A numerical study of soliton solutions of the Boussinesq equation using spectral methods

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Extended Abstract

Numerical schemes using finite differences in time and spectral methods in space have proved very useful in solving numerically non-linear partial differential equations (PDE) describing wave propagation. Recently, we have solved the Korteweg de Vries (KdV) equation and the generalized KdV equation using such combined schemes and have analyzed efficiently unidirectional solitary wave propagation. We have determined that these waves interact elastically in all cases and have computed detailed stability thresholds in the space of physical parameters of the problem. In particular we have obtained specific velocity values beyond which solitary waves break down due to dynamical and not computational instabilities of the equations.

In this study, we apply a combination of spectral methods and finite differences to another famous non-linear PDE describing water waves: the Boussinesq equation. This equation admits bidirectional wave propagation, has closed form solitary wave solutions and, like the KdV, is completely integrable in one space dimension. These solutions, called solitons, exist in arbitrary number and interact completely elastically. We numerically follow their interactions and investigate their stability properties by varying the velocity parameter of the waves appearing in their analytical form.

Let us consider the famous Boussinesq equation (in one space dimension) written as [1]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \qquad (1)$$

Looking for travelling wave solutions of the form

$$u(x,t) = f(x - x_1 - ct) \qquad (2)$$

we obtain an ordinary differential equation (ODE) which can be easily integrated twice. Setting the two integration constants equal to zero it is easy to show that this ODE has the solution

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$$u(x,t) \,=\, 2\,b^2\,sech^2(b\,(x - x_1 - c\,t)), \tag{3}$$

representing a solitary wave, where $c \,=\, \pm\sqrt{1-4\,b^2}$ is the propagation speed and b, x_1 arbitrary constants determining the height and the position of the maximum height of the wave, respectively. From the form of c is apparent that the solution can propagate in either direction (left or right).

The numerical scheme used in the current study is the same as the one employed in [2, 3] and is based on a combination of finite differences and a Fourier pseudospectral method [4]. In order to demonstrate the application of our algorithm we describe it on the Boussinesq equation (1) with the initial condition $u(x,0)$ given by (3) .

The time derivative in (1) is discretised using a finite difference approximation, in terms of central differences

$$u^{n+1} = 2\,u^n - u^{n-1} + (\Delta t)^2\left(u^n_{xx} - 3(u^2)^n_{xx} - u^n_{xxxx}\right) = 0 \tag{4}$$

According to the pseudospectral method, we introduce the approximate solution

$$u(x,t) = \sum_{k=0}^{N} \alpha_k(t)\Phi_k(x) \tag{5}$$

where $\Phi_k(x) = e^{ikx}$ are the Fourier exponentials, and $\alpha_k(t)$ are coefficients to be determined, for $k = 0, 1, \ldots, N$.

The steps used to advance the solution from time step n to $n+1$ are [4]:

(i): Given $u^n_j = u(x_j, t_n)$ evaluate $\alpha^n_k = \alpha_k(t_n)$ from (5).

(ii): Given α^n_k evaluate the derivatives e.g. $\left[\frac{\partial^2 u}{\partial x^2}\right]^n_j$ from (5).

(iii): Evaluate the nonlinear terms e.g. $u^n_j\left[\frac{\partial u}{\partial x}\right]^n_j$.

(iv): Evaluate u^{n+1}_j from (4), at $x = x_j$, $t = t_{n+1}$.

Step (i) is the transformation from physical space to spectral space. This transformation is achieved by the use of a Fast Fourier Transform (FFT) described in [5, 6] with a number of operations $(5/2)N\log_2 N$ (N being the number of polynomials), in contrast to the $2N^2$ operations required for a matrix–vector multiplication. Step (ii) occurs in spectral space and the evaluation of the nonlinear term in step (iii) is in physical space, thus avoiding the expensive multiplication of all coefficients in the expansion (5). Step (iv) occurs again in physical space.

The accuracy of our numerical scheme for the time variable t is $O((\Delta t)^2)$, due to central differences while for the space variable x, where we use the pseudospectral method, the errors are $O(e^{-qN})$, where q is a constant [4]. Numerical calculations were carried out for various numbers of polynomials $N = 128, 256, 512$ and 1024 and time steps $\Delta t = 0.0001$ to 0.002, while the spatial step was chosen to be $\Delta x = 1$.

Our scope, is to examine numerically how the value of the parameter b in the solitary wave solutions (3) affects their stability under evolution. By the term “stable” we mean that a wave solution, when substituted in an equation, retains its initial profile for arbitrarily long times, albeit with some smaller oscillations present as radiation waves, due to unavoidable numerical errors produced mainly by the finite differences in time.

Thus, in order to check stability, one way is to track the residual of the solution in time. For the case of Boussinesq if the approximate solution (5), computed numerically, is substituted into (4) it will not, of course, give zero. Thus we write for it

$$u_{tt} - u_{xx} + 3\,u^2{}_{xx} + u_{xxxx} = R \tag{6}$$

where R is called the residual of the equation. In our case we evaluate $R = R_i$ at each x_i, $i = 1, ..., N$ grid point at specific time moments t_n.

