Pricing and Rate Adaptation in a Non-Cooperative Environment

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***Abstract*— We consider a price-based rate control scheme that has been proposed by Kelly *et al.*. However, rather than assuming that users adapt their transmission rates by mandate, we study the situation where users choose transmission rates to maximize their net benefit at each time step. Extending a result obtained by Ganesh *et al.*, we show that in the single link case the system again converges to the equilibrium bandwidth allocation of Kelly *et al.*. This result suggests that price-based rate control scheme proposed by Kelly *et al.* is also effective when users behave in a selfish manner.**

I. Introduction

An important problem in packet-based computer networks is *rate control*, *i.e.* the regulation of the data transmission rates of individual users. The two main objectives of rate control are:

(a) *Congestion Control:* The aggregated traffic load on each link in the network should not exceed the link capacity.
(b) *Fairness:* The available network bandwidth should be shared fairly among users.

In recent years, price-based rate control schemes have received much attention [3], [5], [6], [7], [8]. Under these schemes, the network charges users a price per unit transmission rate that depends on the current traffic load at individual links. Users then adapt their transmission rates to balance the gain they obtain from accessing the network, which is characterized by a utility function, and the cost they have to pay to the network. Price-based rate control schemes have the attractive feature that they can be implemented in a distributed fashion. Furthermore, they can be used to achieve several bandwidth sharing (fairness) criteria such as proportional fair, max-min fair, and socially optimal allocations. Much of the work on price-based rate control has been motivated by Kelly *et al.* who proposed in [3], [4] a particularly elegant price-based rate allocation mechanism. Kelly *et al.* show that this mechanism converges (under suitable assumptions) to a socially optimal rate allocation [4]. To establish their convergence result, Kelly *et al.* assume that users voluntarily adopt a mandated rate adaptation algorithm similar to the TCP congestion control algorithm. Ganesh *et al.* study in [1], [2] the pricing scheme proposed by Kelly [4] for users motivated by self-interests rather than by mandate. For the single link case, Ganesh *et al.* show that the system converges to the same rate allocation as in [4]. However, the analysis by Ganesh *et al.* assumes a specific class of users' utility and price functions, and requires a certain degree of coordination among users (we will comment on this in more detail in Section II-C).

The contributions of this paper are the following. First, we extend the results by Ganesh *et al.* and show that only some mild assumptions are needed to establish that the pricing scheme proposed by Kelly [4] converges to a socially optimal rate allocation even when users behave selfishly. Second, the proof technique is of independent interest as it can be used for other allocation schemes. Currently, we are applying the technique to a price-based rate control scheme for cellular wireless networks and a price-based priority scheme for wired networks.

The rest of the paper is organized as follows. In Section II, we present the link and user models, and introduce the user adaptation problem. In Section III, we present a numerical case study to illustrate our results. In Section IV, we provide the proof of the main result.

II. Problem Formulation

Consider a discrete-time model of a single link, where time is divided into slots of equal length. The link has a fixed transmission capacity of C units of data traffic per time slot. There is no buffering and data packets which are not transmitted in a given time slot are dropped. Assume that N users share the link and let $x_i(t)$ be the amount of data user i submits in time slot t. We refer to $x_i(t)$ as the *transmission rate* of user i in time slot t. Let $x(t)$ be the *aggregated transmission rate* in time slot t given by

$$x(t) = \sum_{i=1}^{N} x_i(t). \tag{1}$$

The link provider charges users a price $p(t)$ per unit amount of traffic submitted in time slot t given by

$$p(t) = \phi\left(x(t)\right), \tag{2}$$

where $\phi(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a predefined price function which depends on the aggregated transmission rate $x(t)$. We assume that the price $p(t)$ is set and announced to all the users at the end of time slot t. We also make the following assumption.

Assumption 1: The price function $\phi(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing. Furthermore, there exists a constant $L_\phi > 0$ such that $|\phi(x) - \phi(y)| \leq L_\phi |x - y|$, $\forall x, y \in \mathbb{R}_+$.

