SAMPLE CROSS-CORRELATIONS FOR MOVING AVERAGES WITH REGULARLY VARYING TAILS

By Mark M. Meerschaert and Hans-Peter Scheffler

University of Nevada and University of Dortmund

First Version received February 2000

Abstract. We compute the asymptotics of the sample cross-correlation between two scalar moving averages whose IID innovations have regularly varying probability tails with different tail indices.

Keywords. Moving averages, sample cross-correlation.

1. INTRODUCTION

In this paper, we investigate the asymptotics of the sample cross-correlation for linear time series with heavy tails. A probability distribution has heavy tails if some of its moments fail to exist. Roughly speaking, this is because the probability tails fall off like $t^{-\alpha}$ for some $\alpha > 0$. More precisely, we assume that the probability tails are regularly varying (Bingham *et al.*, 1987; Seneta, 1976). Heavy tail probability distributions are important in a wide variety of applications, including electrical engineering (Nickias and Shao, 1995), hydrology (Anderson and Meerschaert, 1998; Hosking and Wallis, 1987) and finance (Fama, 1965; Mandelbrot, 1963). Several additional applications to economics and computer science appear in Adler *et al.* (1998). There is extensive empirical evidence of heavy tail price fluctuations in stock markets, futures markets, and currency exchange rates (Jansen and de Vries, 1991; Loretan and Phillips, 1994; McCulloch, 1996). Mittnik and Rachev (2000) and Nolan *et al.* (1994) discuss multivariable heavy tail models in finance. These models are used for portfolio analysis involving several different stock issues or mutual funds. Since different stocks have different probability tails with different values of α, there is a need for multivariable techniques which allow the tail parameter α to vary. The cross-correlation quantifies the dependence between scalar time series. Our main results give the asymptotic distribution of the sample cross-correlation between two scalar time series whose innovations have heavy tails. We allow one series to have a heavier tail (i.e., a smaller α) than the other. Our results extend those of Davis *et al.* (1985) and Davis and Marengo (1990), who assume that α is the same for both time series. Our results are important in real applications because α is usually different for two different

0143-9782/01/04 481–492 JOURNAL OF TIME SERIES ANALYSIS Vol. 22, No. 4

time series. It is also interesting to note that the asymptotics change significantly when α varies.

Given a pair of scalar moving averages

$$x_t^{(i)} = \sum_{j=0}^{\infty} c_j^{(i)} z_{t-j}^{(i)} \qquad \text{for } i = 1, 2 \tag{1}$$

we define the sample correlation

$$\hat{\rho}_{ij}(h) = \frac{\hat{\gamma}_{ij}(h)}{\sqrt{\hat{\gamma}_{ii}(0)\hat{\gamma}_{jj}(0)}} \tag{2}$$

in terms of the sample covariance

$$\hat{\gamma}_{ij}(h) = n^{-1} \sum_{t=1}^{n} (x_t^{(i)} - \overline{x}^{(i)})(x_{t+h}^{(j)} - \overline{x}^{(j)}) \tag{3}$$

where

$$\overline{x}^{(i)} = n^{-1} \sum_{t=1}^{n} x_t^{(i)} \tag{4}$$

is the sample mean. In this paper, we derive the asymptotic limit of the sample cross-correlation $\hat{\rho}_{12}(h)$ in the case where the IID innovations $z_t^{(i)}$ have probability tails which vary regularly with index $-\alpha_i$. If $\sum_j |c_j^{(i)}|^{\delta} < \infty$ for some $0 < \delta < \alpha_i$ with $\delta \leqslant 1$ then the series (1) converges almost surely; see Proposition 13.3.1 of Brockwell and Davis (1991). In this paper, we will always assume that this condition holds, so that the moving average process (1) is well defined. If $\alpha_i > 4$ then the time series has finite fourth moments and the usual normal asymptotics apply, so we restrict our attention to the case where $0 < \alpha_i < 4$. Earlier results which appear in Davis *et al.* (1985) and Davis and Marengo (1990) assume that $\alpha_1 = \alpha_2$, so in this paper we deal with the case $\alpha_1 \neq \alpha_2$.

Our proof uses the tools of operator stable laws and generalized domains of attraction, which is the general central limit theory for IID random vectors. In the course of the proof, we establish some results on operator stable laws which are of independent interest. In particular, Lemma 2.3 gives a necessary and sufficient condition for a nonnormal operator stable law to have a complete set of one variable stable marginals. This corrects a mistake in Corollary 4.13.12 of Jurek and Mason (1993).

