The beginnings of a logical semantics framework for the integration of thematic map data

NIC WILSON*
Department of Computer Science
University of Keele, Staffordshire, UK

Abstract

The integration of spatial datasets from different sources is becoming an increasingly important issue. It is very desirable to have a rigorous approach to integration, as an *ad hoc* approach can easily lead to incorrect inferences. This paper takes a formal approach, giving the beginnings of a logical semantics framework which allows meaning to be defined mathematically for spatial datasets which represent certain types of thematic map data. The basic idea is to interpret the datasets as summarisations of spatial variables, which are in turn interpreted as sets of possible worlds (ways the world could be). This semantics approach can give a formal meaning to pairs (or sets) of such datasets, which can then be used to determine the valid inferences from an integrated dataset.

1 Introduction

The importance of being able to integrate spatial information from a number of sources is widely acknowledged. Potential advantages include (a) more general conclusions: for example, if together there is greater spatial coverage; (b) more reliable inferences: if two datasets support the same inference then

*Current address: Cork Constraint Computation Centre, Department of Computer Science, University College Cork, Cork, Ireland; e-mail: n.wilson@4c.ucc.ie.

this may suggest that we can trust the inference more; and, most relevant to the situations analysed in this paper, (c) new inferences: together, a pair of datasets can support an inference that neither individually supports; for example, land cover data at different times can suggest change in land cover.

This paper considers a particular kind of spatial dataset: one that can be viewed as being a function from a set of locations (areal units or points) to a set of possibilities representing the values of some variable (or set of variables). We call these functions 'maps', even though they may not be 'drawable'; however (the datasets represented by) many standard thematic maps can be viewed in this way, in particular choropleth maps, and also contour maps and other isarithmic maps (Dent, 1996). Generally the types of 'maps' (spatial datasets) we are talking about are more naturally represented as fields rather than as containing entities or objects (see e.g., discussion in (Mark, 1999)).

This paper focuses on the inferences supported by a dataset or set of datasets. From this perspective, the key issue for integration or fusion is determining what can be deduced from a pair (or, more generally, a set) of datasets, rather than focusing on the form of the representation. This is strongly related to the issue of what the dataset means. For example, suppose a map assigns land cover classes to a set of regions, and that, in particular, it assigns the class 'grassland' to a certain region g. We are interested in a smaller region g' which is fully contained in g. What can we say (if anything) about the land cover for g'? Clearly we cannot hope to adequately answer this question without going into what the map (spatial dataset) means, which depends on how the data was produced; for example, was the label 'grassland' some sort of average value over the region? Or does it mean that all of g is grassland?

The approach taken here is to define a formal semantics, along the same lines as for a formal logic, and a formal definition of what a query is. The basic idea is to imagine a set of possible 'worlds', each one representing a possible way that reality could be. A spatial dataset is generated by some process from the real world. Knowing that map M is produced, as opposed to another map, tells us something about the world: it restricts the set of possible worlds to a set which we write as $[M]$. Understanding precisely the process that was used to generate the dataset can therefore lead to a semantics.

A query asks something about the way the world is, and expects an answer; for example, the query *What is the average height (in metres) above sea level in region g?* expects a number as an answer. So a query can be viewed as a function from the set of worlds to a set of answers. Knowing maps M and N, and so that the world is in $[M] \cap [N]$, may enable us to say that the query has a unique answer, or it may otherwise constrain the possible correct answers to the query.

A map M is assumed to be telling us about some underlying spatial variable, for example the temperature at a given moment in time, which may vary continuously over an area. The map will usually represent a summarised form of the information about the variable. What M tells us is that the underlying variable is such that when summarised it gave the map M (for the case where the map was based on error-free data); M can then be considered as semantically equivalent to the disjunction of all underlying maps of the variable that would have been summarised to M rather than another map.

Such summarisation (generalisation) operators can be complex; however, there are certain simple operators that are used in the production of many spatial datasets, in particular what we call themes-coarsening and various types of spatial coarsening. Themes-coarsening involves generating a map which has a coarser division of the possible values of the variable, e.g. classifying values of temperature into ranges. Spatial-coarsening involves generating a map based on a coarser division of space, for example, dividing the area of the map into larger spatial units, and computing the mean value of the variable over the larger units.

In this paper, we consider the integration of spatial datasets which are assumed to have arisen from *unknown* spatial variables by application of *known* summarisation operators. The focus of the integration is answering queries from the combined dataset. The following example illustrates the approach.

Example

Suppose we have a spatial dataset N giving information about the total rainfall during the year 2002 averaged over particular regions. More precisely, the area of interest D is divided up into three regions g_1, g_2 and g_3 of equal area. N is defined by $N(g_1) = 210$, $N(g_2) = 310$ and $N(g_3) = 260$, where, for example, the latter means that the year's rainfall, averaged over g_3, is 260mm. N is a summary of a particular spatial variable M, which gives the rainfall values over D. For example, $N(g_1)$ is the mean value of M over g_1. N

can be written as $\mu(M)$, where μ is this averaging operator; in this paper we call summarisation operator μ, *mean-coarsening* (see section 4.1.1), which is a particular type of spatial coarsening (section 4): it takes a map M of finer resolution and produces a map N of coarser resolution.

