Strong completeness for non-compact hybrid logics

Barteld Kooi*
Gerard Renardel de Lavalette†
Rineke Verbrugge‡

University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands

Abstract

We provide a strongly complete infinitary proof system for hybrid logic. This proof system can be extended with countably many sequents. Thus completeness proofs are provided for infinitary hybrid versions of non-compact logics like ancestral logic and Segerberg's modal logic with the bounded chain condition. This extends the completeness result for hybrid logics by Blackburn and Tzakova.

Keywords: hybrid logic, strong completeness, non-compact logics, infinitary proof rules

1 Introduction

Hybrid logic is an extension of modal logic. Special propositional variables called *nominals*, which are true in exactly one possible world, are added to the language. Therefore they could equally be taken as names of possible worlds. Hybrid logic was initially developed by Prior in the 1960's [6], but there has been a flurry of activity surrounding hybrid logic in the past decade (see `www.hylo.net`). A textbook introduction to hybrid logic can be found in [1].

One of the pleasant features of hybrid logic is that its correspondence theory is very straightforward. Hybrid logic can be translated into first-order logic, where nominals are interpreted as constants. The link is so strong that it is very easy to obtain complete proof systems for classes of frames that satisfy additional properties. This works not only for the usual properties such as transitivity, reflexivity, and symmetry, but also for irreflexivity, asymmetry, and many others that cannot be characterized by

*Department of Philosophy, `barteld@philos.rug.nl`
†Department of Computing Science, `grl@cs.rug.nl`
‡Department of Artificial Intelligence, `rineke@ai.rug.nl`

modal formulas. The completeness theorem which is proved in [1] (which is a variation of a theorem from [2] by Blackburn and Tzakova) provides complete proof systems for many hybrid logics. It exploits the straightforward correspondence theory. When the base proof system is extended with *pure* axioms (axioms without propositional variables), then the new proof system is automatically complete for the class of corresponding frames.

One would like to prove strong completeness also for logics where the relevant properties are not characterized by axioms, but by infinitary rules, such as the following rule, which characterizes the frame property that any state is reachable from any other state by a (finite) sequence of moves along the accessibility relation:

$$\{\neg @_{\mathsf{i}} \Diamond^n \mathsf{j} \mid n \in \mathbb{N}\} \vdash \bot$$

Although the completeness proof in [1] is very general, it is not applicable to non-compact modal logics, such as propositional dynamic logic PDL, ancestral logic, other modal logics with (reflexive) transitive closure operators, and the 'reachability logic' given by the infinitary rule above.

Let us remind the reader of some relevant definitions. *Strong completeness* (also called extended completeness) with respect to a class of frames S is the following property of a modal logical system S:

$$\Gamma \models_S \varphi \text{ implies } \Gamma \vdash_{\mathsf{S}} \varphi, \text{ for all formulas } \varphi \text{ and all sets of formulas } \Gamma.$$

This generalizes weak completeness, where Γ is empty. Observe that weak completeness implies strong completeness whenever the logic in question is *semantically compact*, i.e. when $\Gamma \models_S \varphi$ implies that there is a finite $\Gamma' \subseteq \Gamma$ with $\Gamma' \models_S \varphi$, hence $\models_S \bigwedge \Gamma' \to \varphi$. This is, for example, the case in modal logics such as K and S5.

Propositional dynamic logic is a well-known example of a non-compact logic: we have for the relevant class of frames S, that $\{[a^n]p \mid n \in \mathbb{N}\} \models_S [a^*]p$ but there is no natural number k with $\{[a^n]p \mid n \leq k\} \models_S [a^*]p$. As a consequence, we do not have strong completeness for any finitary axiomatization, *a fortiori* not for its usual, weakly complete proof system. So strong completeness requires an infinitary proof system. Here, "infinitary" does not refer to the language (all formulas in this paper have finite length), but to the derivation relation (proof sequents may be non-standard in requiring infinitely many premises).