2. ASYMPTOTICS FOR THE SAMPLE COVARIANCE MATRIX

To derive the joint asymptotics of the time series in (1), we consider the vector moving average

$$\mathbf{X}_t = \sum_{j=0}^{\infty} \mathbf{C}_j \mathbf{Z}_{t-j} \tag{5}$$

where $\mathbf{X}_t = (x_t^{(1)}, x_t^{(2)})'$, $\mathbf{C}_j = \text{diag}(c_j^{(1)}, c_j^{(2)})$ and $\mathbf{Z}_t = (z_t^{(1)}, z_t^{(2)})'$. We also let $\hat{\Gamma}(h)$ denote the sample covariance matrix of the moving average (5), which has ij entry equal to $\hat{\gamma}_{ij}(h)$ as defined in (3). We assume that the probability distribution μ of $\mathbf{Z}_t$ varies regularly with exponent $\mathbf{E} = \text{diag}(a_1, a_2)$. This means that

$$nA_n\mu \to \phi \tag{6}$$

where $\mathbf{A}_n$ varies regularly with index $-\mathbf{E}$ and ϕ is a σ-finite Borel measure on $\mathbb{R}^2\backslash\{0\}$ which cannot be supported on any one dimensional subspace. Here $\mathbf{A}\phi(dx) = \phi(\mathbf{A}^{-1}dx)$ and $t^{\mathbf{E}} = \exp(\mathbf{E}\log t)$ where $\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \mathbf{A}^2/2! + \cdots$ is the matrix exponential operator. The convergence in (6) means that $n\mathbf{A}_n\mu(S) \to \phi(S)$ for any Borel set S which is bounded away from the origin, and whose boundary has ϕ-measure zero. This is sometimes called *vague convergence*. The regular variation of $\mathbf{A}_n$ means that $\mathbf{A}_{[n\lambda]}\mathbf{A}_n^{-1} \to \lambda^{-\mathbf{E}}$ for all $\lambda > 0$. It follows easily that $t{\cdot}\phi = t^{\mathbf{E}}\phi$ for every $t > 0$. Regular variation of μ implies that $P(|z_t^{(i)}| > x)$ varies regularly at infinity with index $-\alpha_i = -1/a_i$, along with a balancing condition on the probability tails of $\mathbf{Z}_t$ in every radial direction; see Meerschaert and Scheffler (1999a) for additional information. The spectral decomposition of Meerschaert and Scheffler (1999a) shows that in this case we may always take $\mathbf{A}_n = \text{diag}(a_n^{(1)}, a_n^{(2)})$. It also implies that if μ varies regularly on $\mathbb{R}^2$ with an exponent which has two real distinct eigenvalues, then there is an orthonormal basis of coordinate vectors with respect to which $\mathbf{E} = \text{diag}(a_1, a_2)$; see Meerschaert (1991). Hence, this assumption entails no loss of generality.

Equation (6) with $a_i > \frac{1}{2}$ is necessary and sufficient for $\mathbf{Z}_t$ to belong to the generalized domain of attraction of some operator stable random vector $\mathbf{Y}$ having no normal component, and in fact

$$\mathbf{A}_n(\mathbf{Z}_1 + \cdots + \mathbf{Z}_n) - s_n \xrightarrow{\text{D}} \mathbf{Y} = (y^{(1)}, y^{(2)})$$

for some shifts $s_n \in \mathbb{R}^d$; see for example Meerschaert (1993). The linear operator $\mathbf{E}$ is also the exponent of the operator stable limit $\mathbf{Y}$, meaning that for $\{\mathbf{Y}_n\}$ i.i.d. with $\mathbf{Y}$ we have $n^{\mathbf{E}}\mathbf{Y} + c_n$ identically distributed with $\mathbf{Y}_1 + \cdots + \mathbf{Y}_n$ for each n, for some $c_n \in \mathbb{R}^d$. Jurek and Mason (1993) is a good reference on operator stable laws. Since $\mathbf{A}_n = \text{diag}(a_n^{(1)}, a_n^{(2)})$, we can project onto each component to get $a_n^{(i)}(z_1^{(i)} + \cdots + z_n^{(i)}) - s_n^{(i)} \xrightarrow{\text{D}} y^{(i)}$ where $y^{(i)}$ is nonnormal stable with index $0 < \alpha_i < 2$. In other words, $\{z_t^{(i)}\}$ belongs to the domain of attraction of the stable limit $y^{(i)}$. See Feller (1971) for more information on domains of attraction and also Samorodnitsky and Taqqu (1994) for propoerties of stable laws.

Meerschaert and Scheffler (1999b) show that, for an IID sequence of random vectors $\{\mathbf{Z}_n\}$ on $\mathbb{R}^d$ whose common distribution μ has regularly varying tails with exponent $\mathbf{E}$, where every eigenvalue of $\mathbf{E}$ has real part exceeding $\frac{1}{4}$, we have

$$A_n\left(\sum_{i=1}^{n} Z_i Z_i' - B_n\right) A_n^* \xrightarrow{\mathrm{D}} W \tag{7}$$