Map N is assumed to give entirely accurate information: for example that the mean rainfall over g_1 is exactly 210mm and not 211mm. Throughout most of the paper, it is assumed that such maps (spatial datasets) are entirely accurate. Though this is obviously unrealistic, it can sometimes be a workable approximation. Also one should know how to deal with the case of certain information before one can attempt to deal with uncertain information, as the latter case should extend the certain information case. Uncertainty is briefly discussed in section 8.

Map N restricts the possible ways the world could be. In the terminology of this paper, a *'world'* (a formal object) represents a particular way reality could be. Associated with each world w is then a spatial variable M_w which gives the distribution of rainfall over D, given that w correctly describes reality. World w is said to satisfy N if they are consistent. In particular, if they are indeed consistent then the average rainfall over g_1 in w is 210mm. In fact, w satisfies N if and only if $\mu(M_w) = N$. This idea is developed into a formal semantics in section 5.

We want to be able to deduce things from a dataset. This is expressed using a notion of query, which is formalised in section 6. Fundamentally, a query is a function from the set of worlds to a set of answers A. For example, with the query q_1: *What is the mean rainfall over* $g_1 \cup g_2$*?*, the set of possible answers A is the set of possible values of rainfall. $q_1(w) = 260$ if, given that w correctly describes reality, the average rainfall over $g_1 \cup g_2$ is 260mm, i.e., the mean of M_w over $g_1 \cup g_2$ is 260mm. Queries are applied to maps to ascertain what the map tells us about that issue. $q_1(N)$ is defined to be the set of possible answers to q_1 given that world w satisfies N, i.e., $\{q_1(w) : w \in [N]\}$, where $[N]$ is the set of worlds satisfying N. In this case, the map N uniquely determines the answer to the query: $q_1(N) = \{260\}$ since, for any world w consistent with N, the mean value of rainfall over $g_1 \cup g_2$ is 260mm; this is because N tells us that the mean rainfall over g_1 is 210mm, and over g_2 is 310mm, and so, the mean rainfall over $g_1 \cup g_2$ must be 260mm as g_1 and g_2 have equal area. Using more formal language, if w satisfies N then $\mu(M_w) = N$, so M_w averaged over g_1 is equal to $\mu(M_w)(g_1) = N(g_1) = 210$. Similarly, M_w averaged over g_2 equals 310. Therefore $q_1(w)$, which equals M_w averaged over $g_1 \cup g_2$, equals 260 for any $w \in [N]$.

More generally, a map may not completely determine the answer to a query, but may restrict the possible answers. Let h_1 be a region consisting of g_1 and half of g_3. Consider the query q_2: *What is the mean rainfall over h_1?*. Suppose that we also know that over D the rainfall values are all in the range $[0, 400]$. Map N does not determine a unique answer to this query. It can be shown that $q_2(N)$, the set of possible answers to the query given N, is the range of values $[180, 273\frac{1}{3}]$. (This follows because the range of possible values for the mean rainfall over $h_1 \cap g_3$ is $[120, 400]$, with the lower bound 120 coming from the condition that the mean rainfall over g_3 is 260mm.) So for any number t between 180 and $273\frac{1}{3}$ there exists some world w in which the mean rainfall (the mean of the variable M_w) over h_1 is equal to t.

An *ad hoc* approach might report a value of $226\frac{2}{3}$ to q_2, given map M_1, but this is not justified by the data. It would be valid, for example, if the rainfall values over g_3 were uniform, but we have no reason to assume this.

Now suppose that we receive another entirely accurate map P. Let h_2 be $D - h_1$, so that h_2 consists of g_2 and (the other) half of g_3. Map P gives imprecise information about the mean rainfall values for h_1 and h_2: $P(h_1) = [200, 300)$ and $P(h_2) = [300, 400]$. P tells us that the mean rainfall over h_1 is at least 200mm and less than 300mm, and that the mean rainfall value over h_2 is at least 300mm and at most 400mm.

Map P can be considered as the result of applying two summarisation operators to the underlying (and unknown) spatial variable M. Firstly, a mean-coarsening operator: computing the mean values over h_1 and h_2. Then, classifying the values into ranges; this second operator is called here 'themes-coarsening', and is described in section 3. The worlds w consistent with P are those which result in P, after applying this sequence of two summarisation operators to variable M_w.

The worlds satisfying both maps are those in $[N] \cap [P]$, so the possible answers to query q_2 given both maps is the set of all values $q_2(w)$ for $w \in [N] \cap [P]$, which is equal to the range of values $[200, 220]$; this is because the first map tells us that the mean rainfall over D ($= h_1 \cup h_2 = g_1 \cup g_2 \cup g_3$) is 260, so a value for h_1 of more than 220 would lead to a value for h_2 of less than 300, contradicting the information given by map P.

