapunov functional and the equilibrium

is stable. Since the delay size is not explicit in the conditions and no state-space restriction is made, V is a Lyapunov functional for all r and so the system is globally delay-independent stable. ∎

In order to use Proposition 4 with SOSTOOLS, first construct the polynomials V_0 and V_1 in their arguments. If the degree of V_0 is m, then construct

$$\varphi(x(t)) = \sum_{j=1}^{n} \sum_{i=1}^{m/2} \epsilon_{ij} x_j(t)^{2i}, \qquad \sum_{i=1}^{m/2} \epsilon_{ij} \geq \gamma, \quad (4)$$

for $j = 1, \ldots, n$ with $\epsilon_{ij} \geq 0$ and γ a fixed positive number, to guarantee the positive definiteness of $\varphi(x(t))$. The three conditions in Proposition 4 are then converted into Sum of Squares conditions, for example:

$$V_0(x(t)) - \varphi(x(t)) \text{ is SOS.} \quad (5)$$

The other two conditions are constructed similarly. The resulting SOS program can be solved using SOSTOOLS.

The above procedure generalizes the linear case in which $V_0 = x(t)^T P_0 x(t)$ and $V_1 = x(t+\theta)^T P_1 x(t+\theta)$. Other Lyapunov structures may have better properties.

Remark 5: An alternative structure would be the one introduced by P.-A. Bliman in [1]. Denoting

$$z_k(t) = [x(t), x(t-r), \ldots, x(t-(k-1)r)],$$

then the type of Lyapunov-Krasovskii functionals considered in [1] in the single delay case are

$$V_k(x_t) = V_0(z_k(t)) + \int_{-r}^{0} V_1(z_k(t+\theta)) d\theta$$

where V_0 and V_1 are *quadratic* in their arguments.

Remark 6: Global asymptotic stability can be tested in a similar manner, by constructing a *positive definite* function $\vartheta(x(t))$ as in (4) and requiring that

$$-\frac{dV(x_t)}{dt} - \vartheta(x(t)) \text{ is SOS.}$$

Example 7: Consider a very simple example:

$$\dot{x}_1 = -x_1(t) + x_2^2(t-r), \quad \dot{x}_2 = -x_2(t)$$

This system is delay-independent stable, and we prove this by constructing a Lyapunov functional of the form V with V_0 and V_1 polynomials of bounded degree - note that now $x(t) = [x_1(t), x_2(t)]^T$. When V_0 and V_1 are second order polynomials, no certificate is found. However, when their order is increased, a certificate of stability is obtained. In fact the two conditions become

$$V(x_t) = x_2^2(t) + \frac{3}{4} x_1^2(t) + (0.5 x_1(t) + x_2^2(t))^2 + \int_{-r}^{0} x_2^4(t+\theta) d\theta.$$
$$-\dot{V}(x_t) = (x_1(t) + x_2^2(t) - x_2^2(t-r))^2 + 2 x_2^2(t) +$$
$$+ x_2^2(t) x_2^2(t-r) + 2(x_2^2(t) + \frac{1}{4} x_1(t))^2 + \frac{14}{16} x_1^2(t).$$

To get this certificate, only a handful of SOSTOOLS commands are required.

Nonlinear systems may have more than one equilibria. In this case we have to use the region Ω in Theorem 1. For this, we define the set:

$$\Omega = \{ x_t \in C : \|x_t\| = \sup_{-r \leq \theta \leq 0} |x(t+\theta)| \leq \gamma \}.$$

In particular this means that $|x(t+\theta)| \leq \gamma, \forall \theta \in [-r, 0]$, where $|\cdot|$ is the ∞-norm. This is a set of inequalities which can be adjoined in a way similar to the S-procedure. Suppose for concreteness that one wants to use the Lyapunov functional $V(x_t)$ given by (3) to prove stability for a system described by (1) *locally*, i.e. under the additional constraint that

$$|x(t+\theta)| \leq \gamma, \quad \forall \theta. \quad (6)$$

In particular this gives rise to the following conditions:

$$h_{1i} := (x_i(t) - \gamma)(x_i(t) + \gamma) \leq 0,$$
$$h_{2i} := (x_i(t-r) - \gamma)(x_i(t-r) + \gamma) \leq 0.$$