Infinitary non-hybrid versions of such non-compact modal logics were investigated and strong completeness proofs were given by Goldblatt [3], Segerberg [8] and the present authors [7]. In those cases a strongly complete infinitary proof system can be obtained by adding infinitary rules: one simply makes a rule from an example that shows non-compactness. In [5], Passy and Tinchev investigate hybrid versions of PDL and present an infinitary proof system, which is shown to be strongly complete.

The main goal of this paper is to extend Blackburn and Tzakova's strong completeness result for hybrid logic to hybrid systems that are not semantically compact. Rather than extending the base system with pure axioms, as they do, we allow extensions with countably many pure *sequents*, each with possibly infinitely many premises. We will first prove a general result about a basic hybrid system extended with a countable set of sequents; it turns out that the completeness proof looks considerably different from the usual one. Especially Blackburn's and Tzakova's version of the Lindenbaum Lemma cannot be straightforwardly generalized. Then we show some applications to hybrid versions of specific modal logics.

In Section 2 we briefly introduce the language and semantics of hybrid logic. In Section 3 we provide the infinitary proof system for hybrid logic. In Section 4 we show this proof system is complete. In Section 5 we discuss some specific extensions of the basic proof system. Finally, in Section 6 we draw conclusions and indicate directions for further research.

2 Language and semantics

There are some variants of the language of hybrid logic. We take the minimal language of hybrid logic, where the language of modal logic is extended with nominals and *at*-operators @.

Definition 1 (Language of hybrid logic)
Let a countable set of propositional variables P, and a countably infinite set of nominals I be given. The language of hybrid logic $\mathcal{L}(P, \mathsf{I})$ is given by the following BNF:

$$\varphi ::= \bot \mid p \mid \mathsf{i} \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond\varphi \mid @_{\mathsf{i}}\varphi$$

where $p \in P$, and $\mathsf{i} \in \mathsf{I}$. We use the usual abbreviations. We are usually sloppy and write $\mathcal{L}$ instead of $\mathcal{L}(P, \mathsf{I})$. □

Formulas of the form $@_{\mathsf{i}}\varphi$, are to be read as "φ holds at the world named i." The function $\mathsf{nom} : \mathcal{L} \to \wp(\mathsf{I})$ yields the set of nominals that occur in a formula. We generalize this to sets of formulas, and later to proof sequents and proofs.

The models used in the semantics of hybrid logic are simply models for modal logic, where the valuation of a nominal is a singleton set.

Definition 2 (Models for hybrid logic)
A model for $\mathcal{L}$ is a triple $M = (W, R, V)$ such that:

- $W \neq \emptyset$; a set of possible worlds;
- $R \subseteq W \times W$; an accessibility relation;

- $V : P \cup \mathsf{I} \to \wp(W)$; assigns a set of possible worlds to each propositional variable and a singleton to each nominal.

A frame F is a tuple (W, R), where W and R are as above. $\Box$

Definition 3 (Semantics)
Let a model (M, w) where $M = (W, R, V)$ be given. Let $p \in P$, $\mathsf{i} \in \mathsf{I}$, and $\varphi, \psi \in \mathcal{L}$.

$$
\begin{array}{lll}
(M,w) \not\models \bot & & \\
(M,w) \models p & \text{iff} & w \in V(p) \\
(M,w) \models \mathsf{i} & \text{iff} & V(\mathsf{i}) = \{w\} \\
(M,w) \models \neg\varphi & \text{iff} & (M,w) \not\models \varphi \\
(M,w) \models \varphi \vee \psi & \text{iff} & (M,w) \models \varphi \text{ or } (M,w) \models \psi \\
(M,w) \models \Diamond\varphi & \text{iff} & (M,v) \models \varphi \text{ for some } v \text{ such that } (w,v) \in R \\
(M,w) \models @_{\mathsf{i}}\varphi & \text{iff} & (M,v) \models \varphi \text{ where } V(\mathsf{i}) = \{v\}
\end{array}
$$