Typically, the price function $\phi(\cdot)$ increases sharply as the aggregated transmission rate x approaches C, thus preventing that the aggregated rate $x(t)$ exceeds the capacity.

The Lipschitz constant L_ϕ ensures that the derivative of the price function $\phi(\cdot)$ is bounded. This technical assumption is needed to establish our convergence result.

A. User Optimization Problem

We associate with each user i, $i = 1, ..., N$, a utility $U_i(x_i)$ which depends on the transmission rate x_i of user i. We make the following assumption.

Assumption 2: The utility function $U_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ of user i, $i = 1, ..., N$, is increasing, strictly concave and differentiable. Furthermore, there exists $L_D > 0$ such that

$$|U_i^{'-1}(p) - U_i^{'-1}(q)| \leq L_D |p - q|, \qquad \forall p, q \in \mathbb{R}_+,$$

where $U_i^{'-1} : [0, U_i'(0)] \mapsto \Re_+$ is the inverse of the derivative of U_i.

We note that utility functions with these characteristics are commonly used in the pricing literature (see for example [3]). Assumption 2 does not require all users have the same utility function. We assume that the utility function U_i is only known to user i, and unknown to all other users and the link provider.

Assumption 2 implies that user i can achieve a high utility by choosing a high transmission rate x_i. However, doing so will increase the total cost that user i has to pay to the link provider. Accordingly, we assume that user i chooses its transmission rate x_i such that

$$x_i = \arg\max_{y \geq 0} \{U_i(y) - yp\}, \qquad p \geq 0, \tag{3}$$

where p is the price that link provider charges per unit transmission rate. Let $D_i(p)$ be the solution to the above optimization problem (which exists by Assumption 2) given by

$$D_i(p) = \begin{cases} U_i^{'-1}(p) & p \in [0, U_i'(0)] \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The function D_i can be interpreted as the demand function of user i, *i.e.* $D_i(p)$ is the amount of traffic user i would submit per time slot when the price per unit traffic is equal to p. Then the following lemma holds.

Lemma 1: Under Assumption 2, there exists a constant $L > 0$ such that $|D_i(p) - D_i(q)| \leq L|p - q|$, $\forall p, q \in \mathbb{R}_+$.

The above lemma follows from Assumption 2. We omit a detailed proof.

B. User Adaptation

If user i knew in advance the price $p(t)$ charged during time slot t, then user i would choose transmission rate

$$x_i(t) = D_i(p(t)),$$

as this rate maximizes the net benefit $U_i(x_i(t)) - x_i(t)p(t)$. Unfortunately, this is not possible as price $p(t)$ is set and announced to all users at the end of time slot t.

Alternatively, user i could form an estimate of the price $p(t)$ and use this estimate to determine the transmission rate for time slot t. Here, we consider the following iterative procedure for user i to obtain an estimate $\hat{p}_i(t)$ of the price in time slot t,

$$\hat{p}_i(t) = \hat{p}_i(t-1) + \alpha_i(p(t-1) - \hat{p}_i(t-1)), \tag{5}$$

where $\hat{p}_i(t)$ is the price estimate of user i for time slot t, $\alpha_i > 0$ is a small step size parameter, and $p(t-1)$ is the price of the time slot $t-1$ (which is available at the beginning of time slot t).

Using estimate $\hat{p}_i(t)$, user i sets its transmission rate for time slot t equal to

$$x_i(t) = D_i(\hat{p}_i(t)) = \arg\max_{z \geq 0} \{U_i(z) - z\hat{p}_i(t)\}.$$

Our goal is to analyze this adaptation model. In particular, we wish to determine whether (a) the transmission rate $x_i(t)$ of user i converges to an equilibrium rate x_i^*, (b) the price estimates of all users converge to a common estimate $\hat{p}^*$, and (c) how the link capacity C is shared among users in equilibrium. We have the following result.

