

Dimension Results for Sample Paths of Operator Stable Lévy Processes

Mark M. Meerschaert*
Department of Physics
University of Nevada
Reno, NV 89557

Yimin Xiao†
Department of Statistics and Probability
A-413 Wells Hall
Michigan State University
East Lansing, MI 48824

July 14, 2004

Abstract

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d with exponent B , where B is an invertible linear operator on \mathbb{R}^d . We determine the Hausdorff dimension and the packing dimension of the range $X([0, 1])$ in terms of the real parts of the eigenvalues of B .

Running Title Dimension of Operator Stable Lévy Processes

Key Words Lévy processes, operator stable processes, range, Hausdorff dimension, packing dimension.

AMS Subject Classification Primary 60J25, 28A80; Secondary 60G51, 60G17.

*Research partially supported by NSF grants DES-9980484 and DMS-0139927.

†Research partially supported by NSF grant DMS-0103939.

1 Introduction

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in \mathbb{R}^d , that is, X has stationary and independent increments, $X(0) = 0$ a.s. and such that $t \mapsto X(t)$ is continuous in probability. The finite dimensional distributions of a Lévy process X are completely determined by the distribution of $X(1)$. It is well-known that the class of possible distributions for $X(1)$ is precisely the class of infinitely divisible laws. This implies that for every $t > 0$ the characteristic function of $X(t)$ is given by

$$\mathbb{E}[e^{i\langle \xi, X(t) \rangle}] = e^{-t\psi(\xi)},$$

where, by the Lévy–Khintchine formula,

$$\psi(\xi) = i\langle \mathbf{a}, \xi \rangle + \frac{1}{2}\langle \xi, \Sigma \xi \rangle + \int_{\mathbb{R}^d} \left[1 - e^{i\langle x, \xi \rangle} + \frac{i\langle x, \xi \rangle}{1 + \|x\|^2} \right] \mathbf{L}(dx), \quad \forall \xi \in \mathbb{R}^d, \quad (1.1)$$

and $\mathbf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and \mathbf{L} is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{\|x\|^2}{1 + \|x\|^2} \mathbf{L}(dx) < \infty.$$

The function ψ is called the *Lévy exponent* of X , and \mathbf{L} is the corresponding *Lévy measure*. We refer to the recent books of Bertoin (1996) and Sato (1999) for the general theory of Lévy processes.

There has been considerable interest in studying the sample path properties of Lévy processes. Many authors have investigated the Hausdorff dimension, Hausdorff measure, packing dimension and packing measure of various random sets generated by Lévy processes. See the survey papers of Taylor (1986a) and Xiao (2004) and the references therein for more information. For a stable Lévy process X in \mathbb{R}^d with index $\alpha \in (0, 2]$, many of the results on the sample paths of X can be formulated nicely in terms of α and d . However, when X is a general Lévy process in \mathbb{R}^d , it is often difficult to determine explicitly the Hausdorff dimension of the range $X(E)$, where $E \subset \mathbb{R}_+$ is a Borel set. For $E = [0, 1]$, Pruitt (1969) proved that $\dim_{\mathbb{H}} X([0, 1]) = \gamma$ a.s., where the index γ is defined by

$$\gamma = \sup \left\{ \alpha \geq 0 : \limsup_{r \rightarrow 0} r^{-\alpha} \int_0^1 \mathbb{P}\{\|X(t)\| \leq r\} dt < \infty \right\}. \quad (1.2)$$

However, Pruitt's definition of γ is usually hard to calculate. The natural question of expressing γ in terms of the Lévy exponent ψ was raised by Pruitt (1969) and he obtained some partial results. This problem has recently been solved by Khoshnevisan, Xiao and Zhong (2003) who have shown that

$$\gamma = \sup \left\{ \alpha < d : \int_{\xi \in \mathbb{R}^d: \|\xi\| > 1} \operatorname{Re} \left(\frac{1}{1 + \psi(\xi)} \right) \frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty \right\}. \quad (1.3)$$

The proof of this result relies on the potential theory for multiparameter Lévy processes and the co-dimension argument. For more historical accounts and the latest developments about the Hausdorff dimension and capacity of the range $X(E)$, we refer to Khoshnevisan and Xiao (2004), and Xiao (2004).

The packing dimension of the range of a Lévy process X in \mathbb{R}^d was studied by Taylor (1986b), who proved that $\dim_{\text{p}} X([0, 1]) = \gamma'$ a.s., where the parameter γ' is defined by Hendricks (1983) as

$$\gamma' = \sup \left\{ \alpha \geq 0 : \liminf_{r \rightarrow 0} r^{-\alpha} \int_0^1 \mathbb{P} \{ \|X(t)\| \leq r \} dt < \infty \right\}. \quad (1.4)$$

Similar to (1.2), this definition of γ' is also hard to use. It would be interesting to express γ' in terms of the Lévy exponent ψ . Except for subordinators, this remains to be an open problem.

The objective of this paper is to investigate the Hausdorff and packing dimensions of the range of a large class of Lévy processes, i.e., the operator stable Lévy processes in \mathbb{R}^d ; see Section 2 for the definition and related properties of the latter. For the special case of a Lévy process X with stable components in \mathbb{R}^d , the Hausdorff dimension of the range $X([0, 1])$ was studied by Pruitt and Taylor (1969) and then extended by Hendricks (1972, 1973) who determined the Hausdorff dimension of $X(E)$, where $E \subset \mathbb{R}_+$ is a fixed Borel set. Recently, Becker–Kern, Meerschaert and Scheffler (2003) have obtained $\dim_{\text{H}} X([0, 1])$ for more general operator stable Lévy processes. Their arguments are based on the results of Pruitt (1969) on $\dim_{\text{H}} X([0, 1])$ [cf. (1.2)] and involve several technical probability estimates of operator stable Lévy processes. In addition, they require some restrictions on the transition densities of the processes.

In this paper, by using different methods, we show that the restrictions on the transition densities of the processes in Becker–Kern, Meerschaert and Scheffler (2003) can be removed and thus verify their conjectures on the Hausdorff and packing dimensions of $X([0, 1])$. More specifically, we apply two methods to calculate the Hausdorff dimension of the range of an operator stable Lévy process in \mathbb{R}^d . The first method is based on the covering argument for determining the Hausdorff dimension and is closely related to the arguments of Pruitt and Taylor (1969) and Hendricks (1973). The second method is more analytic and is based on (1.3) and a result of Khoshnevisan and Xiao (2004). Compared to the arguments in Becker–Kern, Meerschaert and Scheffler (2003), our methods in this paper make use of other characteristics of an operator stable Lévy process than its transition densities and hence they are more general. In particular, the covering method allows us to obtain a formula for $\dim_{\text{H}} X(E)$ for every Borel set $E \subset \mathbb{R}_+$.

The rest of the paper is organized as follows. In Section 2, we recall the definitions and some useful properties about operator stable laws, operator self-similar processes, operator stable Lévy processes, Hausdorff dimension and packing dimension. Our main results are stated and proved in Section 3. The key for the proofs is Lemma 3.4, which establishes the estimates on the expected sojourn times of X in the ball $B(0, a)$. In Section 4, we give an analytic proof of the result on $\dim_{\text{H}} X([0, 1])$ by using (1.3) and list some open problems.

Throughout this paper, we will use K to denote unspecified positive finite constants which may not necessarily be the same in each occurrence. More specific constants will be denoted by K_1, K_2, \dots .

2 Preliminaries

A Lévy process $X = \{X(t), t \in \mathbb{R}_+\}$ in \mathbb{R}^d ($d > 1$) is called *operator stable* if the distribution ν of $X(1)$ is *full* [i.e., not supported on any $(d - 1)$ -dimensional hyperplane] and ν is *strictly*

operator stable, i.e., there exists a linear operator B on \mathbb{R}^d such that

$$\nu^t = t^B \nu \quad \text{for all } t > 0, \quad (2.1)$$

where ν^t denotes the t -fold convolution power of the infinitely divisible law ν and $t^B \nu(dx) = \nu(t^{-B} dx)$ is the image measure of ν under the linear operator t^B , which is defined by

$$t^B = \sum_{n=0}^{\infty} \frac{(\log t)^n}{n!} B^n.$$

The linear operator B is called a *stability exponent* of X . The set of all possible exponents of an operator stable law is characterized in Theorem 7.2.11 of Meerschaert and Scheffler (2001).

On the other hand, a stochastic process $X = \{X(t), t \in \mathbb{R}_+\}$ with values in \mathbb{R}^d is said to be *operator self-similar* if there exists a linear operator B on \mathbb{R}^d such that for every $c > 0$,

$$\{X(ct), t \geq 0\} \stackrel{d}{=} \{c^B X(t), t \geq 0\},$$

where $X \stackrel{d}{=} Y$ denotes that the two processes X and Y have the same finite dimensional distributions. Here the linear operator B is called a *self-similarity exponent* of X .