Given a set of formulas Γ we write $(M, w) \models \Gamma$ iff $(M, w) \models \varphi$ for every $\varphi \in \Gamma$. We write $\Gamma \models \varphi$ iff $(M, w) \models \Gamma$ implies $(M, w) \models \varphi$ for every model M and world w. We write $M \models \varphi$ iff $(M, w) \models \varphi$ for every w. We write $M \models \Gamma/\varphi$ iff $(M, w) \models \Gamma$ implies $(M, w) \models \varphi$ for every world w. Given a frame F and a world w, we say that $(F, w) \models \varphi$ iff $((F, V), w) \models \varphi$ for every valuation V. Likewise for $(F, w) \models \Gamma$, $F \models \varphi$ and $F \models \Gamma/\varphi$. $\Box$

3 The proof system Khyb_ω

The proof system is based on *sequents*, i.e. expressions of the form $\Gamma \vdash \varphi$ where Γ is a (possibly infinite) collection of formulas. For technical reasons, only sequents $\Gamma \vdash \varphi$ are allowed in which infinitely many nominals $\mathsf{i} \in \mathsf{I}$ do not occur, i.e. where $(\mathsf{I} - \mathsf{nom}(\Gamma, \varphi))$ is infinite (recall that I is always infinite). This is a weak restriction: any sequent not satisfying it can be transformed by renaming of nominals into a sequent satisfying the restriction.

The proof system consists of *axiom sequents* and *sequent rules*, and derivability is defined inductively as usual: a sequent is derivable when it is an axiom, or when it is the conclusion of a rule with derivable sequents as premises. Observe that, due to the infinitary cut rule, derivations may contain infinitely many sequents. We write $\Box\Gamma$ for $\{\Box\varphi \mid \varphi \in \Gamma\}$, $@_{\mathsf{i}}\Gamma$ for $\{@_{\mathsf{i}}\varphi \mid \varphi \in \Gamma\}$, and $\Gamma \vdash \Delta$ for ($\Gamma \vdash \varphi$ for every $\varphi \in \Delta$).

Definition 4 (Proof system for hybrid logic)
$\Gamma \vdash \varphi$ is defined by the axiom sequents and sequent rules provided in Figure 1.

The soundness of the proof system ($\Gamma \vdash \varphi$, then $\Gamma \models \varphi$) can be shown by induction on the length of the proof. We do not provide an explicit proof.

Taut	$\vdash \varphi$ if φ is an instance of a propositional tautology	
$\mathbf{K}_\Box$	$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(distribution)
$\mathbf{K}_{@_\mathsf{i}}$	$\vdash @_\mathsf{i}(\varphi \to \psi) \to (@_\mathsf{i}\varphi \to @_\mathsf{i}\psi)$	(distribution)
SD	$\vdash @_\mathsf{i}\varphi \to \neg @_\mathsf{i}\neg\varphi$	(self-dual)
Intr	$\vdash \mathsf{i} \wedge \varphi \to @_\mathsf{i}\varphi$	(introduction)
$\mathbf{T}_{@_\mathsf{i}}$	$\vdash @_\mathsf{i}\mathsf{i}$	(reflexivity)
$\mathbf{B}_{@_\mathsf{i}}$	$\vdash @_\mathsf{i}\mathsf{j} \leftrightarrow @_\mathsf{j}\mathsf{i}$	(symmetry)
Nom	$\vdash @_\mathsf{i}\mathsf{j} \wedge @_\mathsf{j}\varphi \to @_\mathsf{i}\varphi$	(nom)
Agree	$\vdash @_\mathsf{i}@_\mathsf{j}\varphi \leftrightarrow @_\mathsf{j}\varphi$	(agree)
Back	$\vdash \Diamond @_\mathsf{i}\varphi \to @_\mathsf{i}\varphi$	(back)
MP	$\varphi, \varphi \to \psi \vdash \psi$	(modus ponens)
$\mathbf{SNec}_\Box$	if $\Gamma \vdash \varphi$, then $\Box\Gamma \vdash \Box\varphi$	(strong necessitation)
$\mathbf{SNec}_{@_\mathsf{i}}$	if $\Gamma \vdash \varphi$ then $@_\mathsf{i}\Gamma \vdash @_\mathsf{i}\varphi$	(strong necessitation)
InfCut	if $\Gamma \vdash \Delta$ and $\Gamma', \Delta \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$	(infinitary cut)
W	if $\Gamma \vdash \varphi$ then $\Gamma, \Delta \vdash \varphi$	(weakening)
Ded	if $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$	(deduction)
Name	if $\Gamma, \mathsf{i} \vdash \varphi$, then $\Gamma \vdash \varphi$, provided $\mathsf{i} \notin \mathsf{nom}(\Gamma, \varphi)$	(name)
Paste	if $\Gamma, @_\mathsf{i}\Diamond\mathsf{j}, @_\mathsf{j}\varphi \vdash \psi$, then $\Gamma, @_\mathsf{i}\Diamond\varphi \vdash \psi$, provided $\mathsf{j} \notin \mathsf{nom}(\Gamma, \varphi, \psi) \cup \{\mathsf{i}\}$	(paste)