Note that both maps together give a much stronger inference than either of the maps individually, with the (inferences from the) whole greater than the sum of (inferences from) the parts: the map M on its own implies just that the mean rainfall over h_1 is in the range $[180, 273\frac{1}{3}]$; map P on its own implies just that it is in the range $[200, 300)$; together the two maps imply that the mean rainfall over h_1 is in the range $[200, 220]$. This illustrates the importance of a rigorous approach to integration; a careless integration of the datasets M and P might only give the inference that the value for h_1 is in the range $[200, 273\frac{1}{3}]$, taking the intersection of the two ranges, when in fact a much stronger upper bound is implied.

Now consider the situation where instead of P we received map P'. Map P' is apparently very similar to P, with $P'(h_1) = [200, 300)$ and $P'(h_2) = [300, 400]$, the same ranges as for P. The difference is in the meaning: $P'(h_1) = [200, 300)$ means that the rainfall value throughout h_1 is in the range $[200, 300)$, and similarly for $P'(h_2) = [300, 400]$: any location in h_2 has a rainfall value between (or equal to) 300mm and 400mm. Map P' might be considered as the result of applying two summarisation operators to the underlying spatial variable M. Firstly, classifying the rainfall values into ranges $[0, 200)$, $[200, 300)$ and $[300, 400]$ (again this is a themes-coarsening, as defined in section 3); then defining h_1 to be the region consisting of all locations associated with range $[200, 300)$, and defining h_2 to be all locations associated with range $[300, 400]$; this latter operator is discussed in section 4.3.

Map P' gives stronger information than that given by P, and gives an even stronger answer to query q_2 when taken in conjunction with map N: the range of rainfall values $[206\frac{2}{3}, 213\frac{1}{3}]$; this follows using the deduction that c, the mean rainfall over $g_3 \cap h_1$ must be in $[200, 220]$: the lower bound comes from $g_3 \cap h_1$ being a subset of h_1 and the upper bound follows using the fact that d, the mean rainfall over $g_3 \cap h_2$, is at least 300, and by map N, we know that $(c + d)/2 =$ the mean rainfall over g_3 equals 260. This illustrates the importance of taking full account of the meaning of a dataset: replacing P by P' makes a difference, even though they are of the same form.

One of the main aims of this paper is to formalise the kind of reasoning used in this example, defining precisely which inferences are valid are which are not, and for certain situations showing how to compute the valid inferences. Each spatial dataset is assumed to have been derived from an unknown spatial variable by the application of a known summarisation operator; this gives us information about the identity of the spatial variable. In a real situation, one may well have to take into account uncertainty about the summarisation processes that led to this dataset; however this uncertainty may be reduced with sufficiently thorough metadata.

The example also illustrates that for certain types of spatial dataset, an objective meaning can be given, leading to precisely defined inferences from one or more datasets. The formal semantics defined in this paper is objective in the sense that the formal meaning of the dataset is independent of the user and of the use.

In section 2, formal definitions of atoms-maps and regions-maps are given; atoms are the finest spatial areas that are of interest; regions are a coarser division of space than atoms. An atoms-map assigns attribute values to each atom; a regions-map assigns attribute values to each region. Section 3 describes themes-coarsening, which can be viewed as classification of attribute values. Section 4 describes some basic summarisation operators for spatial coarsening, which generate a regions-map from an atoms-map. These include the use of weighted means and mode. The formal semantics is described in section 5, interpreting an atoms-maps as a set of worlds, and a regions-map in terms of all the atoms-maps it could be a summarisation of. A general definition of a query is developed in section 6, in order to talk about what can be deduced from such a spatial dataset. Section 7 analyses three cases of integration of pairs of maps. Representation of uncertainty is briefly discussed in section 8, and section 9 concludes and discusses possible extensions.

2 Maps

This section defines the types of spatial entities that we are concerned with. An *atoms-map* is intended as a representation of a spatial variable at a basic spatial granularity; an associated *regions-map* is intended as a representation of the spatial variable at a coarser spatial granularity, so summarises the information given by the atoms-map.

The underlying spatial domain

Our underlying domain D is a subset of a two-dimensional surface, with a well-defined non-zero (and finite) area, e.g. a subset of $\mathbb{R}^2$ (pairs of real numbers), or a subset of the surface of a sphere.

Spatial atoms

Spatial atoms are intended to be the smallest spatial units that we are interested in, corresponding to the finest division of the spatial domain that concerns us. We allow two possibilities for an atoms set X over domain D:

> *either:* X is equal to D, so that spatial atoms are points in D
>
> *or:* X is a set of disjoint subsets of D, where each subset has a well-defined and non-zero area, and the union of all elements of X is equal to D.

The former type will be referred to as *points atoms*, the latter type as *areal atoms*. Note that it is not assumed that areal atoms have the same area. Examples of situations where may one choose to use areal atoms include: pixels in a map generated from remote sensing; parcels in a land ownership map; countries in a political map giving, say, percentage of GNP spent on healthcare.