Proposition 1: Under Assumption 1 and 2 the following holds. If for all users $i = 1, ..., N$, we have that

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\},$$

then the users' price estimates $\hat{p}_i$ converge to the common price p^*, *i.e.*

$$\lim_{t \to \infty} \hat{p}_i(t) = p^*, \qquad i = 1, ..., N,$$

where p^* is a unique price such that

$$p^* = \phi\left(\sum_{i=1}^{N} D_i(p^*)\right).$$

Furthermore, the transmission rate of user i converges to $D_i(p^*)$, *i.e.* $\lim_{t \to \infty} x_i(t) = D_i(p^*)$, $i = 1, ..., N$.

We provide a proof of Proposition 1 in Section IV.

As pointed in [1], [2], [3], Proposition 1 implies that the above price-based allocation scheme approximates a socially optimal rate allocation. For a more detailed discussion of the connection between the above price-based mechanism and concepts of economic theory, we refer to the excellent exposition by Ganesh *et al.* in [1], [2].

C. Related work

Ganesh *et al.* prove Proposition 1 for the special case where the users' utility functions are of the form

$$U_i(x) = k_i \frac{x^\beta - 1}{\beta},$$

where k_i and β, $0 < \beta < 1$, are a positive constants; and the price function $\phi(x)$ is of the form

$$\phi(x) = \left(\frac{x}{K}\right)^k ,$$

where K and k are positive constants. In addition, Ganesh *et al.* require that all users start with the same initial estimate $\hat{p}(0)$, *i.e.*

$$\hat{p}_i(0) = \hat{p}(0), \qquad i = 1, ..., N,$$

and that all users employ the same step size parameter α when updating their price estimates $\hat{p}_i(t)$, *i.e.* $\alpha_i = \alpha$, $i = 1, ..., N$. Under these assumption, the users' price estimates are identical for each time slot t. Note that these assumptions are restrictives; in particular the assumption that the price estimates of individual users are synchronized is not realistic.

In this paper, we extend the results by Ganesh *et al.* by allowing users to have different initial price estimates $\hat{p}_i(0)$ and various step size parameters. In addition, we consider a more general class of utility and price functions as characterized by Assumption 1 and Assumption 2 respectively.

III. Numerical Case Study

Before we provide a proof of Proposition 1, we illustrate in this section the convergence result of Proposition 1 through a case study.

We consider the situation where in total 100 users access a single link with capacity equal to 1. User demand functions are given by

$$D_i(p(t)) = k_i \cdot e^{-b_i \cdot p(t)}, \qquad i = 1, ...N, \tag{6}$$

where k_i, and b_i, are positive constants chosen at random according to a uniform distribution on $[0.05, 0.25]$, and $[0.5, 0.55]$, respectively. The step size parameters α_i, $i = 1, ..., N$, and the initial price estimates $\hat{p}_i(0)$, are chosen at random according to a uniform distribution on $[0.01, 0.11]$, and $[2, 5]$, respectively. The price function $\phi(x)$ (see Figure 1 for an illustration) is given by

$$\phi(x) = \begin{cases} \min\{10, 0.8\cdot(1-x)^{-0.8} - 0.8\}, & \text{if } x<10, \\ 10, & \text{otherwise.} \end{cases} , \tag{7}$$

Figure 2 shows that the price estimates of individual users and price p(t) converge to the equilibrium price $p^* = 5.4$, and the transmission rate $x_i(t)$ of user i converges to $D_i(p^*)$. Figure 2 also illustrates that the convergence rate depends on the users' step size parameters. To highlight this, Figure 3 shows the trajectories of the transmission rates of two users which have the same demand function, but the step size parameter of one user is equal to 0.015, whereas the step size parameter of the other user is equal to 0.1. We note that the transmission rate of the user with the larger step size converges faster,

IV. Convergence Analysis

In this section, we provide a proof of Proposition 1. We start out by deriving two preliminary lemmas.