Figure 1: The axiom sequents and sequent rules of Khyb_ω

4 Strong completeness of Khyb_ω plus countably many sequents

Take the basic system Khyb_ω defined above, and add a *denumerable* set of additional axiom sequents:

$$\mathbf{AS} = \{\Gamma_n \vdash \varphi_n \mid n \in \mathbb{N}\}$$

What **AS** contains may depend on the language, i.e. the parameters P and I. E.g., **AS** may contain (or even consist of) sequents generated by substitution from sequent schema's. When we add all instances of a sequent $\Gamma \vdash \varphi$ to **AS** that are obtained by arbitrary substitutions of formulas for propositional variables and nominals for nominals, then countability is only guaranteed if $\Gamma \vdash \varphi$ is parameter-finite, i.e. if $\Gamma \vdash \varphi$ contains only finitely many nominals and propositional variables.

In this section we provide a completeness proof for $\mathsf{Khyb}_\omega + \mathbf{AS}$. For the completeness proof we follow the completeness proofs for the infinitary logic $\mathcal{L}_{\omega_1\omega}$ presented in [4] and hybrid logic presented in [1]. The completeness proof for hybrid logic in [1] is very general and also shows that if their proof system is extended with extra pure axioms, then this extended proof system is automatically strongly complete with respect to the class of frames defined by these pure axioms. However, it is a finitary proof system, and the completeness proof hinges on a Lindenbaum Lemma where compactness is assumed. So if we were to add an infinitary rule to their system, we would not get a complete proof system. Therefore we follow the completeness proof of [4], which does not depend on compactness. Furthermore we show that extensions of Khyb_ω with extra pure axiom sequents with finitely many nominals are also complete for the class of frames defined by these rules.

Theorem 1 (Completeness)
Every $\mathsf{Khyb}_\omega + \mathbf{AS}$-consistent set of formulas in language $\mathcal{L}$ is satisfiable in a countable named model. □

Proof We prove that if $\Gamma \nvdash \bot$ (i.e. $\Gamma \vdash \bot$ is not derivable in $\mathsf{Khyb}_\omega + \mathbf{AS}$), then there is a named model M satisfying **AS** with $(M, w) \models \Gamma$.