Atoms are intended to be the smallest parts of the domain we are concerned with, and the atoms set needs to be such that every region of interest is a union of atoms. In this paper it is assumed, for simplicity, that when we are combining information from a number of maps, all the maps we're interested in have the same underlying set of spatial atoms.

Regions sets

Regions[1] are areas of the domain which are unions of the atomic areas; they are aggregate areal units.

A *regions set* G on points atoms set X (on domain D) is defined to be a finite set of subsets of D, such that

(i) each $g \in G$ has a well-defined (but possibly zero) area;

(ii) for $g_1, g_2 \in G$ with $g_1 \neq g_2$, $g_1 \cap g_2$ has zero area.

A *regions set* G on areal atoms set X (on domain D) is defined to be a finite set of subsets of D, such that

(i) each $g \in G$ is a union of atoms;

(ii) for $g_1, g_2 \in G$ with $g_1 \neq g_2$, $g_1 \cap g_2 = \emptyset$, so that the elements of G are pairwise disjoint.

[1] This definition of region differs from that of e.g. (Dent, 1996) in that there is no general assumption of internal homogeneity within a region (though see section 4.3).

Theme-statements, theme-elements and themes-divisions

A *theme-statement* associated with a particular location is intended to be a statement about what is at that location. For example at location x, *2002 yearly rainfall is between 500mm and 530mm*; or at x, *land cover = scrub*. In many situations theme-statements will be simple statements such as these. However, we also formally allow theme-statements to be expressed as formulae in some logic, and theme-statements might be compound statements such as *[(land cover = scrub) and (rainfall less than 500mm)] or (rainfall greater than 800mm)*. The intention is that the meaning of a particular theme-statement is the same for any map. This means that, for example if we have two land cover maps which both use a class 'bog', then these should be considered as different theme-statements unless we are sure that the meanings are identical.

Define a *theme-element* to be a set of theme-statements e.g. {*rainfall less than 500mm*, *land cover = scrub*}. A theme-element is interpreted as meaning that all the set of statements hold, so that if the theme-element {*rainfall less than 500mm*, *land cover = scrub*} is associated with a location then it means that at that location, rainfall is less than 500mm *and* land cover = scrub.

A *themes-division*[2] is defined to be a mutually exclusive and exhaustive set of theme-elements. Let t_1 and t_2 be theme-elements in a themes-division T; mutually exclusivity means that it is impossible for the set of theme-statements $t_1 \cup t_2$ to be all simultaneously satisfied, at any single location. Exhaustivity means that for any location, one of the theme-elements $t \in T$ holds, i.e. all theme-statements in t hold. An example of a themes-division is { {*rainfall < 500mm, land cover = scrub*}, {*rainfall < 500mm, land cover ≠ scrub*}, {*rainfall ≥ 500mm, land cover = scrub*}, {*rainfall ≥ 500mm, land cover ≠ scrub*} }; at any particular location, exactly one of these four possibilities must hold. A themes-division reflects a certain level of detail in the description of what the world could be like at a location.

[2]We could instead define a themes-division to be a mutually exclusive and exhaustive set of themes-statements, without being less expressive. The reason for having the extra layer of theme-elements is so that the combination operation $\otimes$ defined in section 7.1 for themes-divisions and atoms-maps is associative as well as commutative.

Atoms-maps

An atoms-map is a function from X to T for some atoms set X and some themes-division T. The set of all such functions is written $F(X, T)$. The atoms-map $M : X \to T$ is intended to mean that for each $x \in X$, $M(x)$ holds at location x. For example, with T defined above, $M(x) = \{$*rainfall* $\geq$ *500mm*, *land cover* = *scrub*$\}$ means that at x the land cover is scrub, and the total rainfall in 2002 at x is at least 500mm. Atoms-maps will be used to represent spatial variables.

Regions-maps

A *regions-map* is defined to be a function from G to T, for some regions set G and some themes-division T. The set of all such functions is written $F(G, T)$.

Since regions are aggregate spatial units, regions-maps are taken in this formal framework to represent derived information: they are assumed to have come from some atoms-map (representing a more fundamental spatial variable) via some summarisation process. Thus a regions-map gives partial information about a spatial variable.

A set of regions may be defined independently of the atoms-map; for example, a regions-map with regions being counties, giving mean population density for each county; this can considered as a be summarisation of a fine resolution population density map. Alternatively, the set of regions may be generated as part of the summarisation process; in particular, regions may be defined so that the spatial variable is close to being homogeneous in each region.

Regions-maps and areal atoms-maps can be considered as *choropleth* maps in that they associate a value to each of a set of locations.