Then we have the following result:

Proposition 8: Let 0 be an equilibrium of system (1), and let there exist SOS multipliers $p_i(x(t))$, $q_{1i}(x(t), x(t-r))$ and $q_{2i}(x(t), x(t-r))$ such that:
1) $V_0(x(t)) - \varphi(x(t)) + \sum_i p_i h_{1i} \geq 0$ with $\varphi(x(t))$ positive definite
2) $V_1(x(t+\theta)) \geq 0$,
3) $-\frac{dV}{dt} + \sum_i (q_{1i} h_{1i} + q_{2i} h_{2i}) \geq 0$.

Then the equilibrium is *delay-independent* stable.

Proof: The proof is similar to the proof of Proposition 4. While $x(t)$ satisfies $h_{1i} \leq 0$ and $p_i(x(t))$ is a SOS we have:

$$V(x_t) = V_0(x(t)) + \int_{-r}^{0} V_1(x(t+\theta)) d\theta \geq \varphi(x(t)) - \sum_i p_i h_{1i} > 0,$$

and so the first Lyapunov condition is satisfied. The same is true for the derivative condition, and so the equilibrium (1) is delay-independent stable. ∎

Let us now turn to delay-dependent stability and derive Lyapunov-based conditions.

B. Delay-dependent stability

In this case the stability of the system changes as the delay, seen as a parameter, varies. Therefore a different type of Lyapunov functionals has to be used to allow for the delay size to appear explicitly in the SOS conditions.

In obtaining stability criteria for linear systems, it is customary to use a *model transformation* [14] to distribute the delays over an interval, which however introduces spurious poles not present in the original system, the dynamics of which may become unstable before the original system does [7]. In this paper we avoid this methodology and therefore our results do not suffer from this conservatism.

The structure that would be adequate for the Lyapunov functional in the linear case is known [19], [9] but it

is difficult to construct [5]. For nonlinear TDS analysis we will use functionals with structures resembling the complete Lyapunov structure but with kernels that are polynomials, using the Sum of Squares decomposition to construct them. Consider the following functional:

$$V(x_t) = V_0(x(t)) + \int_{-r}^{0} V_1(\theta, x(t), x(t+\theta))d\theta +$$
$$+ \int_{-r}^{0} \int_{t+\theta}^{t} V_2(x(\zeta))d\zeta d\theta \qquad (7)$$

for the system of the form (1). The first term is added to impose positive definiteness of V and the last term is added for convenience, as it will be used in the derivative condition to 'complete the squares'. Sufficient conditions for the (global) stability of the zero equilibrium can then be formulated as follows:

Proposition 9: Let 0 be an equilibrium for the system given by (1). Let there exist polynomials V_0, V_1 and V_2 and a positive definite polynomial $\varphi(x(t))$ such that:

1) $V_0(x(t)) - \varphi(x(t)) \geq 0$,
2) $V_1(\theta, x(t), x(t+\theta)) \geq 0$ for $\theta \in [-r, 0]$,
3) $V_2(x(\zeta)) \geq 0$,
4) $r\frac{\partial V_1}{\partial x(t)}f + \frac{dV_0}{dx(t)}f - r\frac{\partial V_1}{\partial \theta} + rV_2(x(t)) - rV_2(x(t+\theta)) + V_1(0, x(t), x(t)) - V_1(-r, x(t), x(t-r)) \leq 0$ for $\theta \in [-r, 0]$.

Then the equilibrium 0 of the system given by (1) is *globally uniformly stable* for all delays in $[0, r]$.