Hudson and Mason (1982) proved that if X is a Lévy process in \mathbb{R}^d such that the distribution of $X(1)$ is full, then X is operator self-similar if and only if $X(1)$ is strictly operator stable. In this case, every stability exponent B of X is also a self-similarity exponent of X . Hence, from now on, we will simply refer B as an exponent of X .

Operator stable Lévy processes are scaling limits of random walks on \mathbb{R}^d , normalized by linear operators; see Meerschaert and Scheffler (2001, Chapter 11). Clearly, all strictly stable Lévy processes in \mathbb{R}^d of index α are operator stable with exponent $B = \alpha^{-1}I$, where I is the identity operator in \mathbb{R}^d . More generally, let X_1, \dots, X_d be independent stable Lévy processes in \mathbb{R} with indices $\alpha_1, \dots, \alpha_d \in (0, 2]$ respectively and define the Lévy process $X = \{X(t), t \geq 0\}$ by

$$X(t) = (X_1(t), \dots, X_d(t)).$$

Then it is easy to verify that X is an operator stable Lévy process with exponent B which has $\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_d^{-1}$ on the diagonal and 0 elsewhere. This class of Lévy processes was first studied by Pruitt and Taylor (1969). Following their terminology, we still call X a *Lévy process with stable components*. This type of Lévy processes is sometimes useful in constructing counterexamples [see Hendricks (1972)] and has been studied by several authors. Examples of operator stable Lévy process with dependent components can be found in Shieh (1998) and Becker–Kern, Meerschaert and Scheffler (2003). For systematic information about operator stable laws and operator stable Lévy processes, we refer to Meerschaert and Scheffler (2001).

Let $X = \{X(t), t \geq 0\}$ be an operator stable Lévy process in \mathbb{R}^d with exponent B . Factor the minimal polynomial of B into $q_1(x) \cdots q_p(x)$, where all roots of $q_i(x)$ have real part a_i and $a_i < a_j$ for $i < j$. Let $\alpha_i = a_i^{-1}$ so that $\alpha_1 > \cdots > \alpha_p$, and note that $0 < \alpha_i \leq 2$ in view of Meerschaert and Scheffler (2001, Theorem 7.2.1). Define $V_i = \text{Ker}(q_i(B))$ and $\dim(V_i) = d_i$. Then $d_1 + \cdots + d_p = d$ and $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of \mathbb{R}^d into B -invariant subspaces. We may write $B = B_1 \oplus \cdots \oplus B_p$, where $B_i : V_i \rightarrow V_i$ and every eigenvalue of B_i has real part equal to a_i . The matrix for B in an appropriate basis is then block-diagonal with p

blocks, the i th block corresponding to the matrix for B_i . Write $X(t) = X^{(1)}(t) + \dots + X^{(p)}(t)$ with respect to this direct sum decomposition, and note that by Corollary 7.2.12 of Meerschaert and Scheffler (2001) we get the same decomposition for any exponent B . Since V_i is an B -invariant subspace it follows easily that $\{X^{(i)}(t), t \in \mathbb{R}_+\}$ is an operator stable Lévy process on the d_i -dimensional vector space V_i with exponent B_i . It follows from (2.1) that $X(t) \stackrel{d}{=} t^B X(1)$ and $X^{(i)}(t) \stackrel{d}{=} t^{B_i} X^{(i)}(1)$ for all $1 \leq i \leq p$. Choose an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d such that $V_i \perp V_j$ for $i \neq j$, and let $\|x\|^2 = \langle x, x \rangle$ be the associated Euclidean norm. Then

$$\|t^B X(1)\|^2 = \|t^{B_1} X^{(1)}(1)\|^2 + \dots + \|t^{B_p} X^{(p)}(1)\|^2. \quad (2.2)$$

The following lemma is a slight variant of Lemmas 3.2 and 3.4 in Becker–Kern, Meerschaert and Scheffler (2003) which can also be proven directly using Corollary 2.2.5 in Meerschaert and Scheffler (2001).

Lemma 2.1 *For every $i = 1, \dots, p$ and every $\epsilon > 0$, there exists a finite constant $K \geq 1$ such that*

$$K^{-1} t^{a_i + \epsilon} \leq \|t^{B_i}\| \leq K t^{a_i - \epsilon} \quad \text{for all } 0 < t \leq 1 \quad (2.3)$$

and

$$K^{-1} t^{-(a_i - \epsilon)} \leq \|t^{-B_i}\| \leq K t^{-(a_i + \epsilon)} \quad \text{for all } 0 < t \leq 1. \quad (2.4)$$

Now we recall briefly the definitions of Hausdorff and packing dimensions and refer to Falconer (1990, 1997), Mattila (1995) for more information.

Let Φ be the class of functions $\varphi : (0, \delta) \rightarrow (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \leq K, \quad \text{for } 0 < s < \frac{1}{2}\delta. \quad (2.5)$$

The inequality (2.5) is usually called a doubling property. A function φ in Φ is often called a *measure function*.

For $\varphi \in \Phi$, the φ -Hausdorff measure of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \rightarrow 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\}, \quad (2.6)$$

where $B(x, r)$ denotes the open ball of radius r centered at x . The sequence of balls satisfying the two conditions on the right-hand side of (2.6) is called an ε -covering of E . It is well-known that $\varphi\text{-}m$ is a metric outer measure and every Borel set in \mathbb{R}^d is $\varphi\text{-}m$ measurable. A function $\varphi \in \Phi$ is called an *exact Hausdorff measure function* for E if $0 < \varphi\text{-}m(E) < \infty$.

The *Hausdorff dimension* of E is defined by

$$\dim_{\text{H}} E = \inf \{ \alpha > 0 : s^\alpha\text{-}m(E) = 0 \} = \sup \{ \alpha > 0 : s^\alpha\text{-}m(E) = \infty \}.$$

Packing dimension and packing measure were introduced by Tricot (1982), Taylor and Tricot (1985) as a dual concept to Hausdorff dimension and Hausdorff measure. For $\varphi \in \Phi$, define the set function $\varphi\text{-}P(E)$ on \mathbb{R}^d by

$$\varphi\text{-}P(E) = \lim_{\varepsilon \rightarrow 0} \sup \left\{ \sum_i \varphi(2r_i) : \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E, r_i < \varepsilon \right\}, \quad (2.7)$$

where \overline{B} denotes the closure of B . A sequence of closed balls satisfying the conditions on the right-hand side of (2.7) is called an ε -packing of E . Unlike φ - m , the set function φ - P is not an outer measure because it fails to be countably subadditive. However, φ - P is a premeasure, so one can obtain an outer measure φ - p on \mathbb{R}^d by defining

$$\varphi\text{-}p(E) = \inf \left\{ \sum_n \varphi\text{-}P(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}. \quad (2.8)$$

φ - $p(E)$ is called the φ -packing measure of E . Taylor and Tricot (1985) proved that φ - $p(E)$ is a metric outer measure; hence every Borel set in \mathbb{R}^d is φ - p measurable. If $\varphi(s) = s^\alpha$, s^α - $p(E)$ is called the α -dimensional packing measure of E . The *packing dimension* of E is defined by

$$\dim_p E = \inf \{ \alpha > 0 : s^\alpha\text{-}p(E) = 0 \} = \sup \{ \alpha > 0 : s^\alpha\text{-}p(E) = \infty \}. \quad (2.9)$$

There is an equivalent definition for $\dim_p E$ which is sometimes more convenient to use. For any $\varepsilon > 0$ and any bounded set $E \subseteq \mathbb{R}^d$, let

$$N(E, \varepsilon) = \text{smallest number of balls of radius } \varepsilon \text{ needed to cover } E.$$

Then the *upper and lower box-counting dimension* of E are defined as

$$\overline{\dim}_B E = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon}$$

and

$$\underline{\dim}_B E = \liminf_{\varepsilon \rightarrow 0} \frac{\log N(E, \varepsilon)}{-\log \varepsilon},$$

respectively. If $\overline{\dim}_B(E) = \underline{\dim}_B(E)$, the common value is called the *box-counting dimension* of E . From the definitions, it is easy to verify that

$$0 \leq \dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E \leq d \quad \text{and} \quad 0 \leq \dim_p E \leq \overline{\dim}_B E \leq d \quad (2.10)$$

for all bounded sets $E \subseteq \mathbb{R}^d$. Hence $\overline{\dim}_B E$ and $\underline{\dim}_B E$ can be used to determine upper bounds for $\dim_H E$ and $\dim_p E$.

The disadvantage of $\overline{\dim}_B$ and $\underline{\dim}_B$ as dimensions is that they are not σ -stable [cf. Tricot (1982), Falconer (1990, p.45)]. One can obtain σ -stable indices $\overline{\dim}_{MB}$ and $\underline{\dim}_{MB}$ by letting

$$\overline{\dim}_{MB} E = \inf \left\{ \sup_n \overline{\dim}_B E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}, \quad (2.11)$$

$$\underline{\dim}_{MB} E = \inf \left\{ \sup_n \underline{\dim}_B E_n : E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

Tricot (1982) has proved that $\dim_p E = \overline{\dim}_{MB}(E)$. Hence, for all sets $E \subseteq \mathbb{R}^d$,

$$0 \leq \dim_H E \leq \underline{\dim}_{MB} E \leq \overline{\dim}_{MB} E = \dim_p E \leq d. \quad (2.12)$$

Thus, if $\dim_H E = \dim_p E$, then all the dimensions in (2.12) coincide.