Maps on atoms to non-mutually exclusive sets of theme-elements. It can sometimes be natural to allow maps $M : X \to T$ where T is an exhaustive but not mutually exclusive set of theme-elements. The intended meaning of M is essentially as before, that for each location x, $M(x)$ holds. The framework developed in this paper can be easily extended to cover this type of map; however, the kinds of examples of spatial datasets discussed here are not of this form, so such maps will not be dealt with further here.

3 Themes-coarsening

An important and very common operator in producing thematic maps is *classification* (see e.g. (Dent, 1996)): dividing up the values of a variable we intend to map into sets of values; in other words, coarsening[3] the themes-division. This might also be described as *thematic generalisation*, e.g., (Petit and Lambin, 2001).

Let T and T' be two themes-divisions. T' is said to be a coarsening of T if it's a coarser division of themes: i.e. for each $t \in T$ there exists some $\tau(t) \in T'$ with t implies $\tau(t)$. (t is a finer division than $\tau(t)$). T' can be thought of as a grouping of the elements of T. A map M allocating theme-elements in T can then be used to produce another map M' which gives thematically coarser information by coarsening the theme-elements assigned by M, defining $M'(x) = \tau(M(x))$.

This type of coarsening operator is also considered in section 3 of (Worboys and Duckham, 2002).

Themes-coarsening of an atoms-map

Let M be an atoms-map from X to T. Define the themes-coarsened map $M' = \tau(M)$ to be an atoms-map $M' : X \to T'$ given by $M'(x) = \tau(M(x))$. In other words, $\tau(M) = M \circ \tau$.

Themes-coarsening has the following transitivity property: if $\tau' : T' \to T''$ is another themes-coarsening and $\tau'' : T \to T''$ is equal to τ followed by τ' then $\tau''(M) = \tau'(\tau(M))$.

Themes-coarsened maps are very common when we have information about a continuous variable, such as elevation, temperature or population density, but divide the possible values of the variable into classes. For example, a continuous scale of elevation is coarsened to a set of intervals such as $T' = \{[0, 100), [100, 200), [200, 300), [300, 400), [400, \infty)\}$. Locations are then assigned a range instead of a precise value. Also, in the introductory example, the map P can be considered as having been produced using a themes-coarsening.

A contour map, or other isarithmic map, can be considered as implicitly a themes-coarsened elevation map (where atoms are points). If the contours

[3] Themes-divisions can be considered as frames of discernment in the sense described in (Shafer, 1976), so the notion of coarsening of themes-divisions is essentially the same as coarsening defined there.

are at 0m, 10m, 20m, ... then the map is saying that points on a contour have that elevation, and that points between two contours have elevations between the two contour values. (All points are either on a unique contour line or are between two consecutive contour lines.) Therefore the map can be viewed as an atoms-map $N : X \to T'$, where X is the set of points and T' consists of alternate contour values and ranges $\{\{0\}, (0, 10), \{10\}, (10, 20), \{20\}, \ldots\}$, with e.g., $(0, 10)$ representing the set of elevations strictly between 0m and 10m. The 20m contour line is then the set of all points x with $N(x) = \{20\}$. N can be considered as the result of themes-coarsening an elevation map $M : X \to T$ which assigns each point its precise elevation, with T being all possible values of elevation.

Themes-coarsening of a regions-map

To themes-coarsen a regions-map we can essentially use the same definition as for an atoms-map. Let $N : G \to T$ be a regions-map, and let $\tau : T \to T'$ be a themes-coarsening. Define the themes-coarsened map $N' = \tau(N)$ by $N' = N \circ \tau$, i.e. for $g \in G$, $N'(g) = \tau(N(g))$.

As a more complex example consider the production of a land cover map such as LCM2000 (Fuller et al., 02) from remote sensing data. Let G be a set of parcels of land. Associated with each parcel $g \in G$ is a number n of spectral values in $[0, 1]$, so we have a function $N : G \to [0, 1]^n$, which can be considered as a regions-map. For example, with LCM2000, $n = 6$: three spectral bands from LANDSAT TM data for each of a Summer and a Winter image. The spectral values for the parcels were computed by using an averaging procedure (a spatial coarsening operator—see section 4) applied to the spectral values over pixels (a finer division of the area).

Any function on $[0, 1]^n$ generates a themes-coarsening. In particular if a maximum likelihood algorithm is used to classify points in $[0, 1]^n$ into land cover classes (based on training data) this gives a function $\tau : [0, 1]^n \to L$ where L is a set of land cover classes; the themes-coarsened map $\tau(N)$ is then a land cover map[4], assigning a land cover class to each parcel. Alternatively, as was done for the LCM2000 metadata, one could use a maximum likelihood

[4]Defining the formal meaning (as defined in section 5) of this land cover map involves consideration of the function τ which associates land cover classes with parts of the n-dimensional 'spectral space'. From this perspective, to fully understand the meaning, one needs to know precisely the processes used: in this case including the details of the maximum likelihood classifier based on that particular set of training data.

algorithm to generate, for any point in $[0,1]^n$, a list of the five land cover classes with highest likelihoods, with their associated normalised likelihoods. This defines a function τ' on $[0,1]^n$ which generates a themes-coarsened map $\tau'(N)$ giving such information for each parcel.