where $\mathbf{A}_n$ is taken from the definition (6) above, $\mathbf{A}_n^*$ is the transpose of $\mathbf{A}$ defined by $\langle \mathbf{A}x, y\rangle = \langle x, \mathbf{A}^* y\rangle$ for all $x, y \in \mathbb{R}^d$, $\mathbf{B}_n = \mathbf{E}\mathbf{Z}_1\mathbf{Z}_1'\mathbf{I}(\|\mathbf{A}_n\mathbf{Z}_1\| \leqslant 1)$, and $\mathbf{W}$ is a nonnormal operator stable random element of the vector space $\mathscr{M}_s^2$ of symmetric 2×2 matrices with real entries. If every eigenvalue of $\mathbf{E}$ has real part exceeding $\frac{1}{2}$ then we may take $\mathbf{B}_n = 0$, and if every eigenvalue of $\mathbf{E}$ has real part between $\frac{1}{4}$ and $\frac{1}{2}$ then we may take $\mathbf{B}_n = \mathbf{E}\mathbf{Z}_i\mathbf{Z}_i'$.

In the following result, we relax the assumption that $\mathbf{A}_n$, $\mathbf{C}_j$ are diagonal matrices. Instead, we assume only that they commute in general. It extends Theorem 2.1 in Davis and Marengo (1990) and Theorem 3.1 of Davis *et al.* (1985) to the case where the tails of the two time series are regularly varying with possibly different indices, i.e., we allow $\alpha_1 \neq \alpha_2$.

THEOREM 2.1. *Suppose* $\{\mathbf{X}_t\}$ *is the moving average (5) and* $\hat{\mathbf{\Gamma}}(h)$ *is the sample covariance matrix at lag h. Suppose that* $\{\mathbf{Z}_n\}$ *are IID with common distribution* μ *which varies regularly with exponent* $\mathbf{E}$*, where every eigenvalue of* $\mathbf{E}$ *has real part exceeding* $\frac{1}{4}$*. If* $\mathbf{A}_n\mathbf{C}_j = \mathbf{C}_j\mathbf{A}_n$ *for all n, j then for all* h_0 *we have*

$$n\mathbf{A}_n\left[\hat{\Gamma}(h) - \sum_{j=0}^{\infty} \mathbf{C}_j\mathbf{B}_n\mathbf{C}_{j+h}^*\right]\mathbf{A}_n^* \xrightarrow{\mathrm{D}} \sum_{j=0}^{\infty} \mathbf{C}_j\mathbf{W}\mathbf{C}_{j+h}^* \tag{8}$$

jointly in $h = 0, \ldots, h_0$.

PROOF. This convergence for a single lag h follows from Theorem 3 of Meerschaert and Scheffler (2000). The joint convergence can be obtained by a simple extension of those arguments. The centering $\mathbf{B}_n$ is the same as in (7), in particular we can omit the centering when every $a_i > \frac{1}{2}$ and if every $a_i < \frac{1}{2}$ then the centering is the covariance matrix $\mathbf{\Gamma}(h) = \mathbf{E}\mathbf{X}_t\mathbf{X}_{t+h}'$.

Now we apply Theorem 2.1 to the case at hand. Since $\mathbf{A}_n$ and $\mathbf{C}_j$ are diagonal, they commute. The limiting matrix in (8) has ij entry

$$l_{ij}(h) = \sum_{k=0}^{\infty} c_k^{(i)} c_{k+h}^{(j)} w_{ij} \tag{9}$$

where w_{ij} is the ij entry of $\mathbf{W}$. Then we have

$$na_n^{(i)}a_n^{(j)}(\hat{\gamma}_{ij}(h) - b_{ijn}(h)) \xrightarrow{\mathrm{D}} l_{ij}(h) \tag{10}$$

jointly in i, j, h. We can take

$$b_{ijn}(h) = 0 \qquad \text{if } a_i + a_j > 1$$

and

$$b_{ijn}(h) = \gamma_{ij}(h) = \mathbf{E}x_t^{(i)}x_{t+h}^{(j)} \qquad \text{if } a_i + a_j < 1$$

To see this, project (7) onto the ij coordinate and apply the standard results on centering for scalar domains of attraction; see for example Feller (1971, XVII.5).

Next, we consider the joint distribution of the matrix elements w_{ij}. Since $\mathbf{W}$ is operator stable, we begin with two simple lemmas on operator stable laws. The first of these may be well known but we could not locate a suitable reference. The second corrects a mistake in Corollary 4.13.12 of Zurek and Mason (1993).

LEMMA 2.2. *If* $\mathbf{Y}$ *is operator stable with exponent* $\mathbf{E}$ *and if* $\mathbf{E}^*\theta = a\theta$ *for some* $\theta \neq 0$ *then* $\langle \mathbf{Y}, \theta \rangle$ *is a stable random variable with index* $1/a$.