3 Main Results

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d with exponent B . Recall from Section 2 the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ and the associated block-diagonal representation $B = B_1 \oplus \cdots \oplus B_p$, where $d_i = \dim V_i$, $B_i : V_i \rightarrow V_i$ and every eigenvalue of B_i has real part equal to $a_i > 0$. We assume that $a_1 < a_2 < \cdots < a_p$, and we let $\alpha_i = a_i^{-1}$ so that $2 \geq \alpha_1 > \cdots > \alpha_p > 0$.

The following are our main results. Theorem 3.1 removes the condition on the density of $X(t)$ in Theorem 2.2 of Becker–Kern, Meerschaert and Scheffler (2003) and extends their results to $X(E)$. This solves the problems in Remarks 3.8 and 3.9 of their paper.

Theorem 3.1 *For any Borel set $E \subset \mathbb{R}_+$, almost surely*

$$\dim_{\text{H}} X(E) = \begin{cases} \alpha_1 \dim_{\text{H}} E & \text{if } \dim_{\text{H}} E \leq d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\text{H}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \quad (3.1)$$

The next result shows that the range $X([0, 1])$ has the same Hausdorff and packing dimensions, which confirms a conjecture of Becker–Kern, Meerschaert and Scheffler (2003, Remark 3.10).

Theorem 3.2 *Let X be an operator stable Lévy process in \mathbb{R}^d . Then*

$$\dim_{\text{H}} X([0, 1]) = \dim_{\text{P}} X([0, 1]) = \begin{cases} \alpha_1 & \text{if } \alpha_1 \leq d_1, \\ 1 + \alpha_2(1 - 1/\alpha_1) & \text{otherwise.} \end{cases} \quad (3.2)$$

We break the proofs of Theorems 3.1 and 3.2 into several parts. The upper bounds in Theorems 3.1 and 3.2 are proved by using Lemmas 3.3 and 3.4 and a covering argument which goes back to Pruitt and Taylor (1969) and Hendricks (1972, 1973); while the lower bounds are proved by using Lemma 3.7 and (2.12).

Let $K_1 > 0$ be a fixed constant. A collection $\Lambda(a)$ of cubes of side a in \mathbb{R}^d is called K_1 -nested if no ball of radius a in \mathbb{R}^d can intersect more than K_1 cubes of $\Lambda(a)$. In this paper, we will let $\Lambda(a)$ be the collection of all cubes of the form $\prod_{j=1}^d [k_j a, (k_j + 1)a]$, where $(k_1, \dots, k_d) \in \mathbb{Z}^d$. Clearly, $\Lambda(a)$ is K_1 -nested with $K_1 = 3^d$. In particular, for each integer $n \geq 1$ and $a = 2^{-n}$, $\Lambda(a)$ is just the collection of dyadic cubes of order n in \mathbb{R}^d . Another example of 3^d -nested collections of cubes is the set of all semi-dyadic cubes of order n in \mathbb{R}^d .

Let

$$T(a, s) = \int_0^s \mathbb{1}_{B(0, a)}(X(t)) dt$$

be the sojourn time of X in $B(0, a)$ up to time s , where $\mathbb{1}_B$ is the indicator function of the set B . The following useful covering lemma is due to Pruitt and Taylor (1969).

Lemma 3.3 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in \mathbb{R}^d and let $\Lambda(a)$ be a fixed K_1 -nested collection of cubes of side a ($0 < a \leq 1$) in \mathbb{R}^d . For any $u \geq 0$, we denote by $M_u(a, s)$ the number of cubes in $\Lambda(a)$ hit by $X(t)$ at some time $t \in [u, u + s]$. Then*

$$\mathbb{E}[M_u(a, s)] \leq 2K_1 s \left[\mathbb{E}(T(a/3, s)) \right]^{-1}.$$

The following lemma gives estimates on the expected sojourn time $T(a, s)$. Even though we only need to use the lower bounds for $\mathbb{E}[T(a, s)]$ in this paper, we also include the upper bounds which may be useful elsewhere. For example, sharp upper bounds for $\mathbb{E}[T(a, s)]$ will be useful for studying the exact Hausdorff measure functions for the range $X([0, 1])$.

Lemma 3.4 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d . For any $0 < \alpha'_2 < \alpha_2 < \alpha''_2 < \alpha'_1 < \alpha_1 < \alpha''_1$, there exist positive and finite constants K_2, \dots, K_5 such that the following hold:*

(i). *If $\alpha_1 \leq d_1$, then for all $0 < a \leq 1$ and $a^{\alpha_1} \leq s \leq 1$,*

$$K_2 a^{\alpha''_1} \leq \mathbb{E}[T(a, s)] \leq K_3 a^{\alpha'_1}. \quad (3.3)$$

(ii). *If $\alpha_1 > d_1$, then for all $a > 0$ small enough, say, $0 < a \leq a_0$, and all $a^{\alpha_2} \leq s \leq 1$,*

$$K_4 a^{\rho''} \leq \mathbb{E}[T(a, s)] \leq K_5 a^{\rho'}, \quad (3.4)$$

where $\rho' = 1 + \alpha'_2(1 - 1/\alpha_1)$ and $\rho'' = 1 + \alpha''_2(1 - 1/\alpha_1)$.

Proof We assume first $\alpha_1 \leq d_1$ and let $\alpha'_1 < \alpha_1$ be fixed. By the operator self-similarity of X and (2.2), we have $\|X(t)\| \stackrel{d}{=} \|t^{B_1} X(1)\| \geq \|t^{B_1} X^{(1)}(1)\|$. Since $\|Ax\| \geq \|x\|/\|A^{-1}\|$ for any vector $x \in \mathbb{R}^{d_1}$ and any invertible linear operator A on \mathbb{R}^{d_1} , we use (2.4) in Lemma 2.1 to derive that

$$\|t^{B_1} X^{(1)}(1)\| \geq K t^{\frac{1}{\alpha'_1}} \|X^{(1)}(1)\| \quad \text{for all } 0 < t \leq 1.$$

Since $X^{(1)}(1)$ has a continuous and bounded density, it follows that

$$\begin{aligned} \mathbb{E}[T(a, s)] &\leq \int_0^s \mathbb{P}(\|X^{(1)}(t)\| < a) dt \\ &\leq \int_0^s \mathbb{P}\left(\|X^{(1)}(1)\| < K a t^{-\frac{1}{\alpha'_1}}\right) dt \\ &\leq \int_0^{a^{\alpha'_1}} dt + \int_{a^{\alpha'_1}}^\infty K \left(a t^{-\frac{1}{\alpha'_1}}\right)^{d_1} dt \\ &\leq K_3 a^{\alpha'_1}, \end{aligned} \quad (3.5)$$

which gives the upper bound in (3.3). To prove the lower bound in (3.3), we fix α''_i ($1 \leq i \leq p$) such that $\alpha''_i > \alpha_i > \alpha''_{i+1}$. It follows from (2.2) and (2.3) in Lemma 2.1 that

$$\begin{aligned} \mathbb{E}[T(a, s)] &\geq \int_0^s \mathbb{P}\left(\|X^{(i)}(t)\| < \frac{a}{\sqrt{p}}, 1 \leq i \leq p\right) dt \\ &\geq \int_0^s \mathbb{P}\left(\|X^{(i)}(1)\| < K \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha''_i}}, 1 \leq i \leq p\right) dt \\ &\geq \int_0^{(\delta a)^{\alpha''_1}} \mathbb{P}\left(\|X^{(i)}(1)\| < K \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha''_i}}, 1 \leq i \leq p\right) dt \\ &\geq K_3 a^{\alpha''_1}, \end{aligned} \quad (3.6)$$

where $0 < \delta < 1$ is a constant such that $\mathbb{P}(\|X(1)\| \leq \frac{K}{\delta\sqrt{p}}) > 0$. Such $\delta > 0$ exists because $X(1)$ is full. So the probability in the last integral is bounded below by a positive constant. Hence (3.3) follows from (3.5) and (3.6).