4 Some summarisation operators for spatial coarsening

Consider a regions-map $N : G \to T$, for regions set G and themes-division T, which represents some physical, social or other variable over the regions. Often this will be summarising information about the variable on a finer spatial resolution, which may be represented as an atoms-map $M : X \to T'$ for some themes-division T'. We can write $N = \sigma(M)$ where σ is the summarisation operator, the procedure that generated N from M. The way we will give meaning to a regions-map N (in section 5.2), produced by known summarisation operator σ from some atoms map, is to consider all the possible atoms-maps M that could have given rise to it.

This notion of a summarisation operator is very close to *model generalisation* (or *database generalisation*), see e.g., (Weibel, 1997), and spatial coarsening can be considered as spatial aggregation, as in (Petit and Lambin, 01).

We describe just some of the most basic ways of spatial coarsening. Section 4.1 discusses mean-coarsening—averaging the values of an atoms-map over each region. Section 4.2 discusses mode-coarsening—associating with region g a value that occurs more than any other value in g. The approaches in 4.1 and 4.2 assume a pre-defined set of regions; alternatively, as described in 4.3, a regions set can be generated by an atoms-map M by grouping together atoms that get allocated the same value by M. This leads to a regions-map conveying the same information as the atoms-map.

4.1 Means and sums

With the spatial summarisation operators discussed in this section, the themes-division T needs to have a particular structure, as we will be taking averages over values in T. For areal atoms this requires an interval scaling; for points atoms, stronger conditions are required so we can integrate values of T over an area. To simplify the presentation we assume T is mathematically equal to $I\!R^n$ where where $I\!R$ is the set of real numbers and n may be $1, 2, \ldots$. (Strictly speaking T is a set of assignments to the variable v associated with the map: $T = \{\{v = r\} : r \in I\!R^n\}$.) Sums and averages of vectors of real numbers are defined co-ordinate-wise: e.g. if $a, b \in I\!R^n$, $a + b$ is defined to be the vector $(a(1) + b(1), a(2) + b(2), \ldots, a(n) + b(n))$.

4.1.1 Mean-coarsening

Suppose we have a map $M : X \to I\!R^n$ assigning a vector of real numbers to atoms. We would like to generate a map $N : G \to I\!R^n$ for some regions set G, that summarises the information given by M. Often the natural way to do this is to define $N(g)$ to be the weighted average of $M(x)$ over $x \subseteq g$, for areal atoms,

$$N(g) = \sum_{x \subseteq g} \frac{|x|}{|g|} M(x),$$

where $|x|$ is the area of atom x. For the case of points atoms,

$$N(g) = \frac{1}{|g|} \int_{x \in g} M(x)\, dx,$$

where $|g| = \int_{x \in g} 1\, dx$ is the area of region g.

We write the mean-coarsening operator to regions set G as μ_G, so that $\mu_G = N$ as just defined.

In the introductory example, the map N can be considered as having been produced by mean-coarsening operator μ_G with regions set $G = \{g_1, g_2, g_3\}$.

Another example is in the production of a land cover map such as LCM2000 (Fuller et al, 02). Associated with each pixel is a vector of (corrected) spectral values, six in the case of LCM2000, which can be represented as an atoms-map $M : X \to [\,0, 1\,]^6$, with atoms being pixels. It can be desirable to spatially coarsen this data to a set of parcels G, where a parcel might, for example, be a single field. A natural way of doing this is to mean-coarsen M to produce $N : G \to [\,0, 1\,]^6$, with $N = \mu_G(M)$ giving the mean spectral values over each parcel.

Other examples where mean-coarsening is appropriate include aggregating population densities or average temperatures.

Weighted-mean-coarsening. The version of mean-coarsening just described is appropriate when we are interested in averaging the values over area. It is sometimes more appropriate to average over other functions. For example, suppose we have an atoms-map $M : X \to T$ giving the average age of people living within each areal unit x, and we are interested in a coarser spatial division represented by regions set G. To compute the average age of people within each region g we need to weight the values of $M(x)$ by (a number proportional to) the population $\gamma(x)$ within each x.

For areal atoms we assume that we have a weighting function $\gamma : X \to [0, \infty)$ (which is the same for each map in its class $F(X, T)$). We define the result of mean-coarsening $N = \mu_G$ by, for $g \in G$, $N(g) = \sum_{x \subseteq g} \frac{\gamma(x)}{\gamma(g)} M(x)$, where $\gamma(g) = \sum_{x \subseteq g} \gamma(x)$; the value $N(g)$ is the weighted average of $M(x)$ over $x \subseteq g$. When $\gamma(x)$ is defined to be the area of x, then this is the same as the original mean-coarsening.

Another example is if $M : X \to T$ gave the mean salary of working adults living in each area x, and we wanted the mean salary over regions $g \in G$; here the weighting function $\gamma(x)$ needs to be proportional to the number of working adults living within x.