PROOF. Since $\mathbf{E}^*\theta = a\theta$ it follows that $n^{-\mathbf{E}^*}\theta = n^{-a}\theta$. Now for $\mathbf{Y}_n$ IID with $\mathbf{Y}$, we have for all n, for some b_n, that $\mathbf{Y} \sim n^{-\mathbf{E}}(\mathbf{Y}_1 + \cdots + \mathbf{Y}_n - b_n)$ meaning that both sides are identically distributed. But then

$$\begin{aligned}
\langle \mathbf{Y}, \theta \rangle &\sim \langle n^{-\mathbf{E}}(\mathbf{Y}_1 + \cdots + \mathbf{Y}_n - b_n), \theta \rangle \\
&= \langle \mathbf{Y}_1 + \cdots + \mathbf{Y}_n - b_n, n^{-\mathbf{E}^*}\theta \rangle \\
&= \langle \mathbf{Y}_1 + \cdots + \mathbf{Y}_n - b_n, n^{-a}\theta \rangle \\
&= n^{-a}\langle \mathbf{Y}_1 + \cdots + \mathbf{Y}_n - b_n, \theta \rangle \\
&= n^{-a}(\langle \mathbf{Y}_1, \theta \rangle + \cdots + \langle \mathbf{Y}_n, \theta \rangle - \langle b_n, \theta \rangle)
\end{aligned}$$

which concludes the proof.

LEMMA 2.3. *Suppose that* $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_k)'$ *is full nonnormal operator stable with exponent* $\mathbf{E}$. *The coordinate axis marginals* Y_i *are independent nondegenerate stable with index* $1/a_i$ *if and only if* $\mathbf{E} = \mathrm{diag}(a_1, \ldots, a_k)$ *and the Lévy measure* ϕ *is concentrated on the coordinate axes.*

PROOF. If ϕ is concentrated on the coordinate axes write $\phi(dx) = \phi_1(dx) + \cdots + \phi_k(dx)$ where ϕ_j is concentrated on the jth coordinate axis. Let $\pi_j(x) = x_j$ and $\tilde{\phi}_j = \pi_j\phi_j$ so that $\tilde{\phi}_j$ is a Lévy measure on $\mathbb{R}^1$, and $\pi_j\phi_k = 0$ for $j \neq k$. Since $\mathbf{Y}$ is full, $\mathbf{Y}_i$ is nondegenerate. If $\mathbf{E} = \mathrm{diag}(a_1, \ldots, a_k)$ then $\pi_j t^{\mathbf{E}} = t^{a_j}\pi_j$. Then for all $t > 0$ we have

$$t\phi = t^{\mathbf{E}}\phi$$

$$\pi_j t\phi = \pi_j t^{\mathbf{E}}\phi$$

$$t\pi_j(\phi_1 + \cdots + \phi_k) = t^{a_j}\pi_j(\phi_1 + \cdots + \phi_k)$$

$$t\tilde{\phi}_j = t^{a_j}\tilde{\phi}_j$$

so that $\tilde{\phi}_j$ is the Lévy measure of a stable law with index $1/a_j$. The log characteristic function of $\mathbf{Y}$ is

$$\begin{aligned}\psi(t) &= i\langle a,\ t\rangle + \int_{x\neq 0} \mathrm{e}^{i\langle x,t\rangle} - 1 - \frac{i\langle x,\ t\rangle}{1+\|x\|^2}\phi(\mathrm{d}x)\\ &= \sum_{j=1}^{k}\left(ia_jt_j + \int_{x\neq 0} \mathrm{e}^{i\langle x,t\rangle} - 1 - \frac{i\langle x,\ t\rangle}{1+\|x\|^2}\phi_j(\mathrm{d}x)\right)\\ &= \sum_{j=1}^{k}\left(ia_jt_j + \int_{x_j\neq 0} \mathrm{e}^{ix_jt_j} - 1 - \frac{ix_jt_j}{1+x_j^2}\tilde{\phi}_j(\mathrm{d}x_j)\right)\end{aligned}$$

which shows that $\mathbf{Y}$ has independent stable marginals $\mathbf{Y}_i$ with index $1/a_i$. Conversely if $\mathbf{Y}_i$ are independent stable laws with index $1/a_i$ let $\tilde{\phi}_j$ denote the Lévy measure of $\mathbf{Y}_i$ as before. Since the log characteristic function of $\mathbf{Y}$ is a sum by virtue of independence, we can reverse the steps above to see that $\mathbf{Y}$ is nonnormal and infinitely divisible with Lévy measure ϕ concentrated on the coordinate axes. Furthermore, since ϕ is not concentrated on any proper subspace of $\mathbb{R}^k$ it follows that $\mathbf{Y}$ has a full distribution. Since $t\tilde{\phi}_j = t^{a_j}\tilde{\phi}_j$ for $j = 1, \ldots, k$ we also have $t\phi = t^{\mathbf{E}}\phi$ where $\mathbf{E} = \mathrm{diag}(a_1, \ldots, a_k)$, hence $\mathbf{Y}$ is nonnormal operator stable with this exponent.