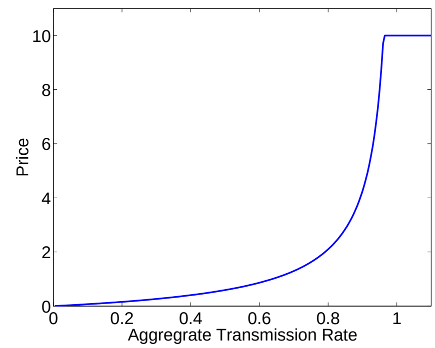

Fig. 1. Price function $\phi(x)$ which depends on the aggregated transmission rate x .

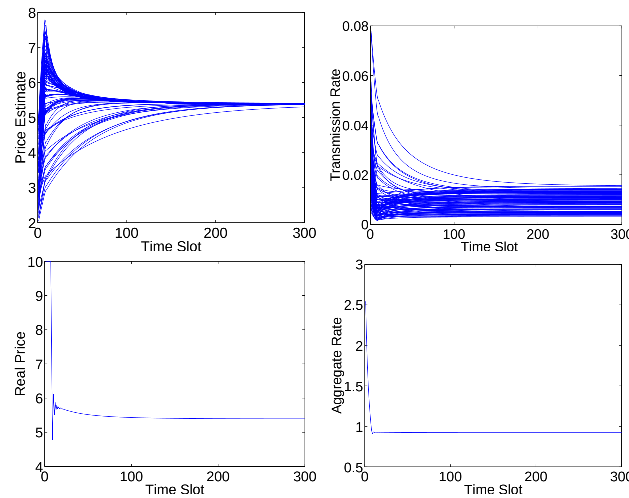

Fig. 2. Trajectories of the users' price estimates $\hat{p}_i(t)$ (top left) and link price $p(t)$ (bottom left), as well as the trajectories of the transmission rates $x_i(t)$ (top right) and the aggregated transmission rate $x(t)$ (bottom right).

A. Preliminary Lemmas

Recall that the transmission rate of user i in time slot t is given by

$$x_i(t) = D_i\big(\hat{p}_i(t)\big),$$

and we can rewrite (2) as

$$p(t) = \phi\left(\sum_{i=1}^{N} D_i(\hat{p}_i(t))\right). \tag{8}$$

Let the function $\Phi(\cdot) : \mathbb{R}_+^N \mapsto \mathbb{R}_+$ be given by

$$\Phi(\hat{\mathbf{p}}(t)) = \phi\left(\sum_{i=1}^{N} D_i(\hat{p}_i(t))\right),$$

where $\hat{\mathbf{p}}(t) = (\hat{p}_1(t), ..., \hat{p}_N(t)) \in \Re_+^N$ is the vector with the price estimates of the individual users. We then have the following lemma.

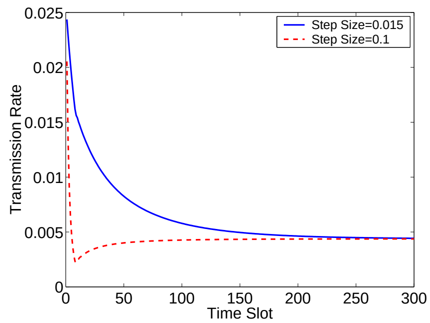

Fig. 3. Trajectories of the sending rates of two users with the same demand function, but different step size parameters.

Lemma 2: Under Assumption 1 and 2, the function $\Phi(\hat{\mathbf{p}})$ has the following properties.

(a) The function $\Phi(\hat{\mathbf{p}})$ is continuous and strictly decreasing in $\hat{p}_i$ on $[0, U_i^{'-1}(0)]$, for all $i \in \{1, ..., N\}$.

(b) We have

$$|\Phi(\mathbf{p}) - \Phi(\mathbf{q})| \leq L_\phi L_D \sum_{i=1}^{N} |p_i - q_i|, \ \mathbf{p}, \mathbf{q} \in \mathbb{R}_+^N,$$

where L_ϕ and L_D are the constants from Assumption 1 and 2.