Assume Γ is consistent, i.e. $\Gamma \nvdash \bot$. We extend the language $\mathcal{L}$ to $\mathcal{L}^+$ by adding the countable set of new nominals J. (Consequently **AS** may now also be extended.) We claim $\Gamma \nvdash_{\mathcal{L}^+} \bot$. For assume $\Gamma \vdash_{\mathcal{L}^+} \bot$ and let Π^+ be the $\mathcal{L}^+$-derivation of this sequent. We shall show that Π^+ can be transformed into a $\mathcal{L}$-derivation Π of $\Gamma \vdash \bot$, contradicting our assumption. Let

$$c : (\mathsf{nom}(\Pi^+) - \mathsf{nom}(\Gamma)) \to (\mathsf{I} - \mathsf{nom}(\Gamma))$$

be an injection (such a c exists, for $(\mathsf{I} - \mathsf{nom}(\Gamma))$ is infinite, by the definition of sequent). Now replace in Π^+ all nominals $\mathsf{i} \in \mathsf{nom}(\Pi^+) - \mathsf{nom}(\Gamma)$ by $c(\mathsf{i})$: this yields the $\mathcal{L}$-derivation Π.

A collection $\Delta \subseteq \mathcal{L}^+$ is called *admissible* if it is $\mathcal{L}^+$-consistent and $\mathsf{J} - \mathsf{nom}(\Delta)$ is infinite. We shall define an increasing sequence $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_n \subseteq \dots$ of admissible sets containing Γ, and use $\Delta_\omega = \bigcup_n \Delta_n$ to construct the model. Let $\mathsf{i} \in \mathsf{J}$. We put $\Delta_0 = \Gamma \cup \{\mathsf{i}\}$, so Δ_0 is indeed admissible (by Name). Now let $\{\varphi_n \mid n \in \omega\}$ be an enumeration of $\mathcal{L}^+$ where every formula occurs infinitely often. This yields an infinite set of numbers $N_\varphi = \{n \mid \varphi_n = \varphi\}$ for every formula $\varphi \in \mathcal{L}^+$. Let $\{\Theta_n \vdash \psi_n \mid n \in \mathbb{N}\}$ be an enumeration of $@\mathbf{AS} = \{@_\mathsf{j}\Theta \vdash @_\mathsf{j}\psi \mid (\Theta \vdash \psi) \in \mathbf{AS}, \mathsf{j} \in \mathsf{J}\}$; and let $M_\varphi = \{m \mid (\Theta_m \vdash \varphi) \in @\mathbf{AS}\}$ be a (possibly empty) set of numbers for every formula φ. Now we define for each φ an *injective* function $f_\varphi : M_\varphi \to N_\varphi$ such that $f_\varphi(m_k) = n_k$ (i.e. the k-th number in M_φ is mapped to the k-th number in N_φ).

Now we define Δ_{n+1} in terms of Δ_n. We distinguish between $\Delta_n \vdash \neg\varphi_n$ and $\Delta_n \cup \{\varphi_n\}$ consistent.
If $\Delta_n \vdash \neg\varphi_n$, then

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\neg\theta\} & \text{if there is an } m \text{ with } f_{\varphi_n}(m) = n \\ & \text{(i.e. there exists a } (\Theta_m, \varphi_n) \in @\mathbf{AS}\text{)} \\ & \text{and } \theta \text{ is the smallest formula in } \Theta_m \\ & \text{such that } \Delta_n \cup \{\neg\theta\} \text{ is consistent.} \\ \Delta_n & \text{otherwise} \end{cases}$$

We claim that the definition is correct, i.e. that, in the first clause, such a θ can always be found: for if not, then we would have $\Delta_n \vdash \theta$ for all $\theta \in \Theta_m$ and hence $\Delta_n \vdash \varphi_n$: with $\Delta_n \vdash \neg\varphi_n$, this contradicts the consistency of Δ_n.
If $\Delta_n \cup \{\varphi_n\}$ is consistent, then

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\varphi_n, @_\mathsf{j}\mathsf{k}\} & \text{if } \varphi_n = \mathsf{k} \text{ with } \mathsf{k} \notin \mathsf{J} \\ & \text{where } \mathsf{j} \in \mathsf{J} - \mathsf{nom}(\Delta_n) \\ \Delta_n \cup \{\varphi_n, \chi_k\} & \text{if } \varphi_n = \chi_0 \vee \chi_1 \\ & \text{where } k \text{ is the least number in } \{0,1\} \\ & \text{such that } \Delta_n \cup \{\chi_k\} \text{ is consistent} \\ \Delta_n \cup \{\varphi_n, @_\mathsf{k}\Diamond\mathsf{j}, @_\mathsf{j}\psi\} & \text{if } \varphi_n = @_\mathsf{k}\Diamond\psi, \\ & \text{where } \mathsf{j} \in \mathsf{J} - \mathsf{nom}(\Delta_n, \varphi_n) \\ \Delta_n \cup \{\varphi_n\} & \text{otherwise} \end{cases}$$

