WATER RESOURCES RESEARCH, VOL. 37, NO. 2, PAGES 427–430, FEBRUARY 2001

A model of water streaking down a wall

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Abstract. Rainwater streaking down a vertical wall exhibited the characteristics of "fingered flow." The wall serves as an analogy to film flow in a relatively smooth and wide aperture fracture. A digital photograph of the wall suggests that the vertical distribution of water is heavy-tailed. A fractional-order advection-dispersion equation is used as a model.

1. Introduction

Highly localized and rapid movement of water in variably saturated media is often characterized as "fingered flow." This type of flow can be difficult to model because of a mixture of diffusive-like flow combined with very long fingers that behave more kinematically [*Glass et al.*, 1989b]. For soil with homogeneous hydraulic properties and well-known boundary and initial conditions, linear perturbation theory can be used to accurately estimate the dominant finger width, spacing, and average velocity [*Parlange and Hill*, 1976; *Glass et al.*, 1989a; *Selker et al.*, 1992]. A wave-like (kinematic) model using Richards' equation in a single finger then predicts average finger velocity. *Nicholl et al.* [1994] extended the linear analyses to flow in relatively homogeneous, bounded, rough glass fractures and found that fingering occurs in much the same way: When the flow supplied to a fracture is less than the saturated hydraulic conductivity of the fracture, gravity forces dominate, the flow becomes gravity-unstable, and fingering occurs. Since gravity-driven infiltration under unsaturated conditions is less than or equal to the effective saturated conductivity of the fracture, unstable and fingering film flow should be considered the normal case along an unsaturated fracture in nonporous rock. *Tokunaga and Wan* [1997] show that film flow will also occur within fractures in porous rock, as long as the matric potential is not too highly negative. The film thickness and flow rate are strongly dependent on the matric potential and fracture surface qualities. When infiltration rates drop low enough, the adjacent matrix suction may eliminate the fracture (film) flow altogether.

The linear approach has not been tested in a heterogeneous system or one without a characteristic system size. Nor has the approach been applied to film flow, which is perhaps more difficult than previous Richards' equation approaches because of the nonlinear terms in the full Navier-Stokes set of equations. Moreover, for steep fractures dipping more than 60°, *Dragila* [1999] shows that films with thickness between roughly 50 and 1000 μm are unstable and can develop solitons (waves) and faster, chaotic flow. The flow rates and thicknesses are highly variable in this regime. Even with a traditional linear theory approach, one might anticipate that spatial heterogeneity of boundary conditions or the properties of the soil or fractures would also lead to heterogeneity in the characteristic finger widths and velocities. Ensemble averages of these different local processes would further complicate the predictions of bulk water and solute transport over a large area.

Paper number 2000WR900309.
0043-1397/01/2000WR900309$09.00

Recent work on two-dimensional flow in "quenched" random velocity fields [e.g., *Compte et al.*, 1997] shows that a mixture of diffusive and kinematic flow can lead to highly nondiffusive transport. If particles (of water or solute) tend to spend relatively long periods of time traveling at a constant velocity, such as that within an individual finger, then the overall system may converge to a "heavy-tailed" process. The heavy tail arises from extremely long distances that a particle might travel in a given time period. If the random velocities and resulting finger lengths also lack a characteristic scale, then they will be well modeled by a power law distribution [*Mandelbrot*, 1983]. Power law and stable distributions are also preferred in a thermodynamic sense (see the review by *Tsallis* [1999]). These motions follow a fractional-order advection-dispersion equation [*Fogedby*, 1994; *Compte*, 1996; *Gorenflo and Mainardi*, 1998; *Benson*, 1998; *Meerschaert et al.*, 1999]. A simple form of a space-and-time fractional-order equation assumes that all of the nonlocal behavior can be modeled by a fractional-order spatial derivative [*Meerschaert et al.*, 1999] implying that the random motions are spatially fractal and have power law (heavy) tails. If the heavy tails are concentrated in the positive z direction (as predicted by *Schumer et al.* [2000]), the equation takes the one-dimensional form:

$$\frac{\partial w}{\partial t} = -v\frac{\partial w}{\partial z} + \mathcal{D}\frac{\partial^{\alpha} w}{\partial z^{\alpha}}, \tag{1}$$

where w is a moving scalar, t is time, v is a drift velocity, $\mathcal{D}$ is a measure of the velocity differences, and the spatial fractional derivative of the order of $0 < \alpha \leq 2$ is defined by its Fourier transform [*Samko et al.*, 1993]: $\mathcal{F}[(d^{\alpha}/dx^{\alpha})f(x)] \equiv \int \exp(-ikx)(d^{\alpha}/dx^{\alpha})f(x)\,dx = (ik)^{\alpha}\mathcal{F}[f(x)]$. For $\alpha > 1$, v is the mean velocity. In the special case that w is solute concentration in steady, saturated flow, *Benson et al.* [2000] show how the parameters $\mathcal{D}$ and α may be estimated from the hydraulic properties of an aquifer.