Proof: Integrating the second and third conditions and adding the first condition to get that $V(x_t) \geq \varphi(x(t))$, where $V(x_t)$ is given by (7): the first Lyapunov condition is satisfied. The time derivative of $V(x_t)$ is:

$$\dot{V}(x_t) = \frac{dV_0}{dx(t)}f + V_1(0, x(t), x(t)) - V_1(-r, x(t), x(t-r))$$
$$+ \int_{-r}^{0} \left(\frac{\partial V_1}{\partial x(t)}f - \frac{\partial V_1}{\partial \theta} + V_2(x(t)) - V_2(x(t+\theta)) \right) d\theta$$
$$= \frac{1}{r} \int_{-r}^{0} \left\{ \begin{array}{c} \frac{dV_0}{dx(t)}f + r\frac{\partial V_1}{\partial x(t)}f - r\frac{\partial V_1}{\partial \theta} + \\ +V_1(0, x(t), x(t)) - V_1(-r, x(t), x(t-r)) + \\ +rV_2(x(t)) - rV_2(x(t+\theta)) \end{array} \right\} d\theta.$$

Condition (4) above states that the kernel of the above integral is nonpositive for $\theta \in [-r, 0]$. So (7) is a Lyapunov functional, and the zero equilibrium is *uniformly stable*. Since there is no constraint on the state-space, the result holds globally. ∎

This proposition can be used in practice in a similar way as described in the delay-independent case. To impose the conditions $\theta \in [-r, 0]$, we use a process similar to the S-procedure. The polynomial $V_1(\theta, x(t), x(t+\theta))$ is required to be non-negative only when $h(\theta) = \theta(\theta + r) \leq 0$ is satisfied. We therefore adjoin this constraint to a, using instead of constant positive multipliers (S-procedure), Sum of Squares multipliers p, and we rewrite condition (2) in Proposition 9 above, as follows:

$V_1(\theta, x(t), x(t+\theta)) + p(\theta, x(t), x(t+\theta))h(\theta)$ is a SOS

Condition (4) can be verified in a similar manner. Then we get four SOS conditions in a relevant Sum of Squares programme which can be solved using SOSTOOLS [18]. We can also consider different structures to resemble the general quadratic form of the *complete* Lyapunov functional.

Remark 10: As remarked earlier, when dealing with nonlinear systems with multiple equilibria or with natural constraints on their state-space, it is useful to use a restricted region for which stability is to be proven, in the same way that it was done in the delay-independent case. We will still need to specify $\Omega = \{x_t \in C : \|x_t\| \leq \gamma\}$, and adjoin the relevant conditions on $x(t)$, $x(t-r)$ and $x(t+\theta) \; \forall \; \theta \in [-r, 0]$ to the relevant kernels of the Lyapunov functionals using the extended S-procedure, in much the same way that the conditions $\theta \in [-r, 0]$ were adjoined in Conditions (2) and (4) of Proposition 9.

Remark 6 on asymptotic stability applies here too.

IV. ROBUST STABILITY ANALYSIS UNDER PARAMETRIC UNCERTAINTY

Robust stability under parametric uncertainty can be treated in a unified way. Consider a time-delay system of the form (1) with an uncertain parameter p:

$$\dot{z}(t) = f(z_t, p), \qquad (8)$$

where $p \in P$ given by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \geq 0, \; i = 1, \ldots, N\}, \qquad (9)$$

i.e. the uncertainty set is captured by certain inequalities. Let $x(t) = z(t) - z_0$. Then we have:

$$\dot{x}(t) = f(x_t + z_0, p) \qquad (10)$$
$$0 = f(z_0, p) \qquad (11)$$

which has the equilibrium x^* at the origin. The stability of this system (which has a DAE form) can be handled by constructing a *Parameter Dependent* Lyapunov functional. Consider the functional (modified from (7)):

$$V(x_t, p) = V_0(x(t), p) + \int_{-r}^{0} \int_{t+\theta}^{t} V_2(x(\zeta), p)d\zeta d\theta +$$
$$+ \int_{-r}^{0} V_1(\theta, x(t), x(t+\theta), p)d\theta. \qquad (12)$$

Then we have:

Proposition 11: Consider the system given by (10), where $p \in P$ as defined by (9). Suppose that there exist polynomials $V_0(x(t), p), V_1(\theta, x(t), x(t+\theta), p)$ and $V_2(x(\zeta), p)$ and a positive definite function $\varphi(x(t))$ such that the following conditions hold for $p \in P$:

1) $V_0(x(t), p) - \varphi(x(t)) \geq 0$,
2) $V_1(\theta, x(t), x(t+\theta), p) \geq 0 \; \forall \; \theta \in [-r, 0]$,
3) $V_2(x(\zeta), p) \geq 0$,
4) $V_1(0, x(t), x(t), p) - V_1(-r, x(t), x(t-r), p) + \frac{dV_0}{dx(t)}f + rV_2(x(t), p) - rV_2(x(t+\theta), p) +$

$+ r\frac{\partial V_1}{\partial x(t)}f - r\frac{\partial V_1}{\partial \theta} \leq 0, \ \forall \ \theta \in [-r, 0]$ and when (11) is satisfied.