THEOREM 2.4. *Suppose $\{Z_n\}$ are IID with common distribution μ which varies regularly with exponent $\mathbf{E} = \mathrm{diag}(a_1, a_2)$, where $a_i > \frac{1}{4}$. Then the matrix entries w_{ij} in the limit $\mathbf{W}$ of (7) are all stable with index $1/(a_i + a_j)$. Both w_{11}, w_{22} are nondegenerate, and $w_{12} = w_{21}$ is degenerate if and only if the limit measure ϕ in (6) is concentrated on the coordinate axes, in which case w_{11}, w_{22} are independent.*

PROOF. We show in Meerschaert and Scheffler (1999b) that the random matrix $\mathbf{W}$ is operator stable on the vector space $\mathcal{M}_s^2$ of symmetric 2×2 matrices, with an exponent ξ which is a linear operator on $\mathcal{M}_s^2$ defined by $\xi(\mathbf{M}) = \mathbf{EM} + \mathbf{ME}$ for any $\mathbf{M} \in \mathcal{M}_s^2$. Let $\mathbf{E}_{ij}$ denote the symmetric matrix with 1 in both the ij and ji position and zeroes elsewhere. It is easy to check that $\xi(\mathbf{E}_{ij}) = (a_i + a_j)\mathbf{E}_{ij}$, and hence by Lemma 2.2 we have w_{ij} stable with index $1/(a_i + a_j)$, but possibly degenerate. The Lévy measure of $\mathbf{W}$ is $T\phi$ where $T\mathbf{x} = \mathbf{xx}'$ and ϕ is the limit measure in (6), and we know from the proof of

Theorem 2 in Meerschaert and Scheffler (1999b) that $\text{supp}(T\phi) = T\,\text{supp}(\phi)$. If for example w_{11} is degenerate then $T\phi$ is supported on the subspace perpendicular to E_{11} in $\mathscr{M}_s^2$, where we use the standard inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i,j} \mathbf{A}_{ij} \mathbf{B}_{ij}$. Since

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix} \tag{11}$$

this means that $x_1^2 = 0$ for any $(x_1, x_2)' \in \text{supp}(\phi)$. But then ϕ is concentrated on the x_2-axis which is a contradiction. Thus w_{ii} is nondegenerate. Now if ϕ is concentrated on the coordinate axes in $\mathbb{R}^2$ then

$$\begin{aligned} \text{supp}(T\phi) &\subset \left\{ T\begin{pmatrix} x_1 \\ 0 \end{pmatrix} : \ x_1 \neq 0 \right\} \cup \left\{ T\begin{pmatrix} 0 \\ x_2 \end{pmatrix} : \ x_2 \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} x_1^2 & 0 \\ 0 & 0 \end{pmatrix} : \ x_1 \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x_2^2 \end{pmatrix} : \ x_2 \neq 0 \right\} \\ &\subset \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : \ u, v > 0 \right\} \end{aligned}$$

and this set is perpendicular to $\mathbf{E}_{12}$, hence w_{12} is degenerate. Also since the Lévy measure $T\phi$ is concentrated on the coordinate axes, w_{11}, w_{22} are independent by Lemma 2.4. Conversely if w_{12} is degenerate then $T\phi$ is supported on the subspace perpendicular to E_{12}. Then if $(x_1, x_2)' \in \text{supp}(\phi)$ we must have $x_1 x_2 = 0$ in (11), showing that ϕ is concentrated on the coordinate axes in $\mathbb{R}^2$.

REMARK 1. If every $a_i > \frac{1}{2}$ then (6) is necessary and sufficient for Z_t to belong to the generalized domain of attraction of some $\mathbf{Y}$ nonnormal operator stable with exponent $\mathbf{E}$ and Lévy measure ϕ. This means that

$$A_n(Z_1 + \cdots + Z_n - s_n) \xrightarrow{\text{D}} Y \tag{12}$$

for some $s_n \in \mathbb{R}^2$. If ϕ is concentrated on the coordinate axes and $\mathbf{E} = \text{diag}(a_1, a_2)$ then by Lemma 2.3 the marginals of $\mathbf{Y}$ are independent stable laws, i.e. $z_t^{(1)}$, $z_t^{(2)}$ are *asymptotically independent*. Then it follows from Theorem 3.2 and Remark 3.3 of Meerschaert and Scheffler (2000) that $x_t^{(1)}$, $x_t^{(2)}$ are also asymptotically independent.

REMARK 2. The papers Davis *et al.* (1985) and Davis and Marengs (1990) assume $a_1 = a_2 = 1/\alpha$ so that each w_{ij} is stable with index $\alpha/2$. Those papers use a very different method of proof, involving point processes. Proposition 3.1 of Resnick (1986) shows that the regular variation (6) is necessary and sufficient for weak convergence of the point processes

$$\sum_{j=1}^{\infty} \varepsilon_{(j/n, A_n Z_j)}$$

to a Poisson random measure on $(0, \infty) \times \mathbb{R}^d$ with mean measure $m \times \phi$ where m is Lebesgue measure on the positive real line. It may also be possible to obtain Theorem 2.4 via point process methods, but we have not pursued this.