Proof: (a) As functions $\phi(x)$ and $D(p)$ are continuous, function $\Phi(\hat{\mathbf{p}})$ is continuous. Assumption 2 implies that the demand function D_i, $i = 1, ..., N$, is strictly decreasing on $[0, U_i^{'-1}(0)]$ and, by Assumption 1, the function $\phi(x)$ is strictly increasing. Combining these facts, we obtain that for every user $i = 1, ..., N$, function $\Phi(\hat{\mathbf{p}})$ is continuous, and strictly decreasing in $\hat{p}_i$ on $[0, U_i^{'-1}(0)]$.

(b) Given two vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^N$, we have

$$\begin{aligned} |\Phi(\mathbf{p}) - \Phi(\mathbf{q})| &\leq \left|\phi(\textstyle\sum_{i=1}^N D_i(p_i)) - \phi(\sum_{i=1}^N D_i(q_i))\right| \\ &\leq L_\phi \left|\textstyle\sum_{i=1}^N D_i(p_i) - \sum_{i=1}^N D_i(q_i)\right| \\ &= L_\phi \left|\textstyle\sum_{i=1}^N (D_i(p_i) - D_i(q_i))\right| \\ &\leq L_\phi L_D \textstyle\sum_{i=1}^N |p_i - q_i| . \end{aligned}$$

■

The next lemma implies that there exists a unique equilibrium price p^*.

Lemma 3: Under Assumption 1 and 2, there exists a unique price $p^* \geq 0$ such that

$$p^* = \Phi\big((p^*, ..., p^*)\big). \tag{9}$$

Furthermore, p^* is positive.

Proof: Let function $\hat{\Phi}(p)$, $p \in \Re_+$, be given by

$$\hat{\Phi}(p) = \Phi((p, ..., p)).$$

In addition, let

$$p_{\max} = \max\left\{U_1^{-1}(0), ..., U_N^{-1}(0)\right\}.$$

Lemma 2 implies that $\hat{\Phi}(p)$ is continuous and strictly decreasing on $[0, p_{\max}]$. Furthermore, $\hat{\Phi}(p)$ is equal to 0 for all $p \geq p_{\max}$. Combining these results, we obtain that there exists a unique p^* such that $p^* = \Phi\big((p^*, ..., p^*)\big)$, and p^* is positive. ■

B. Proof of Proposition 1

Let $\hat{p}_{min}(t)$ and $\hat{p}_{max}(t)$ be given by

$$\hat{p}_{min}(t) = \min_i \hat{p}_i(t),$$

and

$$\hat{p}_{max}(t) = \max_i \hat{p}_i(t).$$

Furthermore, let $P_{min}(t)$ and $P_{max}(t)$ be given by

$$P_{min}(t) = \min\{\hat{p}_{min}(t), p(t)\}$$

and

$$P_{max}(t) = \max\{\hat{p}_{max}(t), p(t)\}.$$

To prove Proposition 1, it suffices to show that

$$\lim_{t\to\infty} \Big(P_{max}(t) - P_{min}(t)\Big) = 0. \tag{10}$$

Indeed, by combining the above equation with Lemma 2 and Lemma 3 we obtain that

$$\lim_{t\to\infty} p(t) = p^*$$

and

$$\lim_{t\to\infty} \hat{p}_i(t) = p^*, \qquad i = 1, ..., N,$$

where p^* is the unique price such that $p^* = \Phi\big((p^*, ..., p^*)\big)$. Proposition 1 then follows. The next lemma establishes Eq. (10).