Again we claim that the definition is correct. For the first clause, this comes down to showing that $\Delta_n \cup \{\mathsf{k}, @_\mathsf{j}\mathsf{k}\}$ is consistent. Assume $\Delta_n, \mathsf{k}, @_\mathsf{j}\mathsf{k} \vdash \bot$, then (by propositional reasoning with the axioms on nominals) $\Delta_n, \mathsf{k}, \mathsf{j} \vdash \bot$, so with the rule **Name** we get $\Delta_n, \mathsf{k} \vdash \bot$, contradicting the consistency of $\Delta_n \cup \{\varphi_n\}$. For the second clause, the reasoning is standard. For the third clause, the reasoning is as for the first, but now we use the rule **Paste**.

We observe that Δ_ω satisfies the following properties:

- Δ_ω is maximal: for all $\varphi \in \mathcal{L}^+$, either $\varphi \in \Delta_\omega$ or $\neg\varphi \in \Delta_\omega$, but not both;

- Δ_ω is finitely $\vdash$-closed: if $\Delta' \subseteq \Delta_\omega$ finite and $\Delta' \vdash \varphi$, then $\varphi \in \Delta_\omega$;
- Δ_ω is closed under all rules $(\Theta, \varphi) \in @\mathbf{AS}$: if $\Theta \subseteq \Delta_\omega$, then $\varphi \in \Delta_\omega$.

We shall use these properties frequently below, without explicit mention.

Before we construct the model from Δ_ω, we define $\sim$ on J:

$$\mathsf{j} \sim \mathsf{k} \text{ iff } @_\mathsf{j}\mathsf{k} \in \Delta_\omega$$

and observe that, by the three axioms on nominals, $\sim$ is an equivalence relation on J. We define

$$[\mathsf{k}] = \{\mathsf{j} \in \mathsf{J} \mid @_\mathsf{j}\mathsf{k} \in \Delta_\omega\}$$

so, for $\mathsf{j} \in \mathsf{J}$, we have $[\mathsf{j}] = [\mathsf{j}]_\sim \in \mathsf{J}/\sim$, the $\sim$-equivalence class of j. By the definition of Δ_{n+1} and the axioms on nominals, we have $[\mathsf{k}] \in \mathsf{J}/\sim$, also if $\mathsf{k} \notin \mathsf{J}$.

Now we construct the model $M = (W, R, V)$ from Δ_ω:

- $W = \{[\mathsf{j}] \mid \mathsf{j} \in \mathsf{J}\}(= \mathsf{J}/\sim)$,
- $R = \{([\mathsf{j}], [\mathsf{k}]) \mid @_\mathsf{j}\Diamond\mathsf{k} \in \Delta_\omega\}$,
- $V(p) = \{[\mathsf{j}] \mid @_\mathsf{j} p \in \Delta_\omega\}$
- $V(\mathsf{j}) = \{[\mathsf{j}]\}$

We claim that for all $\varphi \in \mathcal{L}^+$ and all $\mathsf{j} \in \mathsf{J}$, the following analogue of the Truth Lemma holds:

$$@_\mathsf{j}\varphi \in \Delta_\omega \Leftrightarrow (M, [\mathsf{j}]) \models \varphi$$