3. LIMIT THEOREMS FOR THE SAMPLE CROSS-CORRELATION

Now we come to the main results of this paper, in which we derive the asymptotic distribution of the sample cross-correlation (2) between two scalar time series. We begin with the case where $a_i > \frac{1}{2}$ so that $\mathbf{E}|z_t^{(i)}|^2 = \infty$. In this case, the cross-correlation

$$\rho_{12}(h) = \frac{\gamma_{12}(h)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} \tag{13}$$

is undefined. The following result shows that, even in this case, the sample cross-correlation provides some useful information.

THEOREM 3.1. *Suppose $x_t^{(i)}$ are scalar moving averages as in (1) where the innovations vectors $(z_t^{(1)}, z_t^{(2)})'$ are IID with common distribution μ which varies regularly with exponent $\mathbf{E} = \operatorname{diag}(a_1, a_2)$, where $a_i > \frac{1}{2}$. Then*

$$\hat{\rho}_{12}(h) \xrightarrow{\mathrm{D}} \frac{l_{12}(h)}{\sqrt{l_{11}(0) l_{22}(0)}} \tag{14}$$

where $l_{ij}(h)$ is defined in (9).

PROOF. Apply the continuous mapping theorem along with (10) to obtain

$$\hat{\rho}_{12}(h) = \frac{n a_n^{(1)} a_n^{(2)} \hat{\gamma}_{12}(h)}{\sqrt{n(a_n^{(1)})^2 \hat{\gamma}_{11}(0) n(a_n^{(2)})^2 \hat{\gamma}_{22}(0)}} \xrightarrow{\mathrm{D}} \frac{l_{12}(h)}{\sqrt{l_{11}(0) l_{22}(0)}}$$

and note that $f(x, y, z) = x/\sqrt{yz}$ is continuous at $(x, y, z) = (l_{12}(h), l_{11}(0), l_{22}(0))$ with probability one by Theorem 2.4 since $(l_{11}(0), l_{22}(0))$ is full and operator stable and hence it has a density. Note also that $l_{ii}(0)$ is stable with index $1/(2a_i) < 1$ and skewness 1, so the expression under the square root is almost surely positive.

REMARK 3. If $z_t^{(1)}, z_t^{(2)}$ are asymptotically independent then so are $x_t^{(1)}, x_t^{(2)}$ in view of Remark 2. Then Theorem 2.4 shows that $w_{12} = 0$ almost surely (in this case no centering is required in (7) and we obtain a limit $\mathbf{W}$ which is centered at zero) and so $l_{12}(h) = 0$ almost surely. Then (14) yields $\hat{\rho}_{12}(h) \to 0$ in probability. Otherwise $l_{12}(h)$ is nondegenerate stable, hence it has a density, and so the limit in (14) is almost surely nonzero. Thus the sample cross-correlation indicates asymptotic independence even when the cross-correlation is undefined.

THEOREM 3.2. *Suppose* $x_t^{(i)}$ *are scalar moving averages as in (1) where the innovations vectors* $(z_t^{(1)}, z_t^{(2)})'$ *are IID with common distribution* μ *which varies regularly with exponent* $\mathbf{E} = \text{diag}(a_1, a_2)$*, where* $\frac{1}{4} < a_1 < a_2 < \frac{1}{2}$*. Then*

$$n(a_n^{(2)})^2(\hat{\rho}_{12}(h) - \rho_{12}(h)) \Rightarrow -\frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{22}(0)} l_{22}(0) \tag{15}$$

where $l_{ij}(h)$ *is defined in (9).*

PROOF. Define $\mathbf{v} = (\hat{\gamma}_{12}(h), \hat{\gamma}_{11}(0), \hat{\gamma}_{22}(0))'$ and $\mathbf{v}_0 = (\gamma_{12}(h), \gamma_{11}(0), \gamma_{22}(0))'$ and let $f(x, y, z) = x(yz)^{-1/2}$. Since $a_1 < a_2$ and $a_n^{(i)}$ varies regularly with index $-a_i$ we have $a_n^{(i)} a_n^{(j)}/(a_n^{(2)})^2 \to 0$ unless $i = j = 2$. Then it follows from (10) that

$$n(a_n^{(2)})^2(\mathbf{v} - \mathbf{v}_0) \xrightarrow{\text{D}} (0, 0, l_{22}(0))'$$

Since

$$Df(\mathbf{v}_0) = \left(\frac{\rho_{12}(h)}{\gamma_{12}(h)}, -\frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{11}(0)}, -\frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{22}(0)}\right)'$$

we have

$$\begin{aligned} n(a_n^{(2)})^2(\hat{\rho}_{12}(h) - \rho_{12}(h)) &= n(a_n^{(2)})^2(f(\mathbf{v}) - f(\mathbf{v}_0)) \\ &= Df(\mathbf{v}_0) n(a_n^{(2)})^2(\mathbf{v} - \mathbf{v}_0) + o_P(1) \\ &\xrightarrow{\text{D}} -\frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{22}(0)} l_{22}(0) \end{aligned}$$

which completes the proof.