Lemma 4: Under Assumption 1 and 2, if

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\}, \qquad i = 1, ..., N,$$

then there exists a constant γ, $\gamma < 1$, such that for every $t \geq 0$, we have

$$P_{max}(t+1) - P_{min}(t+1) < \gamma\Big(P_{max}(t) - P_{min}(t)\Big).$$

Proof: To prove the above lemma, we distinguish two separate cases. First, we consider the case where all users either under-estimate or over-estimate, price $p(t)$, *i.e.* we have

$$p(t) \geq \hat{p}_{max}(t) \quad \text{ or } \quad p(t) \leq \hat{p}_{min}(t).$$

Second, we consider the situation where some users underestimate and some users over-estimate price $p(t)$, *i.e.* we have

$$\hat{p}_{min}(t) < p(t) < \hat{p}_{max}(t).$$

CASE 1: Suppose that at time slot t we have

$$p(t) \geq \hat{p}_{max}(t).$$

This implies that

$$p(t) - \hat{p}_i(t) \geq 0, \qquad i = 1, ..., N,$$

and we obtain

$$\hat{p}_i(t+1) = \hat{p}_i(t) + \alpha_i\big(p(t) - \hat{p}_i(t)\big) \geq \hat{p}_i(t).$$

Since for every user $i = 1, ..., N$ the step size parameter α_i is strictly less than 1, we obtain from the above equation that

$$\hat{p}_i(t+1) < p(t). \tag{11}$$

Combining this result with Lemma 2, which states that $\Phi((\hat{p}_1, ..., \hat{p}_N))$ is non-increasing in $\hat{p}_i$, $i = 1, ...N$, we obtain that

$$p(t+1) \leq p(t). \tag{12}$$

Furthermore, by Lemma 2 we have

$$\begin{aligned} &|p(t+1) - p(t)| \\ &\leq \Big|\phi(\textstyle\sum_{i=1}^N D_i(\hat{p}_i(t+1))) - \phi(\sum_{i=1}^N D_i(\hat{p}_i(t)))\Big| \\ &= L_\phi L_D \textstyle\sum_{i=1}^N |\alpha_i(p(t) - \hat{p}_i(t))| \\ &< L_\phi L_D \alpha_{max} \textstyle\sum_{i=1}^N |p(t) - \hat{p}_i(t)| \\ &< L_\phi L_D N \alpha_{max} |p(t) - \hat{p}_{min}(t)|, \end{aligned} \tag{13}$$

where

$$\alpha_{max} = \max\{\alpha_1, ..., \alpha_N\}.$$

Combining (11) and (12), we obtain that

$$P_{max}(t+1) \leq p(t).$$

Furthermore, if

$$p(t+1) < \hat{p}_{min}(t+1),$$

then we obtain from (13) and Lemma 2 that

$$\frac{P_{max}(t+1) - P_{min}(t+1)}{P_{max}(t) - P_{min}(t)} < \frac{p(t) - p(t+1)}{p(t) - \hat{p}_{min}(t)} < L_\phi L_D N \alpha_{max}.$$

If

$$p(t+1) \geq \hat{p}_{min}(t+1),$$

then we have

$$\begin{aligned} p(t) &- \hat{p}_i(t+1) \\ &= p(t) - \Big[\hat{p}_i(t) + \alpha_i(p(t) - \hat{p}_i(t))\Big] \\ &\leq p(t) - \Big[\hat{p}_{min}(t) + \alpha_{min}\big(p(t) - \hat{p}_{min}(t)\big)\Big] \\ &= (1 - \alpha_{min})\big(p(t) - \hat{p}_{min}(t)\big), \end{aligned}$$

and

$$\frac{P_{max}(t+1) - P_{min}(t+1)}{P_{max}(t) - P_{min}(t)} \leq \frac{p(t) - \hat{p}_{min}(t+1)}{p(t) - \hat{p}_{min}(t)} \leq 1 - \alpha_{min}.$$

Combining the above results, we obtain that for

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\}, \qquad i = 1, ..., N,$$

and

$$p(t) \geq \hat{p}_{max}(t),$$

that there exists a constant γ', $\gamma' < 1$, such that

$$P_{max}(t+1) - P_{min}(t+1) \leq \gamma'\Big(P_{max}(t) - P_{min}(t)\Big).$$

Similarly, one can show that if

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\}, \qquad i = 1, ..., N,$$

and

$$p(t) \leq \hat{p}_{min}(t),$$

then there exists a positive constant γ'', $\gamma'' < 1$, such that

$$P_{max}(t+1) - P_{min}(t+1) \leq \gamma''\Big(P_{max}(t) - P_{min}(t)\Big).$$

Let

$$\gamma_1 = \max\{\gamma', \gamma''\}.$$

Note that γ_1 is strictly less than 1.