Now we consider the case when $\alpha_1 > d_1 = 1$. Note that $(X^{(1)}(1), X^{(2)}(1))$ has a continuous bounded density. Similar to (3.5), we have for any $\alpha'_2 < \alpha_2$,

$$\begin{aligned} \mathbb{E}[T(a, s)] &\leq \int_0^s \mathbb{P}(\|X^{(1)}(t)\| < a, \|X^{(2)}(t)\| < a) dt \\ &\leq \int_0^s \mathbb{P}(\|X^{(1)}(1)\| < a t^{-\frac{1}{\alpha_1}}, \|X^{(2)}(1)\| < K a t^{-\frac{1}{\alpha'_2}}) dt \\ &\leq \int_0^{a^{\alpha'_2}} a t^{-\frac{1}{\alpha_1}} dt + \int_{a^{\alpha'_2}}^\infty K a^{1+d_2} t^{-\frac{1}{\alpha_1} - \frac{d_2}{\alpha'_2}} dt \\ &\leq K_5 a^{\rho'}. \end{aligned}$$

On the other hand, similar to (3.6) we have

$$\mathbb{E}[T(a, s)] \geq \int_0^s \mathbb{P}\left(\|X^{(1)}(1)\| < \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha_1}}, \|X^{(i)}(1)\| < K_6 \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha'_i}}, 2 \leq i \leq p\right) dt \quad (3.7)$$

for some constant $K_6 > 0$. Denote by $g(x_1, \dots, x_p)$ the density function of $X(1)$. Then the density function of $X^{(1)}(1)$ is given by

$$g_1(x_1) = \int_{\mathbb{R}^{d-1}} g(x_1, x_2, \dots, x_p) dx_2 \cdots dx_p.$$

Since $X^{(1)}(1)$ is a strictly stable random variable with index $\alpha_1 > 1$, by Theorem 1 of Taylor (1967) its distribution is of *type A*, i.e., $g_1(0) > 0$. Combining this with the continuity of g , we see that there exist a super-rectangle $I = [-m, m] \times J$ in \mathbb{R}^d , where $m > 0$ is a constant and J is a cube in \mathbb{R}^{d-1} , and a constant $\ell > 0$ such that $g(x_1, \dots, x_p) \geq \ell$ for all $(x_1, \dots, x_p) \in I$. Now we choose a constant $\delta \in (0, 1)$ such that

$$J \subset \left\{ (x_2, \dots, x_p) \in \mathbb{R}^{d-1} : \|x_i\| \leq \frac{K_6}{\delta\sqrt{p}}, 2 \leq i \leq p \right\}$$

and let $\eta = 1/(m\sqrt{p})$. Note that $t \geq (\eta a)^{\alpha_1}$ implies $\frac{a}{\sqrt{p}} t^{-1/\alpha_1} \leq m$. Furthermore, since $\alpha_1 > \alpha''_2$, there exists a constant $0 < a_0 \leq 1$ such that for all $0 < a \leq a_0$, we have $(\eta a)^{\alpha_1} < (\delta a)^{\alpha''_2}$. Hence, it follows from (3.7) that

$$\begin{aligned} \mathbb{E}[T(a, s)] &\geq \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha''_2}} \mathbb{P}\left(\|X^{(1)}(1)\| < \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha_1}}, \|X^{(i)}(1)\| < K_6 \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha'_i}}, 2 \leq i \leq p\right) dt \\ &\geq \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha''_2}} \int_{[-\frac{a}{\sqrt{p}} t^{-1/\alpha_1}, \frac{a}{\sqrt{p}} t^{-1/\alpha_1}] \times J} g(x_1, x_2, \dots, x_p) dx_1 \cdots dx_p dt \\ &\geq K \ell \int_{(\eta a)^{\alpha_1}}^{(\delta a)^{\alpha''_2}} \frac{a}{\sqrt{p}} t^{-\frac{1}{\alpha_1}} dt \\ &\geq K_4 a^{\rho''} \end{aligned}$$

for some constant $K_4 > 0$ that may depend on the constants m, δ, p and the cube J . This finishes the proof of (3.4). \square

Now we can prove the upper bounds in Theorems 3.1 and 3.2.

Lemma 3.5 *For any Borel set $E \subset \mathbb{R}_+$, almost surely*

$$\dim_{\mathbb{H}} X(E) \leq \begin{cases} \alpha_1 \dim_{\mathbb{H}} E & \text{if } \dim_{\mathbb{H}} E \leq d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathbb{H}} E - 1/\alpha_1) & \text{otherwise} \end{cases} \quad (3.8)$$

and

$$\dim_{\mathbb{P}} X(E) \leq \begin{cases} \alpha_1 \dim_{\mathbb{P}} E & \text{if } \dim_{\mathbb{P}} E \leq d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathbb{P}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \quad (3.9)$$

Remark 3.6 It should be pointed out that, unlike (3.8), the upper bounds for $\dim_{\mathbb{P}} X(E)$ in (3.9) may not be sharp even when X is a Brownian motion, cf. Talagrand and Xiao (1996). The problem for determining $\dim_{\mathbb{P}} X(E)$ for operator stable Lévy processes is still open (cf. Problem 4.3).

Proof We only prove (3.8). A similar argument also yields that for every bounded set $E \subset \mathbb{R}_+$, almost surely

$$\overline{\dim}_{\mathbb{B}} X(E) \leq \begin{cases} \alpha_1 \overline{\dim}_{\mathbb{B}} E & \text{if } \overline{\dim}_{\mathbb{B}} E \leq d_1/\alpha_1, \\ 1 + \alpha_2(\overline{\dim}_{\mathbb{B}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \quad (3.10)$$

Then (3.9) follows from (2.11) and (3.10).

Assume first that $\dim_{\mathbb{H}} E \leq d_1/\alpha_1$. For any $\gamma > \dim_{\mathbb{H}} E$, we choose $\alpha_1'' > \alpha_1$ such that $\gamma' \triangleq 1 - \frac{\alpha_1''}{\alpha_1} + \gamma > \dim_{\mathbb{H}} E$. Then for every $\epsilon > 0$, there exists a sequence $\{I_i\}$ of intervals in \mathbb{R}_+ with length $|I_i| < \epsilon$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} |I_i|^{\gamma'} < 1. \quad (3.11)$$

For each interval I_i , let $s_i = |I_i|$ and $b_i = |I_i|^{1/\alpha_1}$. It follows from Lemmas 3.3 and 3.4 that $X(I_i)$ can be covered by M_i cubes $C_{i,j} \in \Lambda(b_i)$ of sides b_i in \mathbb{R}^d and

$$\mathbb{E}(M_i) \leq K |I_i| \cdot |I_i|^{-\alpha_1''/\alpha_1}. \quad (3.12)$$

Note that

$$X(E) \subset \bigcup_i \bigcup_{j=1}^{M_i} C_{i,j}$$

and the diameter of $C_{i,j}$ is $\sqrt{d} b_i$. That is, $\{C_{i,j}\}$ is a $(\sqrt{d} \epsilon^{1/\alpha_1})$ -covering of $X(E)$. It follows from (3.11) and (3.12) that

$$\begin{aligned} \mathbb{E} \left(\sum_{i=1}^{\infty} M_i b_i^{\alpha_1 \gamma} \right) &\leq K \sum_{i=1}^{\infty} |I_i|^{1-\alpha_1''/\alpha_1} \cdot |I_i|^{\gamma} \\ &= K \sum_{i=1}^{\infty} |I_i|^{\gamma'} < K. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and using Fatou's lemma, we have $\mathbb{E}(s^{\alpha_1 \gamma - m}(X(E))) \leq K$. Thus $s^{\alpha_1 \gamma - m}(X(E)) < \infty$ a.s. which implies that $\dim_{\mathbb{H}} X(E) \leq \alpha_1 \gamma$ a.s. Since $\gamma > \dim_{\mathbb{H}} E$ is arbitrary, we obtain (3.8) in the case when $\dim_{\mathbb{H}} E \leq d_1/\alpha_1$.

Now we consider the case when $\dim_{\mathbb{H}} E > d_1/\alpha_1$. This implies that $\alpha_1 > 1$ and $d_1 = 1$. For any $\gamma > \dim_{\mathbb{H}} E$, we choose $\alpha_2'' > \alpha_2$ such that

$$\gamma' \triangleq 1 - \frac{\alpha_2''}{\alpha_2} + \frac{\alpha_2''}{\alpha_2} \gamma > \dim_{\mathbb{H}} E. \quad (3.13)$$

So there exists a sequence $\{I_i\}$ of intervals in \mathbb{R}_+ such that (3.11) holds. Let $s_i = |I_i|$ and $b_i = |I_i|^{1/\alpha_2}$. Denote by M_i the number of cubes $C_{i,j}' \in \Lambda(b_i)$ of side b_i in \mathbb{R}^d that meet $X(I_i)$. Then by Lemmas 3.3 and 3.4,

$$\mathbb{E}(M_i) \leq K |I_i| \cdot |I_i|^{-\rho''/\alpha_2}, \quad (3.14)$$

where we recall that $\rho'' = 1 + \alpha_2''(1 - \frac{1}{\alpha_1})$. It follows from (3.14) and (3.11) that

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{\infty} M_i b_i^{1+\alpha_2''(\gamma-\alpha_1^{-1})}\right) &\leq K \sum_{i=1}^{\infty} |I_i|^{1-\rho''/\alpha_2} \cdot |I_i|^{\alpha_2''\gamma/\alpha_2+(1-\alpha_2''\alpha_1^{-1})/\alpha_2} \\ &= K \sum_{i=1}^{\infty} |I_i|^{1-\alpha_2''/\alpha_2+\alpha_2''\gamma/\alpha_2} \\ &= K \sum_{i=1}^{\infty} |I_i|^{\gamma'} < K. \end{aligned}$$

The same argument as in the first part yields $\dim_{\mathbb{H}} X(E) \leq 1 + \alpha_2(\dim_{\mathbb{H}} E - 1/\alpha_1)$ a.s. Thus we have proven (3.8). \square

Lemma 3.7 below proves the lower bounds of $\dim_{\mathbb{H}} X(E)$ in Theorem 3.1. Similar results under more restrictive conditions [such as either $d = 1$ or independence among the components of X] can be found in Falconer (1991) and Lin and Xiao (1994). By taking $E = [0, 1]$ and using (2.12), we obtain the desired lower bound for $\dim_{\mathbb{P}} X([0, 1])$ in Theorem 3.2.