REMARK 4. Since $a_n^{(2)}$ varies regularly with index $-a_2$ the norming sequence $n(a_n^{(2)})^2$ varies regularly with index $1 - 2a_2 > 0$ and hence this sequence tends to infinity, which agrees with the fact that $\hat{\rho}_{12}(h) \to \rho_{12}(h)$ in probability. If $\frac{1}{4} < a_1 = a_2 < \frac{1}{2}$ and $a_n = a_n^{(i)}$ for $i = 1, 2$ then an argument similar to that of Theorem 3.2 yields

$$na_n^2(\hat{\rho}_{12}(h) - \rho_{12}(h)) \Rightarrow \frac{\rho_{12}(h)}{\gamma_{12}(h)} l_{12}(h) - \frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{11}(0)} l_{11}(0) - \frac{1}{2}\frac{\rho_{12}(h)}{\gamma_{22}(0)} l_{22}(0) \tag{16}$$

which agrees with the result in (Davis and Marengo, 1990). If $z_t^{(1)}$, $z_t^{(2)}$ are independent then $\rho_{12}(h) = 0$ and $n^{1/2}\hat{\rho}_{12}(h) \xrightarrow{\text{D}} N$ normal with mean zero; see, for example, Brockwell and Davis (1991, Theorem 11.2.2). In this case, the weak limit in (15) or (16) is zero, and the norming sequence tends to infinity slower than $n^{1/2}$, so there is no contradiction.

THEOREM 3.3. *Suppose $x_t^{(i)}$ are scalar moving averages as in (1) where the innovations vectors $(z_t^{(1)}, z_t^{(2)})'$ are IID with common distribution μ which varies regularly with exponent* $\mathbf{E} = \mathrm{diag}(a_1, a_2)$*, where $\frac{1}{4} < a_1 < \frac{1}{2} < a_2$. If $a_1 + a_2 < 1$ then*

$$\frac{\hat{\rho}_{12}(h)}{n^{1/2} a_n^{(2)}} \xrightarrow{\mathrm{D}} \frac{\gamma_{12}(h)}{\sqrt{\gamma_{11}(0) l_{22}(0)}} \tag{17}$$

and if $a_1 + a_2 > 1$ then

$$n^{1/2} a_n^{(1)} \hat{\rho}_{12}(h) \xrightarrow{\mathrm{D}} \frac{l_{12}(h)}{\sqrt{\gamma_{11}(0) l_{22}(0)}} \tag{18}$$

where $l_{ij}(h)$ is defined in (9).

PROOF. Since $2a_1 < 1$ we know that $\gamma_{11}(0)$ exists and (10) implies that $\hat{\gamma}_{11}(0) \xrightarrow{\mathrm{P}} \gamma_{11}(0)$. Since $2a_2 > 1$ we know that $\gamma_{22}(0)$ is undefined, and in this case (10) yields $n(a_n^{(2)})^2 \hat{\gamma}_{22}(0) \Rightarrow l_{22}(0)$. Now if $a_1 + a_2 < 1$ then $\gamma_{12}(h)$ exists and $\hat{\gamma}_{12}(h) \xrightarrow{\mathrm{P}} \gamma_{12}(h)$, so continuous mapping yields

$$\frac{\hat{\rho}_{12}(h)}{n^{1/2} a_n^{(2)}} = \frac{\hat{\gamma}_{12}(h)}{\sqrt{\hat{\gamma}_{11}(0) n (a_n^{(2)})^2 \hat{\gamma}_{22}(0)}} \xrightarrow{\mathrm{D}} \frac{\gamma_{12}(h)}{\sqrt{\gamma_{11}(0) l_{22}(0)}}$$

as desired. If $a_1 + a_2 > 1$ then $\gamma_{12}(h)$ is undefined and $n a_n^{(1)} a_n^{(2)} \hat{\gamma}_{12}(0) \Rightarrow l_{12}(h)$, so

$$n^{1/2} a_n^{(1)} \hat{\rho}_{12}(h) = \frac{n a_n^{(1)} a_n^{(2)} \hat{\gamma}_{12}(h)}{\sqrt{\hat{\gamma}_{11}(0) n (a_n^{(2)})^2 \hat{\gamma}_{22}(0)}} \xrightarrow{\mathrm{D}} \frac{l_{12}(h)}{\sqrt{\gamma_{11}(0) l_{22}(0)}}$$

by another application of the continuous mapping theorem. Note that $l_{22}(0)$ is stable with index $1/(2a_2) < 1$ and skewness 1, so the denominator in the limit is almost surely positive.