For the case of points atoms, we assume a function $\rho : X \to [0, \infty)$ which represents the relative densities. $N(g) = \frac{1}{\gamma(g)} \int_{x \in g} \rho(x) M(x)\, dx$, where $\gamma(g) = \int_{x \in g} \rho(x)\, dx$. A type of example where this is natural is if the area-weighted-mean-coarsening was appropriate but where we wanted to compensate for the area-distortion of a map projection of a part of the Earth onto a flat surface.

Mean-coarsening of regions-maps. We can also apply mean-coarsening to regions-maps to produce coarser regions-maps. Suppose G' is a coarser regions set than G, with each $g' \in G'$ a union of regions $g \in G$. Let $N : G \to I\!R^n$ be a regions-map. We can define the appropriate regions-map on G', $N' = \mu^G_{G'}(N)$, for both areal and points atoms cases, using essentially the earlier definition for areal atoms: $N'(g') = \sum_{g \subseteq g'} \frac{\gamma(g)}{\gamma(g')} N(g)$. The following transitivity property holds: for any $M : X \to I\!R^n$, $\mu^G_{G'}(\mu_G(M)) = \mu_{G'}(M)$, so that $\mu_{G'} = \mu_G \circ \mu^G_{G'}$; mean-coarsening to G and then from G to G' gives the same result as mean-coarsening directly to G'.

4.1.2 Sum-coarsening

For atoms-map $M : X \to T$ and regions set G on X we can define $N : G \to T$, the result of sum-coarsening M, by $N(g) = \sum_{x \subseteq g} M(x)$ for the areal atoms case. If X is a points atoms set then $N(g)$ is defined to be $\int_{x \in g} M(x)\,dx$.

Examples include total population maps, where the value associated with an (areal) atom or region is a total population figure. In geographical maps this is much less common than mean-coarsening; however it is an important fundamental operator for spatial data.

Mean-coarsened maps can be considered as the ratio of two sum-coarsened maps.

4.2 Mode-coarsening and proportional-coarsening

The mean-coarsenings described above required at least an interval scale for T. The approaches in this section can be applied for any T, even with only a nominal scale (essentially just a set of labels with no ordering information).

Suppose we have an atoms-map $M : X \to T$ giving land cover types associated with each atom x. An obvious way to produce a map $N : G \to T$ is to let $N(g)$ be the theme-element that appears over the greatest area in g. We can consider g as partitioned into parts g_t, for $t \in G$, where g_t is the part of g which is assigned theme-element t. $N(g)$ is then assigned a theme t which has greatest associated area $|g_t|$. A general problem with the mode is what to do if there is a tie; one way to resolve ties is to choose a total order $\succ$ on the themes-division T and, if $|g_{t_1}| = |g_{t_2}| > |g_t|$ for all $t \in T$ with $t \neq t_1, t_2$, then assign $N(g)$ to be t_1 if $t_1 \succ t_2$, and $N(g) = t_2$ if $t_2 \succ t_1$.

Mode-coarsening can be decomposed into another type of coarsening, *proportional-coarsening*, followed by a themes-coarsening, where proportional-coarsening can be viewed as an application of mean-coarsening.

4.2.1 Proportional-coarsening

The idea of proportional-coarsening is to compute, for each region g and each theme-element t, the measure of parts of g with theme-element t; essentially, a relative frequency histogram of the different measures of each theme is associated with each region. For example, suppose we have a land cover map on a set of pixels, associating a land cover class (such as 'grassland' or 'bog') to each 25m by 25m pixel. We would like to generate a land cover map on a coarser spatial division G, where regions in G are parcels of land which are often around the size of a typical field. For each region g and land cover class t we can compute the fraction of the area of g which has land cover t, thus generating a histogram for each g.

For finite themes-division T define T^* to be the set of all probability distributions on T, i.e. the set of all functions $p : T \to [0,1]$ such that $\sum_{t \in T} p(t) = 1$. Let $M : X \to T$ be a map to a finite themes-division T, and let G be a regions set over areal atoms set X. One natural spatial coarsening operator produces $\mu^o_G(M)$ from M, where $\mu^o_G(M)$ gives, for each $g \in G$, the fraction of the area of g which is labelled by each theme. More precisely, for each $g \in G$ we can consider the union g_t of atoms x which M assigns to theme-element t, so $g_t = \bigcup\{x \subseteq g : M(x) = t\}$. If we write the result of applying $\mu^o_G(M)$ to g, i.e. $\mu^o_G(M)(g)$, as V_g, we have, for each $t \in T$, $V_g(t) = \frac{|g_t|}{|g|}$.

In fact, this proportional-coarsening can be seen to be essentially a special case of mean-coarsening. Define $M^* : X \to T^*$ by $M^*(x)(t) = 1$ if $M(x) = t$, and $M^*(x)(t) = 0$ otherwise. Then $V_g(t) = \sum_{x \subseteq g} \frac{|x|}{|g|} M^*(x)(t)$, so V_g, i.e., $\mu^o_G(M)(g)$, is equal to $\sum_{x \subseteq g} \frac{|x|}{|g|} M^*(x)$. Therefore we can write $\mu^o_G(M) = \mu_G(M^*)$, the mean-coarsening of M^*.