This is proved with formula induction. The base cases (for propositional variables and nominals) follow immediately from the definition of the model. The cases for the Boolean connectives follow straightforwardly from the induction hypothesis (for the negation case, we use that Δ_ω is maximal). For the $\Box$ step, we use the property

$$@_\mathsf{j}\Box\varphi \in \Delta_\omega \Rightarrow \forall \mathsf{k} \in \mathsf{J}(@_\mathsf{j}\Diamond\mathsf{k} \in \Delta_\omega \Rightarrow @_\mathsf{k}\varphi \in \Delta_\omega)$$

This property follows from the fact that Δ_ω is finitely $\vdash$-closed. Since $@_\mathsf{j}\Box\varphi, @_\mathsf{j}\Diamond\mathsf{k} \vdash @_\mathsf{j}\Diamond(\mathsf{k}\wedge\varphi)$, and by **Intr**, **Back** and **Agree** in $@_\mathsf{j}\Diamond\mathsf{k}\wedge\varphi \vdash @_\mathsf{k}\varphi$.

As a consequence, we have (by the definition of Δ_0):

$$\varphi \in \Delta_\omega \Leftrightarrow (M, [\mathsf{i}]) \models \varphi$$

Moreover, we have that all $(\Theta, \varphi) \in \mathbf{AS}$ hold in M:

$$\text{if } (M, [\mathsf{j}]) \models \Theta, \text{ then } (M, [\mathsf{j}]) \models \varphi$$

To see this observe that:

$$\begin{array}{rcl}(M,[\mathsf{j}]) \models \Theta & \equiv & \text{for all } \theta \in \Theta, (M,[\mathsf{j}]) \models \theta \\ & \equiv & \text{for all } \theta \in \Theta, @_{\mathsf{j}}\theta \in \Delta_\omega \\ & \Rightarrow & @_{\mathsf{j}}\varphi \in \Delta_\omega \\ & \equiv & (M,[\mathsf{j}]) \models \varphi\end{array}$$

So if $\Gamma \not\vdash \bot$, then there is a named model M satisfying **AS** with $(M,[\mathsf{i}]) \models \Gamma$. This ends the proof □

Although the model which is constructed in this proof satisfies Γ, it is not necessarily the case that the underlying *frame* satisfies the additional sequents, i.e. we do not show canonicity. However if we restrict the additional axiom sequents to those generated by pure sequents (where no propositional variables occur), we do get canonicity, due to the following lemma. Thus, for named models and pure formulas containing only finitely many nominals, truth in a model and validity in a frame coincide

Lemma 1
Let $M = (F, V)$ be a named model and $\Gamma \vdash \varphi$ be a pure sequent. Suppose that for all pure instances $\Delta \vdash \psi$ of $\Gamma \vdash \varphi$, $M \models \Delta$ implies $M \models \psi$. Then $F \models \Gamma/\varphi$, i.e. for all V', w we have that $((F, V'), w) \models \Gamma$ implies $((F, V'), w) \models \varphi$. □

Proof (sketch) $V : \mathsf{I} \to \{\{w\} | w \in W\}$ is surjective, so it has a right inverse $V^{-1} : W \to \mathsf{I}$ with $V(V^{-1}(w)) = \{w\}$. Define the nominal substitution $\sigma : \mathsf{I} \to \mathsf{I}$ by $\sigma(\mathsf{i}) = V^{-1}(V'(\mathsf{i}))$. Now we can prove, with straightforward formula induction, that for all pure formulas θ:

$$((F, V), w) \models \sigma(\theta) \Leftrightarrow ((F, V'), w) \models \theta$$

This implies the lemma. □

5 Application to non-compact modal logics

We provide a number of interesting instances of Theorem 1. These are examples of cases where pure axioms do not suffice to obtain completeness for the relevant class of models, but pure sequents do. Since pure axioms are a special case of pure sequents, these are generalizations of Blackburn and Tzakova's result. Thus, if **AS** contains only pure formulas and finitely many nominals, not only the model provided by Theorem 1, but also the frame underlying it, validates **AS**. In this section by $\mathsf{Khyb}_\omega + \mathbf{AS}$ we mean Khyb_ω extended with all pure instances of **AS**.