Lemma 3.7 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d . Then for any Borel set $E \subset \mathbb{R}_+$, almost surely*

$$\dim_{\mathbb{H}} X(E) \geq \begin{cases} \alpha_1 \dim_{\mathbb{H}} E & \text{if } \dim_{\mathbb{H}} E \leq d_1/\alpha_1, \\ 1 + \alpha_2(\dim_{\mathbb{H}} E - 1/\alpha_1) & \text{otherwise.} \end{cases} \quad (3.15)$$

Proof For the proof of (3.15), we use a standard capacity argument; see e.g., Kahane (1985, Chapter 10), Falconer (1990, Chapter 4), Mattila (1995, Chapter 8), Taylor (1986a) or Xiao (2004). Note that Frostman's lemma and theorem are only proved for compact sets in Kahane (1985). Both of them are still valid for all Borel sets as shown in Falconer (1990, Chapter 4) and Mattila (1995, Chapter 8).

First consider the case when $\dim_{\mathbb{H}} E \leq d_1/\alpha_1$. If $\dim_{\mathbb{H}} E = 0$, there is nothing to prove. So we assume $\dim_{\mathbb{H}} E > 0$. For any $0 < \gamma < \alpha_1 \dim_{\mathbb{H}} E$, we choose $0 < \alpha_1' < \alpha_1$ such that

$\gamma < \alpha'_1 \dim_{\mathbb{H}} E$. Then, it follows from Frostman's lemma [cf. Kahane (1985) or Mattila (1995)] that there exists a probability measure σ on E such that

$$\int_E \int_E \frac{\sigma(ds)\sigma(dt)}{|s-t|^{\gamma/\alpha'_1}} < \infty. \quad (3.16)$$

By Frostman's theorem [cf. Kahane (1985) or Mattila (1995)], we know that, in order to prove $\dim_{\mathbb{H}} X(E) \geq \gamma$ almost surely, it suffices to show

$$\int_E \int_E \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) \sigma(ds)\sigma(dt) < \infty. \quad (3.17)$$

It follows from (2.2) that for all $s, t \in \mathbb{R}_+$ such that $|s-t| \leq 1$,

$$\begin{aligned} \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) &= \mathbb{E}(\| |s-t|^{B_1} X(1) \|^{-\gamma}) \\ &\leq \mathbb{E}(\| |s-t|^{B_1} X^{(1)}(1) \|^{-\gamma}) \\ &\leq K |s-t|^{-\gamma/\alpha'_1}, \end{aligned} \quad (3.18)$$

where in deriving the last inequality, we have used Lemma 2.1 and the elementary fact that if a random variable X in \mathbb{R}^{d_1} has a bounded density, then for any $0 < \gamma < d_1$, $\mathbb{E}(\|X\|^{-\gamma}) < \infty$. Also, a simple argument using Lemma 2.1 shows that

$$\sup_{|s-t| \geq 1} \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) < \infty.$$

Now it is clear that (3.17) follows from (3.18) and (3.16).

Now we consider the case when $\dim_{\mathbb{H}} E > 1/\alpha_1$ and $d_1 = 1$. Let $1 < \gamma < 1 + \alpha_2(\dim_{\mathbb{H}} E - 1/\alpha_1)$ be fixed. Note that since $\rho = \gamma/\alpha_2 - (1/\alpha_2 - 1/\alpha_1) < \dim_{\mathbb{H}} E$, we can choose $0 < \alpha'_2 < \alpha_2$ such that $\rho' = \gamma/\alpha'_2 - (1/\alpha'_2 - 1/\alpha_1) < \dim_{\mathbb{H}} E$. Then there exists a probability measure σ on E such that

$$\int_E \int_E \frac{\sigma(ds)\sigma(dt)}{|s-t|^{\rho'}} < \infty. \quad (3.19)$$

Similar to (3.18), we use (2.2) to deduce that for all $s, t \in \mathbb{R}_+$ such that $|s-t| \leq 1$,

$$\begin{aligned} \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) &= \mathbb{E}(\| |t-s|^{B_1} X(1) \|^{-\gamma}) \\ &\leq \mathbb{E}([\|t-s\|^{\frac{2}{\alpha_1}} |X^{(1)}(1)|^2 + \| |t-s|^{B_2} X^{(2)}(1) \|^2]^{-\gamma/2}) \\ &\leq K \int_{\mathbb{R}^{1+d_2}} \frac{1}{|s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + \| |s-t|^{B_2} x_2 \|^{\gamma}} g(x_1, x_2) dx_1 dx_2 \\ &\leq K \int_{\mathbb{R}^{1+d_2}} \frac{1}{|s-t|^{\frac{\gamma}{\alpha_1}} |x_1|^\gamma + |s-t|^{\frac{\gamma}{\alpha_2}} \|x_2\|^\gamma} g(x_1, x_2) dx_1 dx_2 \\ &= K |s-t|^{-\frac{\gamma}{\alpha_1}} \int_{\mathbb{R}^{1+d_2}} \frac{1}{|x_1|^\gamma + |s-t|^{\frac{\gamma}{\alpha_2} - \frac{\gamma}{\alpha_1}} \|x_2\|^\gamma} g(x_1, x_2) dx_1 dx_2, \end{aligned} \quad (3.20)$$

where $g(x_1, x_2)$ is the density function of $(X^{(1)}(1), X^{(2)}(1))$ which is bounded and continuous. We will use integration by parts to derive an upper bound for the integral J in (3.20). To this end, let

$$F(r_1, r_2) = \mathbb{P}(|X^{(1)}(1)| \leq r_1, \|X^{(2)}(1)\| \leq r_2).$$

Then by using spherical coordinates, we can write

$$\begin{aligned}
F(r_1, r_2) &= \int_{|x_1| \leq r_1} \int_{\|x_2\| \leq r_2} g(x_1, x_2) dx_2 dx_1 \\
&= \int_{-r_1}^{r_1} \int_0^{r_2} \int_{S_{d_2-1}} \tilde{g}(\rho_1, \rho_2 \theta) \rho_2^{d_2-1} \mu(d\theta) d\rho_2 d\rho_1,
\end{aligned} \tag{3.21}$$

where $\tilde{g}(y_1, y_2 \theta)$ is bounded and continuous in $(y_1, y_2, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times S_{d_2-1}$ and μ is the surface measure on the unit sphere S_{d_2-1} in \mathbb{R}^{d_2} . Note that there also exists a finite constant $K_7 > 0$ such that

$$F(r_1, r_2) \leq (1 \wedge K_7 r_1)(1 \wedge K_7 r_2) \quad \text{for all } r_1, r_2 \geq 0. \tag{3.22}$$

For simplicity of notation, we denote $c = |s - t|^{\frac{1}{\alpha_2} - \frac{1}{\alpha_1}}$. By using Fubini's theorem and integration by parts when integrating dr_1 , we deduce

$$\begin{aligned}
J &= \int_0^\infty \int_0^\infty \frac{1}{r_1^\gamma + c^\gamma r_2^\gamma} F(dr_1, dr_2) \\
&= \int_0^\infty dr_2 \int_0^\infty \left[\frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^2} \int_0^{r_1} \int_{S_{d_2-1}} \tilde{g}(\rho_1, r_2 \theta) r_2^{d_2-1} \mu(d\theta) d\rho_1 \right] dr_1 \\
&= \int_0^1 dr_2 \int_0^\infty [\dots] dr_1 + \int_1^\infty dr_2 \int_0^\infty [\dots] dr_1 \\
&\doteq J_1 + J_2.
\end{aligned} \tag{3.23}$$

Now we estimate J_1 and J_2 separately. Since \tilde{g} is bounded, we have

$$\begin{aligned}
J_1 &\leq K \int_0^1 r_2^{d_2-1} dr_2 \int_0^\infty \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^2} r_1 dr_1 \\
&\leq K \int_0^1 r_2^{d_2-1} \frac{1}{(c r_2)^\gamma} dr_2 \int_0^\infty \frac{\gamma s_1^\gamma}{(s_1^\gamma + 1)^2} ds_1 \\
&= \frac{K_8}{c^{\gamma-1}} = K_8 |s - t|^{-(\gamma-1)(\frac{1}{\alpha_2} - \frac{1}{\alpha_1})},
\end{aligned} \tag{3.24}$$

In getting the second inequality above, we have used the change of variable $r_1 = c r_2 s_1$. Also note that since $1 < \gamma < \alpha_1 \leq 2 \leq d_2 + 1$, the last two integrals are convergent and K_8 is a positive and finite constant.