CASE(2): Suppose that

$$\hat{p}_{min}(t) < p(t) < \hat{p}_{max}(t).$$

For this case, we consider the following three sub-cases. First, if

$$\hat{p}_{min}(t+1) < p(t+1) < \hat{p}_{max}(t+1),$$

then we obtain that

$$\hat{p}_{max}(t+1) \leq \hat{p}_{max}(t) + \alpha_{min}\Big(p(t) - \hat{p}_{max}(t)\Big)$$

and

$$\hat{p}_{min}(t) \geq \hat{p}_{min}(t) + \alpha_{min}\Big(p(t) - \hat{p}_{min}(t)\Big).$$

Combining the above results, we obtain that

$$\begin{aligned} &\frac{P_{max}(t+1) - P_{min}(t+1)}{P_{max}(t) - P_{min}(t)} \\ &\leq \frac{\hat{p}_{max}(t) + \alpha_{min}\Big(p(t) - \hat{p}_{max}(t)\Big)}{\hat{p}_{max}(t) - \hat{p}_{min}(t)} - ... \\ &\quad - \frac{\hat{p}_{min}(t) + \alpha_{min}\Big(p(t) - \hat{p}_{min}(t)\Big)}{\hat{p}_{max}(t) - \hat{p}_{min}(t)} \\ &= (1 - \alpha_{min})\frac{\hat{p}_{max}(t) - \hat{p}_{min}(t)}{\hat{p}_{max}(t) - \hat{p}_{min}(t)} \\ &= 1 - \alpha_{min}. \end{aligned}$$

Next, if

$$p(t+1) \geq \hat{p}_{max}(t+1),$$

then we have

$$p(t+1) > p(t).$$

Let the sets N_+ and N_- be given as follows,

$$N_+ = \{i : \hat{p}_i(t) \geq p(t)\}$$

and

$$N_- = \{i : \hat{p}_i(t) < p(t)\}.$$

Note that this implies that

$$\hat{p}_i(t+1) - \hat{p}_i(t) \geq 0, \qquad i \in N_+,$$

and

$$\hat{p}_i(t+1) - \hat{p}_i(t) < 0, \qquad i \in N_-.$$

Therefore, we have

$$\begin{aligned} p(t+1) - p(t) \\ &= \phi\left(\sum_{i=1}^{N} D_i(\hat{p}_i(t+1))\right) - \phi\left(\sum_{i=1}^{N} D_i(\hat{p}_i(t))\right) \\ &< L_\phi \left|\sum_{i=1}^{N}\Big(D_i(\hat{p}_i(t+1)) - D_i(\hat{p}_i(t)\Big)\right| \\ &< L_\phi \left|\sum_{i\in N_+}\Big(D_i(\hat{p}_i(t+1)) - D_i(\hat{p}_i(t)\Big) + \ldots \right. \\ &\left. \sum_{i\in N_-}\Big(D_i(\hat{p}_i(t+1)) - D_i(\hat{p}_i(t)\Big)\right| \\ &\leq L_\phi \left|\sum_{i\in N_+}\Big(D_i(\hat{p}_i(t+1)) - D_i(\hat{p}_i(t)\Big)+\right| \\ &< L_\phi L_D \sum_{i\in N_+}\Big|\alpha_i\big(\hat{p}_{max}(t) - p(t)\big)\Big| \\ &< L_\phi L_D N \alpha_{max}\big|\hat{p}_{max}(t) - p(t)\big|, \end{aligned}$$

where we used the fact that for every user $i = 1, ..., N$, the demand function D_i is non-increasing. Recall that $P_{max}(t+1) = p(t+1)$. Combining the above results, we obtain that

$$\begin{aligned} P_{max}(t+1) - P_{min}(t+1) \\ &\leq \big|p(t+1) - p(t)\big| + \big|p(t) - P_{min}(t+1)\big| \\ &< L_\phi L_D N \alpha_{max}\big|p(t) - \hat{p}_{max}(t)\big| + \ldots \\ &\quad +(1-\alpha_{min})\big|p(t) - P_{min}(t)\big|. \end{aligned}$$