5.1 Hybrid ancestral logic

Ancestral logic is the modal logic with two modalities $\Box$ and $\Box^*$, where the accessibility relation associated with the latter is the reflexive transitive closure of the accessibility relation associated with the former. Ancestral logic is non-compact, and in fact a counterexample to compactness gives the inspiration for a suitable hybrid version of this logic, namely Khyb_ω extended with a countable set of pure sequents containing only finitely many nominals. Let $\Box^n\varphi$ stand for φ preceded by n $\Box$-operators.

$$\textbf{AS1} \quad \{@_{\mathsf{i}}\Box^n\neg\mathsf{j} \mid n \in \mathbb{N}\} \vdash @_{\mathsf{i}}\Box^*\neg\mathsf{j}$$

It is clear that **AS1** is valid exactly in those frames in which the accessibility relation of $\Box^*$ is the reflexive transitive closure for the accessibility relation of $\Box$. Thus, by Theorem 1 and Lemma 1, Khyb_ω + **AS1** is strongly complete with respect to such frames.

5.2 Hybrid reachability logic

Let us define Hybrid reachability logic as the hybrid logic given by Khyb_ω + **AS2**, as follows, where $\Diamond^n\varphi$ stands for φ preceded by n $\Diamond$-operators:

$$\textbf{AS2} \quad \{\neg@_{\mathsf{i}}\Diamond^n\mathsf{j} \mid n \in \mathbb{N}\} \vdash \bot$$

It is clear that **AS2** is valid exactly in those frames in which the accessibility relation R is reachable, in the sense that for any two states i, j in the model either $i = j$ or there is a sequence $s_0R \ldots s_n$ where $s_0 = i$ and $s_n = j$, where $n \geq 1$. Thus, by Theorem 1 and Lemma 1, Khyb_ω + **AS2** is strongly complete with respect to reachable frames.

5.3 Hybrid no-cycle logic

Let us define Hybrid no-cycle logic as the hybrid logic given by Khyb_ω + **AS3**, as follows:

$$\textbf{AS3} \quad \{\neg@_{\mathsf{i}}\Diamond^n\mathsf{i} \mid n \in \mathbb{N}, n \geq 1\} \vdash \bot$$

It is clear that **AS3** is valid exactly in those frames in which the accessibility relation R contains no cycles, in the sense that for any state i in the model, there is no sequence $s_0R \ldots s_n$ where $s_0 = i$ and $s_n = i$, where $n \geq 1$. Thus, by Theorem 1 and Lemma 1, Khyb_ω + **AS3** is strongly complete with respect to frames without cycles.

5.4 Hybrid BCC logic

BCC-logic is the logic of the bounded chain condition, as defined in [8]: for all states i, there is a bound $n \in \mathbb{N}$ such that for any j there are only chains $iR \dots j$ of length smaller than n from i to j.

Let us define Hybrid BCC logic as the hybrid logic given by Khyb_ω + **AS4**, the infinitary rule of [8], which turns out to be pure and does not contain any nominals:

$$\textbf{AS4} \quad \{\Diamond^n \top \mid n \in \mathbb{N}\} \vdash \bot$$

Thus, by Theorem 1 and Lemma 1, Khyb_ω + **AS4** is strongly complete with respect to frames with the bounded chain condition.

Note that the BCC-condition is stronger than converse wellfoundedness (no infinite ascending chains), for which we did not find a characterizing countable set of sequents containing only finitely many nominals.

6 Conclusion and further research

In this paper we provided a strongly complete infinitary proof system for hybrid logic. The completeness proof worked in such a way that we immediately got completeness for logics that extend the proof system with countably many axiom sequents. This allowed us to obtain strongly complete proof systems for non-compact hybrid logics. If the additional axiom sequents are pure and contain only finitely many nominals, then we automatically have canonicity.

In the future we hope to attain similar results for hybrid logic with uncountably many pure rules with countably many nominals. This would yield strongly complete proof systems for many more interesting classes of frames.

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