On the other hand, it follows from Fubini's theorem and integration by parts for dr_2 that

$$\begin{aligned}
J_2 &= - \int_{S_{d_2-1}} \mu(d\theta) \int_0^\infty dr_1 \frac{\gamma r_1^{\gamma-1}}{(r_1^\gamma + c^\gamma)^\gamma} \left[\int_0^1 \int_0^{r_1} \tilde{g}(\rho_1, \rho_2 \theta) \rho_2^{d_2-1} d\rho_1 d\rho_2 \right] \\
&+ \int_{S_{d_2-1}} \mu(d\theta) \int_0^\infty dr_1 \int_1^\infty \frac{2\gamma^2 c^\gamma r_1^{\gamma-1} r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3} \left[\int_0^{r_2} \int_0^{r_1} \tilde{g}(\rho_1, \rho_2 \theta) \rho_2^{d_2-1} d\rho_1 d\rho_2 \right] dr_2 \\
&\leq \int_0^\infty dr_1 \int_1^\infty \frac{2\gamma^2 c^\gamma r_1^{\gamma-1} r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3} \left[\int_0^{r_2} \int_0^{r_1} \int_{S_{d_2-1}} \tilde{g}(\rho_1, \rho_2 \theta) \rho_2^{d_2-1} \mu(d\theta) d\rho_1 d\rho_2 \right] dr_2.
\end{aligned}$$

Note that the triple integral in the brackets is $F(r_1, r_2)$, thus (3.22) together with a change of variables $r_1 = cr_2 s_1$ implies that

$$\begin{aligned} J_2 &\leq K \int_1^\infty dr_2 \int_0^\infty \frac{c^\gamma r_1^{\gamma-1} r_2^{\gamma-1}}{(r_1^\gamma + c^\gamma r_2^\gamma)^3} r_1 dr_1 \\ &\leq \frac{K}{c^{\gamma-1}} \int_1^\infty \frac{1}{r_2^\gamma} dr_2 \int_0^\infty \frac{s_1^\gamma}{(s_1^\gamma + 1)^3} ds_1 \\ &= K_9 |s - t|^{-(\gamma-1)(\frac{1}{\alpha_2} - \frac{1}{\alpha_1})}. \end{aligned} \quad (3.25)$$

Here we have used again the fact that $\gamma > 1$.

Combining (3.20), (3.23), (3.24) and (3.25), we have proven that for $|s - t| \leq 1$

$$\mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) \leq K |s - t|^{(\frac{1}{\alpha_2} - \frac{1}{\alpha_1}) - \frac{\gamma}{\alpha_2}} = K |s - t|^{-\rho'}. \quad (3.26)$$

Again a simple argument using (2.2) and Lemma 2.1 shows that

$$\sup_{|s-t| \geq 1} \mathbb{E}(\|X(s) - X(t)\|^{-\gamma}) \leq \mathbb{E}\left[\left(\|X^{(1)}(1)\|^2 + \|X^{(2)}(1)\|^2\right)^{-\gamma/2}\right] < \infty.$$

Therefore, it follows from (3.19) and (3.26) that (3.17) holds. Using Frostman's theorem again, we have $\dim_{\mathbb{H}} X(E) \geq \gamma$ a.s. This finishes the proof of Lemma 3.7. \square

4 Further Remarks and Open Questions

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in \mathbb{R}^d with Lévy exponent ψ . Recently, Khoshnevisan, Xiao and Zhong (2003) have proved the following formula for $\dim_{\mathbb{H}} X([0, 1])$ in terms of ψ : almost surely

$$\dim_{\mathbb{H}} X([0, 1]) = \sup\left\{\alpha < d : \int_{\xi \in \mathbb{R}^d: \|\xi\| > 1} \operatorname{Re}\left(\frac{1}{1 + \psi(\xi)}\right) \frac{d\xi}{\|\xi\|^{d-\alpha}} < +\infty\right\}. \quad (4.1)$$

This gives a different, analytic way to study the Hausdorff dimension of $X([0, 1])$ for Lévy processes. We refer to Khoshnevisan and Xiao (2004) for further developments on Hausdorff dimension and capacity. The following result is an extension of Proposition 7.7 (see also Remark 7.8) of Khoshnevisan and Xiao (2004), as well as the result of Pruitt and Taylor (1969) for Lévy processes with stable components.

Proposition 4.1 *Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a Lévy process in \mathbb{R}^d with Lévy exponent ψ . If ψ satisfies the following condition: there are constants $2 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_d > 0$ such that for every $\varepsilon > 0$, there exists a constant $\tau > 1$ such that*

$$\frac{K^{-1}}{\|\xi\|^\varepsilon \sum_{j=1}^d |\xi_j|^{\beta_j}} \leq \operatorname{Re}\left(\frac{1}{1 + \psi(\xi)}\right) \leq \frac{K \|\xi\|^\varepsilon}{\sum_{j=1}^d |\xi_j|^{\beta_j}}, \quad \forall \xi \in \mathbb{R}^d \text{ with } \|\xi\| \geq \tau, \quad (4.2)$$

where $K \geq 1$ is a constant which may depend on ε and τ . Denote $n_1 = \max\{j : \beta_j = \beta_1\}$. Then almost surely,

$$\dim_{\mathbb{H}} X([0, 1]) = \begin{cases} \beta_1 & \text{if } \beta_1 \leq n_1, \\ 1 + \beta_2(1 - 1/\beta_1) & \text{otherwise.} \end{cases} \quad (4.3)$$

Proof The proof, based on (4.1), is a slight modification of that of Proposition 7.7 of Khoshnevisan and Xiao (2004). Hence it is omitted. \square

Proposition 4.1 leads to a completely different proof of the Hausdorff dimension of $X([0, 1])$ for operator stable Lévy processes.

Theorem 4.2 *Let X be an operator stable Lévy process in \mathbb{R}^d as in Theorem 3.2. Then (3.2) holds almost surely.*

Proof In the notations of Section 3, we will show that for every $\epsilon > 0$, there exists a constant $K \geq 1$ such that (4.2) holds for $\beta_1 \geq \dots \geq \beta_d$ defined by $\beta_j = \alpha_\ell$ if $\sum_{i=0}^{\ell-1} d_i < j \leq \sum_{i=\ell-1}^{\ell} d_i$, where $d_0 = 0$. Once this is proved, the theorem will follow from Proposition 4.1 with $n_1 = d_1$.

The proof is based on asymptotic inverses, a method first used in Meerschaert (1990) to get sharp bounds on the probability tails of operator stable random vectors. Use the Jordan decomposition [see, e.g., Theorem 2.1.16 in Meerschaert and Scheffler (2001)] to obtain a basis b_1, \dots, b_d for \mathbb{R}^d in which B is block-diagonal where every block is of the form

$$\begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & a & 0 & \cdots & 0 \\ 0 & 1 & a & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 1 & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & 0 & 0 & \cdots & 0 \\ I & C & 0 & \cdots & 0 \\ 0 & I & C & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & I & C \end{pmatrix}, \quad (4.4)$$

where a is a real eigenvalue of B in the first case, and in the second case

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.5)$$

where $a \pm ib$ is a complex conjugate pair of eigenvalues of B . Define $\|x\| = \sqrt{\langle x, x \rangle}$ using the inner product associated with this basis [see, e.g., Proposition 1.1.20 (b) in Meerschaert and Scheffler (2001)], so that $\langle b_i, b_j \rangle = I(i = j)$. In these coordinates, the matrix power t^B can be explicitly computed [see Lemma 2.2.3 in Meerschaert and Scheffler (2001)] as well as the norm $\|t^B x\|$ for every $x \in \mathbb{R}^d$ [see the proof of Theorem 2.2.4 in Meerschaert and Scheffler (2001)]. This follows easily from the (unique) decomposition $B = S + N$ where S is semisimple (diagonalizable over the complex numbers) and N is nilpotent ($N^m = 0$ for some positive integer m). In the first case [i.e., B is a $(k+1) \times (k+1)$ block as the first matrix in (4.4)], if $x = (x_1, \dots, x_{k+1})$ are the coordinates for one block and $z(t) = t^B x = (z_1(t), \dots, z_{k+1}(t))$, then

$$z_j(t) = \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} x_{j-n} \quad (4.6)$$

for all $j = 1, \dots, k+1$. In the second case, if $u = (x_1, y_1, \dots, x_{k+1}, y_{k+1})$ are coordinates for one block and $t^B u = (z_1(t), w_1(t), \dots, z_{k+1}(t), w_{k+1}(t))$, then

$$\begin{aligned} z_j(t) &= \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} (\cos(b \log t) x_{j-n} - \sin(b \log t) y_{n-j}), \\ w_j(t) &= \sum_{n=0}^{j-1} \frac{t^a (\log t)^n}{n!} (\sin(b \log t) x_{j-n} + \cos(b \log t) y_{n-j}) \end{aligned} \quad (4.7)$$