Due to the fact that the wave solutions are computed for sufficiently large values of N (128 to 1024), the spatial error of the pseudospectral method, is in agreement with the $O(e^{-qN})$ estimate mentioned above, is practically zero. The maximum absolute residual, which we refer to as the error, $E = \max_i |R_i|$, will increase due to the central differencing in time, but cannot be greater than $O((\Delta t)^2)$. Several tests have been made verifying that for various values of N (128 to 1024) and time step $\Delta t = 0.0001$ to 0.02, $E < (\Delta t)^2$ at least for a time period of 1000 time units.

Therefore, a practical way to verify that a wave solution is stable is to check if the error remains, for long times, less than $O((\Delta t)^2)$. If E increases above this value already from the outset, oscillations will soon grow and become unbounded after relatively short times, not because of the numerical scheme, but due to the nonlinear nature of the equations, thus suggesting that the initial wave solution has become unstable. *This is also supported by the fact that blowup occurs nearly at the same times, irrespective of the values of the Δx and Δt step sizes used in the numerical scheme.*

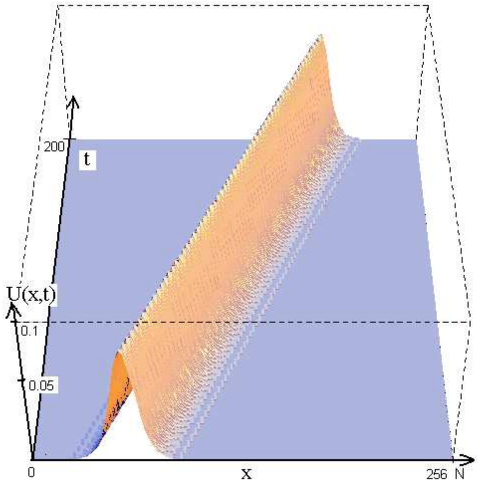

Fig. 1 Propagation of one wave.

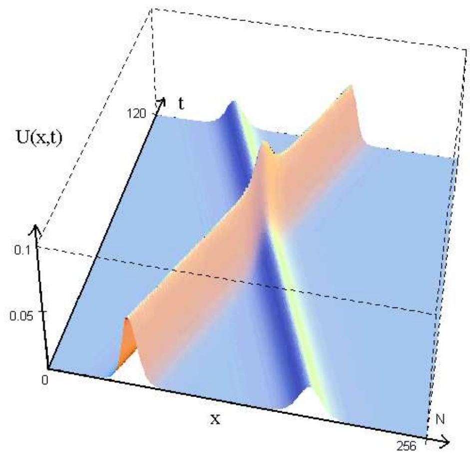

Fig. 2 Two wave interaction.

We begin our investigation by taking as initial condition the solitary wave (3) at $t = 0$ with $\beta = 0.2$, $x_1 = 30$. We observe that our wave moves along the spatial direction retaining its initial profile for a long time period, at least for $t = 2.5 \times 10^6$ time units (see Figure 1). The propagation of the wave is pictured in Figure 1 for $dt = 0.001$ and $N = 128$. As b increases, the stability of the wave propagating in time breaks down and for values of $|b|$ close to 0.5 the wave blows up. For stability, we have found that the maximum value of b is 0.4, i.e. if b exceeds this value the wave blows up after 250 time units.

For a study of multiple wave interaction we consider initially two soliton solutions

$$u_i(x,t) \;=\; 2\,b_i^2\,sech^2(b_i\,(x - x_i - c_i\,t), \quad i = 1, 2 \tag{7}$$

sufficiently far from each other at $t = 0$. Thus we start with

$$u(x,0) = \begin{cases} u_1(x,0)\, for\, [0, x_k] \\ u_2(x,0)\, for\, (x_k, x_N] \end{cases} \tag{8}$$

where x_N is the x corresponding to the Nth element (for $dx = 1$, $x_N \equiv N$), $b_1 = 0.15$, $b_2 = 0.1$, $x_1 = 40$, $x_2 = 180$ and $N = 256$. The interaction of the two waves is shown in Figure 2. We then examine the stability of the interaction of the two waves, by fixing the value of b_1 and varying b_2 until it reaches a value where the wave blows up. The results are shown here in Figure 3.

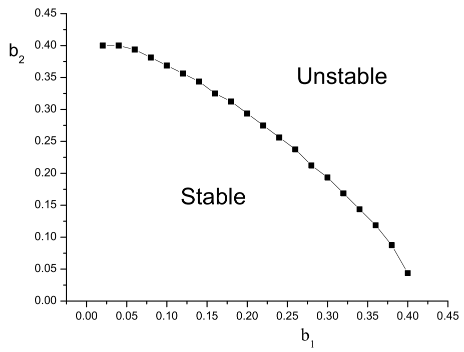

Fig. 3 Stability region for two wave interaction.

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