2. An Analog to Film Flow in a Half Fracture

During a light rain event in which the prevailing breeze was from the north, water ponded atop a relatively smooth, south facing concrete wall (Figure 1). A digital photograph was converted to a binary (wet/dry) field, and the number of wet pixels at a certain depth was counted (Figure 2). Each pixel is approximately 0.5 cm on a side. We did not measure (1) the relationship between the number of wet pixels and water mass fraction, (2) the uniformity or timing of rainwater loading atop the wall, or (3) the surface characteristics of the smoothed concrete wall. Rather, this analysis is based on analogy to recharge in fractures, which also have largely unmeasured

Figure 1. Photograph of rainwater-streaked, smooth concrete wall. Prevailing breeze is from behind the wall.

properties and boundary conditions. We define the relative saturation $w(z,t)$ as the fraction of wet pixels at any depth along the wall.

A telltale characteristic of stable Lévy motion following (1) is the appearance of heavy, power law tails. Consistent with the solution of (1), $w(z,t) \sim z^{-1.3}$ for large z (Figure 3). The depth-averaged water content along the entire profile is well modeled by the α-stable distribution solution to (1) for the initial condition $w(z,0) = 0$ and ponded boundary condition $w(0,t) = 1$ (see *Benson* [1998] for the solution). The two fitted quantities $vt = 120$ pixels and $(\mathcal{D}t)^{1/\alpha} = 140$ pixels are the mean and scale parameters [*Samorodnitsky and Taqqu*, 1994] of the α-stable distribution. The power law tail means that successive magnification of the wall, or successive upscaling of measurement or observation, reveals a larger number of longer tails or streaks. In the case of this wall each doubling of the scale or observation window would, on average, reveal a maximum streak length that is longer than the maximum at the smaller scale by roughly $2^{1/\alpha} = 1.7$ times. For a discussion of maxima of power law or α-stable data in a hydrologic context, see *Anderson and Meerschaert* [1998].

3. Discussion

A fractional-order model (1) is only subtly different from previous linear models. It is based on a mixture of kinematic (constant velocity) fingers but assumes that the system heterogeneity or unboundedness leads to a heavy-tailed distribution of finger velocities. Linear theories seek to estimate a single characteristic finger velocity. This velocity is then used as piston flow analog for the whole system. This "step-function" approach will overestimate transport at shallow depths and underestimate it at depths beyond the average finger length. The linear theories, which find a characteristic finger width, speed, and length, will underestimate solute flux at the bottom of deep unsaturated zones.