for all $j = 1, \dots, k+1$. Recall from Section 2 the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \dots \oplus V_p$ and the associated block-diagonal representation $B = B_1 \oplus \dots \oplus B_p$, where $B_i : V_i \rightarrow V_i$ and every eigenvalue of B_i has real part equal to a_i . Now apply Theorem 3.1 in Meerschaert and Veeh (1993) to obtain a further direct sum decomposition $V_i = U_{i1} \oplus \dots \oplus U_{iq(i)}$ where U_{ij} is a B -invariant subspace and every nonzero vector $x \in U_{ij}$ is of order j , so that $N^j x = 0$ and $N^{j-1} x \neq 0$. Note that every basis element b_1, \dots, b_d lies in one of these subspaces. Write $x = \sum_i \sum_j x_{ij}$ with respect to this direct sum decomposition, so that $x_{ij} \in U_{ij}$. Then it follows from (4.6) and (4.7) that

$$\|t^{-B^*} x\|^2 = \sum_{i=1}^p \sum_{j=1}^{q(i)} \frac{t^{-2a_i} (\log t)^{2(j-1)}}{((j-1)!)^2} \|x_{ij}\|^2 + o_{ij}(t, x), \quad (4.8)$$

where B^* is the transpose of B , $o_{ij}(t, x)$ is a linear combination of terms of the form $t^{-2a_i} (\log t)^k \times |x_{ijr}| |x_{ijs}|$ with $k < 2(j-1)$, x_{ijr} is one of the coordinates of x_{ij} in the basis b_1, \dots, b_d , and the coefficients of this linear combination are independent of both x and t . Then clearly $R(t) = 1/\|t^{-B^*} x\|$ is a regularly varying function of $t > 0$ at infinity with index $a = \min\{a_i : x_i \neq 0\}$ where $x = \sum_i x_i$ with respect to the direct sum decomposition $\mathbb{R}^d = V_1 \oplus \dots \oplus V_p$. Of course the function $R(t) = 1/\|t^{-B^*} x\|$ (as well as $t(r)$ below) also depends on x . We have suppressed x so that the notation will not get too heavy.

Since $a > 0$, the function $R(t)$ has an asymptotic inverse $t(r)$, regularly varying at infinity with index $\alpha = 1/a$, such that $R(t(r)) \sim r$ as $r \rightarrow \infty$ [see, e.g., Bingham et al (1987), p.28 or Seneta (1976), p.21]. In fact, we can take

$$t(r) = \sum_{i=1}^p \sum_{j=1}^{q(i)} K_{ij} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i}, \quad (4.9)$$

where $K_{ij} = (\alpha_i^{j-1}/(j-1)!)^{\alpha_i}$, and the convergence of $R(t(r))/r \rightarrow 1$ as $r \rightarrow \infty$ is uniform in x on compact sets of $\mathbb{R}^d \setminus \{0\}$. To see this, let i be the index such that $a_i = a$, and let $j = k+1$ be the order of x_i . Then by (4.8) we can write

$$\|t^{-B^*} x\|^2 = \frac{t^{-2a} (\log t)^{2k}}{(k!)^2} \|x_{ij}\|^2 + o(t^{-2a} (\log t)^{2k}) \quad \text{as } t \rightarrow \infty, \quad (4.10)$$

and the convergence is uniform in x on compact sets of $\mathbb{R}^d \setminus \{0\}$. Similarly, it follows from (4.9) that as $r \rightarrow \infty$

$$t(r) = K_{ij} r^\alpha (\log r)^{\alpha k} \|x_{ij}\|^\alpha + o(r^\alpha (\log r)^{\alpha k}) \quad (4.11)$$

uniformly for x on compact sets of $\mathbb{R}^d \setminus \{0\}$.

Now it suffices to show that $R(t(r))^{-2} = \|t(r)^{-B^*} x\|^2 \sim r^{-2}$ as $r \rightarrow \infty$ uniformly for x on compact sets of $\mathbb{R}^d \setminus \{0\}$. This follows from (4.10), (4.11) and an elementary computation:

$$\begin{aligned} R(t(r))^{-2} &= \frac{\alpha^{2k}}{(k!)^2} \left[K_{ij} r^\alpha (\log r)^{\alpha k} \|x_{ij}\|^\alpha \right]^{-2a} (\log r)^{2k} \|x_{ij}\|^2 + \dots \\ &= r^{-2} + o(r^{-2}) \quad \text{as } r \rightarrow \infty, \end{aligned} \quad (4.12)$$

where the convergence is uniform in x on compact sets of $\mathbb{R}^d \setminus \{0\}$. This establishes our claim.

Since $X(t)$ and $t^B X(1)$ are identically distributed we have

$$t\psi(\xi) = \psi(t^{B^*} \xi) \quad (4.13)$$

for all $\xi \in \mathbb{R}^d$ and all $t > 0$. Let $F(\xi) = \text{Re}(\psi(\xi))$ so that $tF(\xi) = F(t^{B^*} \xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^d$. Moreover, $F(\xi)$ is bounded away from zero and infinity on compact subsets of $\mathbb{R}^d \setminus \{0\}$ since $X(t)$ is full [see, e.g., Corollary 7.1.12 in Meerschaert and Scheffler (2001)].

Given $x \in \mathbb{R}^d \setminus \{0\}$ and $r > 0$, we define $\theta_r = t(r)^{-B^*}(rx)$. Then it follows from the above that as $r \rightarrow \infty$, $\|\theta_r\| = r\|t(r)^{-B^*} x\| = r/R(t(r)) \rightarrow 1$ uniformly for x on compact sets in \mathbb{R}^d . Consequently for every $0 < \eta < 1$, there exists some $r_0 > 0$ such that $1 - \eta < \|\theta_r\| < 1 + \eta$ for all $r \geq r_0$ and all $x \in S_d$. Here $S_d = \{x : \|x\| = 1\}$ is the unit sphere in \mathbb{R}^d .

For any $\xi \in \mathbb{R}^d \setminus \{0\}$, let $r = \|\xi\|$ and $x = \xi/r \in S_d$ so that $\xi = rx$. It follows that

$$F(rx) = F(t(r)^{B^*} \theta_r) = t(r)F(\theta_r) \quad (4.14)$$

and $F(\theta_r)$ is bounded away from zero and infinity for all $r \geq r_0$ and $x \in S_d$. On the other hand, for any $\epsilon > 0$, there is a constant $\tau \geq \max\{r_0, e\}$ such that for all $r \geq \tau$, $(\log r)^{\alpha_i(q(i)-1)} \leq r^{\epsilon/2}$ for every $1 \leq i \leq p$. Therefore it follows from (4.14) and (4.9) that

$$\begin{aligned} F(\xi) &\leq K \sum_{i=1}^p \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \\ &\leq K r^{\epsilon/2} \sum_{i=1}^p (r \|x_i\|)^{\alpha_i} = K \|\xi\|^{\epsilon/2} \sum_{i=1}^p \|\xi_i\|^{\alpha_i} \end{aligned} \quad (4.15)$$

for all $\|\xi\| \geq \tau$. Similarly, we derive from (4.14) and (4.9) that for all $\|\xi\| \geq \tau$,

$$\begin{aligned} F(\xi) &\geq K' \sum_{i=1}^p \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \\ &\geq K' \sum_{i=1}^p \|\xi_i\|^{\alpha_i}. \end{aligned} \quad (4.16)$$

Now we consider $G(\xi) = \text{Im}(\psi(\xi))$. Note that (4.13) implies $tG(\xi) = G(t^{B^*} \xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^d$. By the continuity of $\psi(\xi)$, $G(\xi)$ is bounded on compact subsets of \mathbb{R}^d . Hence, similar to (4.15), we have that for all $\|\xi\| \geq \tau$,

$$|G(\xi)| \leq K \sum_{i=1}^p \sum_{j=1}^{q(i)} r^{\alpha_i} (\log r)^{\alpha_i(j-1)} \|x_{ij}\|^{\alpha_i} \leq K \|\xi\|^{\epsilon/2} \sum_{i=1}^p \|\xi_i\|^{\alpha_i}. \quad (4.17)$$

Combining (4.15), (4.16) and (4.17) with the following identity

$$\operatorname{Re}\left(\frac{1}{1+\psi(\xi)}\right) = \frac{1 + \operatorname{Re}\psi(\xi)}{(1 + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi))^2}, \quad (4.18)$$

we obtain

$$\frac{K_{10}}{\|\xi\|^\varepsilon \sum_{i=1}^p \|\xi_i\|^{\alpha_i}} \leq \operatorname{Re}\left(\frac{1}{1+\psi(\xi)}\right) \leq \frac{K_{11}}{\sum_{i=1}^p \|\xi_i\|^{\alpha_i}}, \quad \forall \xi \in \mathbb{R}^d \text{ with } \|\xi\| \geq \tau. \quad (4.19)$$

Thus (4.2) holds. This completes our proof. \square

We end this section with some open questions. Since $\dim_{\mathbb{H}}X([0, 1]) = \dim_{\mathbb{P}}X([0, 1])$ is given by Theorem 3.2, it would be interesting to further investigate the following natural question.