Then the equilibrium 0 of the system given by (10–11) is *robustly globally uniformly stable* for all $p \in P$ and for all delays in the interval $[0, r]$.

The proof is similar to the one given earlier, it is based on functional (12) and is omitted. The equality constraints given by (11) that may arise during the transformation process can be adjoined using appropriate polynomial multipliers. See [15] for more details on how this is done. State-space constraints for local stability analysis can also be adjoined in a unified manner.

V. EXAMPLE: STABILITY ANALYSIS OF A PREDATOR-PREY MODEL

A simple model of predator-prey interactions is

$$\dot{x} = bx - k_1 xy, \quad \dot{y} = k_2 xy - \sigma y,$$

where x and y are the prey and predator populations, b is the rate of increase of prey, k_1 and k_2 are the coefficients of the effect of predation on x and y and σ is the death rate of y. So the cause of death of the prey is due to predation alone, and the growth of the predator population has as the only limitation the number of prey. These equations give rise to Lotka-Volterra predator-prey cycles, but the model is not biologically meaningful because it is *conservative* giving rise to a family of closed trajectories rather than a single limit cycle [13].

The above equations describe ideal populations that can react instantaneously to any change in the environment; in real populations this change comes with a delay that represents *maturation* of the predator population. A more realistic set of equations is [21]:

$$\dot{x}(t) = x(t)[b - ax(t) - k_1 y(t)],$$
$$\dot{y}(t) = -\sigma y(t) + k_2 x(t-r)y(t-r),$$

where $-ax(t)^2$ limits the growth of the prey, and $r \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators.

Assumption 12: a, b, k_1, k_2 and σ are positive.
The equilibria (x^*, y^*) of the above system are:

$$(x^*, y^*) = (0, 0), \quad (x^*, y^*) = (b/a, 0),$$
$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1 k_2} \right). \tag{13}$$

We are only interested in the equilibrium given by (13).
Assumption 13: $(bk_2 - a\sigma) > 0$.
Assumption 13 ensures that the equilibrium (13) is in the first quadrant. We now shift the coordinates to $(x_1, x_2) = (x - \frac{\sigma}{k_2}, y - \frac{bk_2-a\sigma}{k_1 k_2})$ to get:

$$\dot{x}_1(t) = \left[x_1(t) + \frac{\sigma}{k_2} \right][-ax_1(t) - k_1 x_2(t)] \tag{14}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-r) + \frac{bk_2 - a\sigma}{k_1}x_1(t-r)+$$
$$+ k_2 x_1(t-r)x_2(t-r) \tag{15}$$

We can linearize the above system about $(0, 0)$ to get:

$$\dot{x}_1(t) = \frac{\sigma}{k_2}[-ax_1(t) - k_1 x_2(t)] \tag{16}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-r) + \frac{bk_2 - a\sigma}{k_1}x_1(t-r) \tag{17}$$

For the linearised system, we have the following result:
Proposition 14: Consider the system (16–17) under the assumptions (12,13). Then if $(bk_2 - 3a\sigma) < 0$ the zero equilibrium is stable independent of the delay. If $(bk_2 - 3a\sigma) > 0$ the zero equilibrium is stable if the delay satisfies $r < r^*$ and is unstable otherwise, where r^* is given by:

$$r^* = \frac{1}{\omega} \ \text{atan} \left[\omega \frac{(a\sigma^2 - \omega k_2)k_2 - \sigma(2a\sigma + bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega k_2)} \right].$$