This implies that for

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\}, \qquad i = 1, ..., N,$$

and $p(t+1) \geq \hat{p}_{max}(t+1)$, there exists a constant γ', $\gamma' < 1$, such that

$$\begin{aligned} P_{max}(t+1) - P_{min}(t+1) \\ &< \gamma'\Big(\big|p(t) - \hat{p}_{max}(t)\big| + \big|p(t) - P_{min}(t)\big|\Big). \end{aligned}$$

Note that when

$$\hat{p}_{min}(t) < p(t) < \hat{p}_{max}(t),$$

then we have

$$P_{max}(t) - P_{min}(t) = \Big(\hat{p}_{max}(t) - p(t)\Big) + \Big(p(t) - P_{min}(t)\Big).$$

Combining the two equations above, we obtain that

$$P_{max}(t+1) - P_{min}(t+1) < \gamma'\Big(P_{max}(t) - P_{min}(t)\Big).$$

Similarly, one can show that for

$$0 < \alpha_i < \min\left\{1, \frac{1}{L_\phi L_D N}\right\}, \qquad i = 1, ..., N,$$

and

$$p(t+1) \leq \hat{p}_{min}(t+1),$$

there exists a constant γ'', $\gamma'' < 1$, such that

$$P_{max}(t+1) - P_{min}(t+1) \leq \gamma''\Big(P_{max}(t) - P_{min}(t)\Big).$$

Let

$$\gamma_2 = \max\big\{1 - \alpha_{min}, \gamma', \gamma''\big\}.$$

Combining the results that we obtained for Case 1 and Case 2, the lemma follows for

$$\gamma = \max\big\{\gamma_1, \gamma_2\big\}.$$

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V. REFERENCES

[1] A. Ganesh, K. Laevens, and R. Steinberg *"Dynamics of Congestion Pricing,"* Technical Report, Microsoft Research, June 2000.

[2] A. Ganesh, K. Laevans, and R. Steinberg *"Congestion Pricing and User Adaptation,"* In Proceedings of IEEE INFOCOM, pages 959–965, Anchorage, AK, April 2001.

[3] F. Kelly, *"Charging and Rate Control for Elastic Traffic,"* European Transactions on Telecommunications, vol. 8, no. 1, 1997, pp. 33-37.

[4] F. Kelly, A. Maulloo, and D. Tan *"Rate Control in Communication Networks: Shadow Prices, Proportional Fairness and Stability,"* Journal of the Operational Research Society 49 (1998) 237-252.

[5] R. J La and V. Anantharam, *"Utility-Based Rate Control in the Internet for Elastic Traffic,"* IEEE/ACM Transactions on Networking, vol. 10, no. 2, 272-286, 2002.

[6] S. H. Low and D. E. Lapsley, *"Optimization Flow Control, I: Basic Algorithm and Convergence,"* IEEE/ACM Transactions on Networking, vol. 7, no. 6, pp. 861–875.

[7] J. Mo and J. Walrand, *"Fair end-to-end window-based congestion control,"* In Proceedings of SPIE '98 Performance and Control of Network Systems II, pages 55-63, November 1998.

[8] H. Yaiche, R. Mazumdar, and C. Rosenberg, *"A Game-Theoretic Framework for Rate Allocation and Charging of Elastic Connections in Broadband Networks,"* IEEE/ACM Transactions on Networking, vol. 8, no. 5, pp. 667-678, 2000.