Problem 4.3 Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d . Find exact Hausdorff and packing measure functions for the range $X([0, 1])$.

We also mention that, if X is a Lévy process with stable components in \mathbb{R}^d or an operator stable Lévy process in \mathbb{R}^d , no general formula for the packing dimension of $X(E)$ has yet been established. When X is a one-dimensional Brownian motion, the packing dimension of $X(E)$ was studied by Talagrand and Xiao (1996) who showed that the inequality $\dim_{\mathbb{P}}X(E) < 2\dim_{\mathbb{P}}E$ holds for some Cantor-type set $E \subset [0, 1]$. Hence the formula analogous to that for $\dim_{\mathbb{H}}X(E)$ does not hold for the packing dimension $\dim_{\mathbb{P}}X(E)$. Xiao (1997) proved a formula for $\dim_{\mathbb{P}}X(E)$ in terms of the packing dimension profile of E introduced by Falconer and Howroyd (1997). We believe a result analogous to that in Xiao (1997) for Brownian motion still holds for all stable Lévy processes in \mathbb{R} with stability index $\alpha > 1$ [This is the only remaining problem for $\dim_{\mathbb{P}}X(E)$, where X is a stable Lévy process X in \mathbb{R}^d with index α , since Perkins and Taylor (1987) have shown that if $\alpha \leq d$, then a.s. $\dim_{\mathbb{P}}X(E) = \alpha\dim_{\mathbb{P}}E$ for all Borel sets $E \subset \mathbb{R}_+$]. However, for Lévy processes with stable components in \mathbb{R}^d or operator stable Lévy processes, the packing dimension profile introduced by Falconer and Howroyd (1997) does not seem to be appropriate for characterizing $\dim_{\mathbb{P}}X(E)$. One may need to introduce a corresponding concept of packing dimension profile that can capture different growths in different directions.

Shieh (1998) has investigated the Hausdorff dimension of the multiple points of a class of operator stable processes including Lévy processes with stable components. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d with exponent B which has $a_1 \leq \dots \leq a_d$ on its diagonal and 0 elsewhere. Let

$$L_k = \{x \in \mathbb{R}^d : \exists \text{ distinct } t_1, \dots, t_k \text{ such that } X(t_1) = \dots = X(t_k) = x\}.$$

be the set of k -multiple points of X . Under certain conditions, Shieh (1998) proved that for $k \geq 2$ almost surely,

$$\dim_{\mathbb{H}}L_k = \min\left\{\alpha_1(k - (k - 1)\sum_{i=1}^d \alpha_i^{-1}), d - k\alpha_d(\sum_{i=1}^d \alpha_i^{-1} - 1)\right\}, \quad (4.20)$$

where $\alpha_i = a_i^{-1}$ ($i = 1, \dots, d$) and negative dimension means that the set L_k is empty. We believe his result may still be true for all operator stable Lévy processes, where now a_i are the real parts of the eigenvalues of B as described at the beginning of Section 3 and each $\alpha_i = a_i^{-1}$ is repeated $d_i = \dim V_i$ times. It would be interesting to solve the following problem:

Problem 4.4 Let $X = \{X(t), t \in \mathbb{R}_+\}$ be an operator stable Lévy process in \mathbb{R}^d . Let L_k be the set of k -multiple points. Show that (4.20) holds.

Acknowledgements. We thank Professors H. P. Scheffler and D. Khoshnevisan for helpful discussions. We are grateful to an anonymous referee for his/her careful reading of our paper and for his/her thoughtful comments that have led to several improvements of the manuscript.

References

- [1] P. Becker-Kern, M. M. Meerschaert and H. P. Scheffler (2003), Hausdorff dimension of operator stable sample paths. *Monatsh. Math.* **14**, 91–101.
- [2] J. Bertoin (1996), *Lévy Processes*. Cambridge University Press, Cambridge.
- [3] N. H. Bingham, C. M. Goldie and J. L. (1987), *Regular Variation*. Cambridge University Press, Cambridge.
- [4] K. J. Falconer(1990), *Fractal Geometry – Mathematical Foundations and Applications*. Wiley & Sons, Chichester.
- [5] K. J. Falconer (1991), The horizon problem for random surfaces. *Math. Proc. Cambridge Philos. Soc.* **109**, 211–219.
- [6] K. J. Falconer (1997), *Techniques in Fractal Geometry*. Wiley & Sons, Chichester.
- [7] K. J. Falconer and J. D. Howroyd (1997), Packing dimensions for projections and dimension profiles. *Math. Proc. Camb. Philos. Soc.* **121**, 269–286.
- [8] W. J. Hendricks (1972), Hausdorff dimension in a process with stable components—an interesting counterexample. *Ann. Math. Statist.* **43**, 690–694.
- [9] W. J. Hendricks (1973), A dimension theorem for sample functions of processes with stable components. *Ann. Probab.* **1**, 849–853.
- [10] W. J. Hendricks (1983), A uniform lower bound for Hausdorff dimension for transient symmetric Lévy processes. *Ann. Probab.* **11**, 589–592.
- [11] W. N. Hudson and J. D. Mason (1982), Operator-self-similar processes in a finite-dimensional space. *Trans. Amer. Math. Soc.* **273**, 281–297.
- [12] J.-P. Kahane (1985), *Some Random Series of Functions*. 2nd edition. Cambridge University Press, Cambridge.
- [13] D. Khoshnevisan and Y. Xiao (2004), Lévy processes: capacity and Hausdorff dimension. *Ann. Probab.* to appear.
- [14] D. Khoshnevisan, Y. Xiao and Y. Zhong (2003), Measuring the range of an additive Lévy processes. *Ann. Probab.* **31**, 1097–1141.
- [15] H. Lin and Y. Xiao (1994), Dimension properties of sample paths of self-similar processes. *Acta Math. Sinica (N.S.)* **10**, 289–300.
- [16] P. Mattila (1995), *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, Cambridge.

- [17] M. M. Meerschaert (1990), Moments of random vectors which belong to some domain of normal attraction. *Ann. Probab.* **18**, 870–876.
- [18] M. M. Meerschaert and J. Veeh (1993), The structure of the exponents and symmetries of an operator stable measure. *J. Theoret. Probab.* **6**, 713–726.
- [19] M. M. Meerschaert and H.-P. Scheffler (2001), *Limit Distributions for Sums of Independent Random Vectors*. John Wiley & Sons, New York.
- [20] E. A. Perkins and S. J. Taylor (1987), Uniform measure results for the image of subsets under Brownian motion. *Probab. Th. Rel. Fields* **76**, 257–289.
- [21] W. E. Pruitt (1969), The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. Mech.* **19**, 371–378.
- [22] W. E. Pruitt and S. J. Taylor (1969), Sample path properties of processes with stable components. *Z. Wahrsch. Verw. Gebiete* **12**, 267–289.
- [23] K. Sato (1999), *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.
- [24] E. Seneta (1976), Regularly Varying Functions. *Lecture Notes in Math.* **508**, Springer-Verlag, Berlin.
- [25] N.-R. Shieh (1998), Multiple points of dilation-stable Lévy processes. *Ann. Probab.* **26**, 1341–1355.
- [26] M. Talagrand and Y. Xiao (1996), Fractional Brownian motion and packing dimension. *J. Theoret. Probab.* **9**, 579–593.
- [27] S. J. Taylor (1967), Sample path properties of a transient stable process. *J. Math. Mech.* **16**, 1229–1246.
- [28] S. J. Taylor (1986a), The measure theory of random fractals. *Math. Proc. Camb. Philos. Soc.* **100**, 383–406.
- [29] S. J. Taylor (1986b), The use of packing measure in the analysis of random sets. In: *Stochastic Processes and Their Applications (Nagoya, 1985)*, pp. 214–222, Lecture Notes in Math. **1203**, Springer-Verlag, Berlin.
- [30] S. J. Taylor and C. Tricot (1985), Packing measure and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.* **288**, 679–699.
- [31] C. Tricot (1982), Two definitions of fractional dimension. *Math. Proc. Camb. Philos. Soc.* **91**, 57–74.
- [32] Y. Xiao (1997), Packing dimension of the image of fractional Brownian motion. *Statist. Probab. Lett.* **33**, 379–387.
- [33] Y. Xiao (2004), Random fractals and Markov processes. To appear in: *Proceedings of the Conference on Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*. AMS.

MARK M. MEERSCHAERT

Department of Physics, University of Nevada, Reno, NV 89557

E-mail: mcubed@unr.edu

URL: <http://unr.edu/homepage/mcubed>

YIMIN XIAO

Department of Statistics and Probability, A-413 Wells Hall, Michigan State University,

East Lansing, MI 48824
E-mail: xiao@stt.msu.edu
URL: <http://www.stt.msu.edu/~xiaoyimi>