Proof: In the absence of delay, and under the two assumptions, the system is *asymptotically stable*. Substituting $s = j\omega$ in the characteristic equation and separating real and imaginary parts we get:

$$-\omega^2 + \frac{a\sigma^2}{k_2} = \sigma\omega\sin(\omega r) + \sigma\left(\frac{2a\sigma}{k_2} - b \right)\cos(\omega r)$$

$$\sigma\left[1 + \frac{a}{k_2}\right]\omega = \sigma\omega\cos(\omega r) - \sigma\left(\frac{2a\sigma}{k_2} - b \right)\sin(\omega r)$$

Squaring the two equations and adding we get:

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{18}$$

Denoting $p_1 = \frac{a^2\sigma^2}{k_2^2}$ and $p_2 = \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2)$ the roots of this equation are:

$$\omega^2 = -\frac{p_1}{2} \pm \frac{\sqrt{p_1^2 - 4p_2}}{2}. \tag{19}$$

Under assumption 13, if $(bk_2 - 3a\sigma) < 0$ (i.e $p_2 > 0$), then there are no real solutions to (18). Since the equilibrium is stable when the delay is zero, and there is no ω for which poles cross to the RHP, we conclude that (16–17) is delay-independent stable.

Under assumption 13 and $(bk_2 - 3a\sigma) > 0$ then $p_2 < 0$ and one of the two roots of (19) is positive and the other one is negative. Therefore the poles cross the imaginary axis at only one ω — there is no possibility for *stability reversal*. If ω is the solution to the above equation, then at $r = r^*$ given in the statement of the Proposition a Hopf bifurcation occurs; the system is stable for $r < r^*$ and unstable for $r > r^*$. ∎

We now analyze the nonlinear description of the system (14–15) using the methodology that was developed in the previous sections. We choose as nominal values for the parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$.

A. Delay-independent stability analysis

The system (14–15) has many equilibria, and so we need to define a region around the zero equilibrium

to obtain a stability condition (this is the region Ω in Theorem 1). We let:

$$|x_{1_t}| \le \gamma_1 x^*, \quad |x_{2_t}| \le \gamma_2 y^*, \quad (20)$$

where the Equilibrium (x^*, y^*) is given by (13). We consider b to be a parameter in the problem. From Proposition 14, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of a, σ and k_2, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating (x^*, y^*) we use a value of $b = 20/3$. The equilibrium $(0, 0)$ of system (14–15) does not move as b changes, however the other two equilibria cross through the region defined by (20). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other equilibrium enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = V_0(x_1(t), x_2(t), b) + \int_{-r}^{0} V_1(x_1(t+\theta), x_2(t+\theta), b) d\theta.$$

We use a variant of Proposition 11 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When the order of V_0 is second order and V_1 is 4th order, we can construct $V(x_t)$ for $4.56 \le b \le 7.11$. When they are 4th order and 6th order respectively, then this region becomes $3.67 \le b \le 9.95$, which is essentially the full interval.

B. Delay-dependent stability analysis

Now we will try to obtain a bound for r for which stability is retained using the same parameters as before and fixed $b = 15$. Given these parameters $r^* = 0.0541$. The system has several equilibria and so we use the same constraints on x_1 and x_2 on the state-space given by (20) with $\gamma_1 = \gamma_2 = 0.97$.

We can construct the Lyapunov functional $V(x_t)$ given by (7) with V_1 0th order with respect to θ and 2nd order with respect to the rest of the variables for $0 \le r \le 0.04$. When V_1 is quartic with respect to all variables but θ (which is kept at 0 order) then we can construct this $V(x_t)$ for $0 \le r \le 0.053$. The corresponding SDP is bigger as the functional is more complicated, but we can see that stability for almost the full delay interval can be proven this way.

VI. Concluding Remarks

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is done through the solution of a set of Linear Matrix Inequalities (LMIs).

There is increasing interest in the effect of time delays on congestion control schemes for the Internet and we hope that this methodology will find application in this very active area of research. Moreover, the above methods can be easily extended to systems with many delays, either commensurate or not. Still a judicious choice for the structure of the Lyapunov functional would be required. Functional differential equations of neutral type can also be treated in a unified way. The case in which the vector field is not polynomial can be handled in a way similar to ODEs [15].

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