REMARK 5. The norming sequence $n^{1/2} a_n^{(i)}$ varies regularly with index $\frac{1}{2} - a_i$. Since $\frac{1}{2} < a_2$, we have $n^{1/2} a_n^{(2)} \to \infty$ and since $a_1 < \frac{1}{2}$ we have $n^{1/2} a_n^{(1)} \to 0$. Define $\beta_{ij} = \mathbf{E} z_t^{(i)} z_t^{(j)}$ if $a_i + a_j < 1$ so that this mean exists, and $\beta_{ij} = w_{ij}$ if $a_i + a_j > 1$, so that $\mathbf{E} z_t^{(i)} z_t^{(j)}$ does not exist. Then each of the limits in (14), (17), and (18) can be written in the form

$$\frac{\beta_{12} \sum_{k=0}^{\infty} c_k^{(1)} c_{k+h}^{(2)}}{\beta_{11} \sum_{k=0}^{\infty} (c_k^{(1)})^2 \beta_{22} \sum_{k=0}^{\infty} (c_k^{(2)})^2}$$

which is also equal to $\rho_{12}(h)$ if $a_i < \frac{1}{2}$, in which case $\hat{\rho}_{12}(h)$ converges in probability to this limit.

4. DISCUSSION

The time series models treated in this paper are linear and stationary. In some applications, it is more natural to employ periodically stationary time series models; see, for example, the river flow model in Anderson and Meerschaert (1998). There is also mounting evidence that, at least in finance, the most promising time series models are nonlinear; see, for example, McCulloch (1996) and the article by Resnick in (Adler *et al.*, 1998). It would be quite interesting to explore the asymptotics of the sample cross-correlation for nonstationary or nonlinear time series.

REFERENCES

ADLER, R., FELDMAN, R. and TAQQU, M. (1998) *A Practical Guide to Heavy Tails*. Boston: Birkhauser.

ANDERSON, P. and MEERSCHAERT, M. (1998) Modeling river flows with heavy tails. *Water Resour. Res.* 34, 2271–80.

BINGHAM, N., GOLDIE, C. and TEUGELS, J. (1987) *Regular variation, Encyclopedia of Mathematics and its Applications*, 27, Cambridge University Press.

BROCKWELL, P. and DAVIS, R. (1991) *Time Series: Theory and Methods* (2nd edn). New York: Springer-Verlag.

DAVIS, R. and MARENGO, J. (1990) Limit theory for the sample covariance and correlation matrix functions of a class of multivariate linear processes. *Commun. Statist. Stochastic Models* 6, 483–97.

DAVIS, R., MARENGO, J. and RESNICK, S. (1985) Extremal properties of a class of multivariate moving averages, *Proceedings of the 45th International Statistical Institute*, Vol. 4, Amsterdam.

FAMA, E. (1965) The behavior of stock market prices. *J. Business* 38, 34–105.

FELLER, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. 2, (2nd edn). New York: Wiley.

HOSKING, J. and WALLIS, J. (1987) Parameter and quantile estimation for the generalized Pareto distribution. *Technometrics* 29, 339–49.

JANSEN, D. and DE VRIES, C. (1991) On the frequency of large stock market returns: Putting booms and busts into perspective. *Review of Econ. and Statist.* 23, 18–24.

JUREK, Z. and MASON, J. D. (1993) *Operator-Limit Distributions in Probability Theory.* New York: Wiley.

LORETAN, M. and PHILLIPS, P. (1994) Testing the covariance stationarity of heavy tailed time series. *J. Empirical Finance* 1, 211–48.

MANDELBROT, B. (1963) The variation of certain speculative prices. *J. Business* 36, 394–419.

MCCULLOCH, J. (1996) Financial applications of stable distributions. In *Statistical Methods in Finance: Handbook of Statistics* 14 (eds G. S. Madfala and C. R. Rao). Amsterdam: Elsevier, 393–425.

MEERSCHAERT, M. (1991) Spectral decomposition for generalized domains of attraction. *Ann. Probab.* 19, 875–92.

—— (1993) Regular variation and generalized domains of attraction in $\mathbb{R}^k$. *Statis. Probab. Lett.* 8, 233–9.

MEERSCHAERT, M. and SCHEFFLER, H. P. (1999a) Multivariable regular variation of functions and measures. *J. Applied Anal.* 5, 125–46.

—— and —— (1999b) Sample covariance matrix for random vectors with heavy tails. *J. Theoret. Probab.* 12, 821–38.

—— and —— (2000) Moving averages of random vectors with regularly varying tails. *J. Time Series Anal.* 23, 297–328.

MITTNIK, S. and RACHEV, S. (2000) *Stable Paretian Markets in Finance*. New York: Wiley.

NIKIAS, C. and SHAO, M. (1995) *Signal Processing with Alpha Stable Distributions and Applications*. New York: Wiley.

NOLAN, J., PANORSKA, A. and MCCULLOCH, J. H. (1994) Estimation of stable spectral measures, unpublished paper.

RESNICK, S. (1986) Point processes, regular variation and weak convergence. *Adv. Appl. Prob.* 18, 66–138.

SAMORODNITSKY, G. and TAQQU, M. (1994) *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. London: Chapman and Hall.

SENETA, E. (1976) *Regularly varying functions*, *Lectures Notes in Mathematics*, 508, Berlin: Springer.