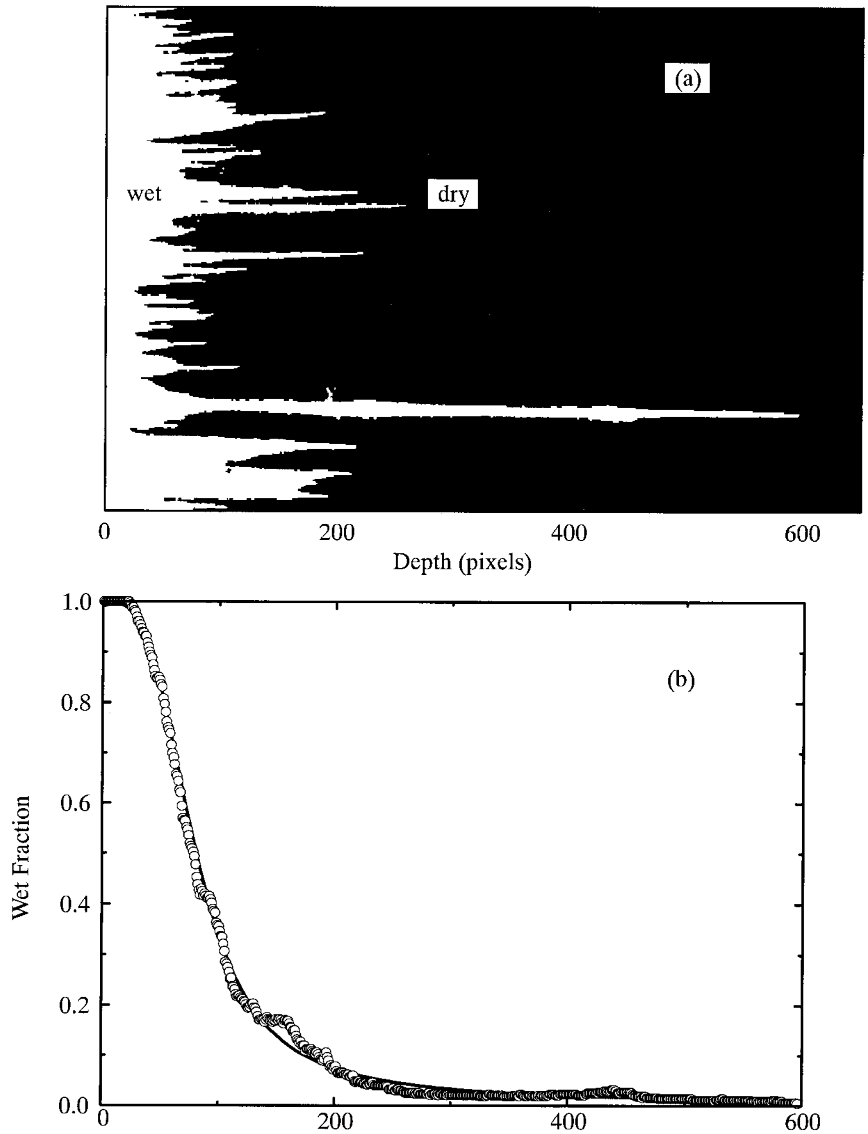

Figure 2. (a) Binary (wet/dry) image of Figure 1. (b) Fraction of wet pixels $w(z,t)$ versus depth (circles) and α-stable solution (solid curve).

The fractional equation is a depth (or ensemble) averaged equation that combines diffusive and kinematic motions. The smaller $1 < \alpha < 2$, the more kinematic and heavier the leading tail or wetting "front." This suggests that at high average flow near the saturated hydraulic conductivity [see *Glass et al.*, 1989b, Figure 2], fewer and fatter fingers may have less variation in speed and thus behave in a more diffusive manner that is better modeled by Richards' equation. *Nicholl et al.* [1994] show that as the input of water into a fracture approaches the saturated conductivity, the system becomes more gravitationally stable, so that gravity-unstable fingering should disappear. At low flow, water should partition into fewer fingers with more heterogeneity of velocities, leading to a smaller value of α and poorer performance of Richards' diffusion equation. At very low flows, gravity is no longer the dominant force, and capillary forces quash finger development [*Yao and Hendrickx*, 1996].

The parameter $\mathcal{D}$ is most sensitive to the heterogeneity of the system, since it measures the spread rate of the process or

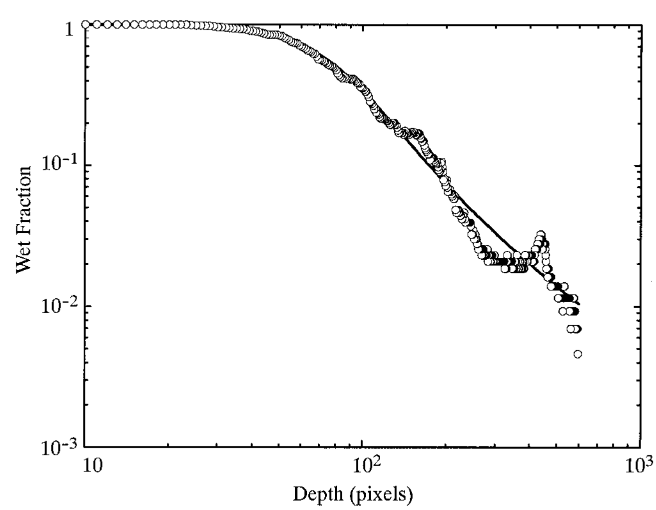

Figure 3. Log-log plot of water distribution and α-stable solution showing power law leading edge.

differences in velocity magnitude. *Benson et al*. [2000] show that all of the parameters in (1) can be estimated directly from the statistical properties of the heavy-tailed velocity distribution. In saturated media the hydraulic conductivity measurements serve as a surrogate for velocity, so v, $\mathcal{D}$, and α can be accurately estimated to predict solute transport in steady flow. Since v is the mean water front velocity when $\alpha > 1$, it is directly related to average film thickness and water application rate. A major open question is whether the parameters α and $\mathcal{D}$ can be rigorously related to measurable fracture and infiltration data. It is also unknown whether a fractional-order equation can reproduce the entire parameter space of fingered flow in homogeneous media. For example, the channeling that may accompany steady flow in unsaturated columns leads to heavy-tailed and fractional-order transport [*Pachepsky et al*., 2000]. We note that the experiments in relatively homogeneous media also showed wide variability in finger lengths and velocities at low flows [*Glass et al*., 1989b; *Selker et al*., 1992] and a failure of the linear (gravity dominated) theories at very low flows [*Yao and Hendrickx*, 1996].

Presently, the fractional-order model for fingered film flow is purely phenomenological. It captures two important features of film flow, nonuniformity and fingering, but relies on fitting two parameters. We do not know how the parameters change with time or water application rate. Yet if the model is robust enough to be applied to real-world data and if $\mathcal{D}$ and α can be related to fracture and flow conditions, then simple and accurate predictions of water and solute mass loading versus depth in unsaturated fractured rock are possible.

Acknowledgments. K. Boc and C. Cooper at the Desert Research Institute (DRI) helped procure and process the data. I thank D. Decker at DRI and M. Dragila at Oregon State University for helpful discussions and T. Tokunaga at Lawrence Berkeley National Laboratory for helpful criticism. The author was supported by DOE-BES grant DE-FG03-98ER14885 and NSF grant AR-9980489.

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(Received April 24, 2000; revised August 25, 2000; accepted September 26, 2000.)