Similarly, suppose we have a map $U : X \to T^*$ giving, for each atom and theme-element, the fraction of the area of the atom having that theme-element, and a regions set G on X. Then we can compute the new map $V = \mu_G(U)$, where $V : G \to T^*$ gives, for each region and theme-element, the fraction of the area of the region associated to that theme-element.

If we wish to weight the atoms by some other function γ instead of area, we can apply the weighted mean-coarsening instead; similarly for the points atoms case. In all cases we have a new map $V = \mu^o_G(M) = \mu_G(M^*)$, if we start with a map $M : X \to T$, or a new map $V = \mu_G(U)$ if we start with a map $U : X \to T^*$.

Proportional-coarsening of regions-maps We can also apply proportional-coarsening to regions-maps to produce coarser regions-maps. Suppose G' is a coarser regions set than G. For $N : G \to T$ and $V : G \to T^*$, define new maps $\mu^G_{G'}(N^*)$ (where N^* is defined analogously to M^*) and $\mu^G_{G'}(V)$ giving the proportions of various themes within regions in G'. As observed earlier we have the transitivity property $\mu_{G'} = \mu_G \circ \mu^G_{G'}$, and so also $\mu^o_{G'} = \mu^o_G \circ \mu^G_{G'}$: proportional-coarsening to G and then from G to G' gives the same result as proportional-coarsening directly to G'.

4.2.2 Mode-coarsening

To define the mode-coarsened map $N : G \to T$ of atoms-map $M : X \to T$, for each region g, we consider for each theme-element t, the total area of atoms x in g assigned value t by map M. The value of $N(g)$ is chosen to be a theme-element t that appears in map M over the largest area, where we break ties using a given total order $\succ$ on T.

For atoms-map $U : X \to T^*$ define regions-map $N = \eta_G(U)$ by $N(g) = t$ where t is such that for all $t' \in T$ with $t' \neq t$, $\mu_G(U)(t) \geq \mu_G(U)(t')$ and if $\mu_G(U)(t) = \mu_G(U)(t')$ then $t \succ t'$. For atoms-map $M : X \to T$, define $\eta^o_G(M) = \eta_G(M^*)$. If we want to use a weighting γ instead of the area weighting, then we use the same definitions but with the appropriate definition of weighted-mean-coarsening μ_G.

Define $\tau^* : T^* \to T$ by $\tau^*(p) = t$ if for all $t' \in T$ with $t' \neq t$, $p(t) \geq p(t')$ and if $p(t) = p(t')$ then $t \succ t'$. $\tau^*(p)$ is the theme-element given highest value by p, where ties are broken using the total order $\succ$. τ^* defines a themes-coarsening. Mode-coarsening can be broken down into proportional-coarsening followed by this special type of themes-coarsening: $\eta^o_G = \mu^o_G \circ \tau^*$, and $\eta_G = \mu_G \circ \tau^*$.

One can also define mode-coarsening of a regions-map $N : G \to T$ to give a regions-map $N' : G' \to T$, where G' is a coarser regions set than G. Define this operator $\eta^G_{G'}$ by $N' = \eta^G_{G'}(N) = \mu^G_{G'}(N^*) \circ \tau^*$. However, as is well known,[5] the transitivity property does not always hold: we can have $\eta^o_{G'} \neq \eta^o_G \circ \eta^G_{G'}$; mode-coarsening to G and then from G to G' can give a different result from mode-coarsening directly to G'.

4.3 Theme-generated maps: regions defined from themes

The spatial coarsenings discussed so far have been based on a pre-determined set of regions G which is not dependent on the atoms-map M that we wish to spatially coarsen. However, regions can be created from an atoms-map as homogeneous parts of the map.

Let $M : X \to T$ be an atoms-map. Define regions set G on X in such a way that for any region g, M is constant over x in g. In other words, if x and x' are atoms in the same region then $M(x) = M(x')$. Naturally, $N : G \to T$ is then defined by $N(g) = t_g$, where t_g is the value of $M(x)$ for all x in g.

[5] For example, gerrymandering uses this fact.

This summarisation operator is different from the ones discussed above in that no information is lost; indeed M can be recovered from N. Regions-map N is, in a sense, just a relabelling of atoms-map M.

There will typically be many ways to define the regions set G from atoms-map M. One way is to make the regions as large as possible. For the points atoms case this is done by making each region of the form $M^{-1}(t) = \{x \in X : M(x) = t\}$ for some $t \in T$. Then $G = \{M^{-1}(t) : t \in T\} - \{\emptyset\}$. The areal atoms case is similar except that each region is the union of a set of atoms $M^{-1}(t)$ for some $t \in T$.

This approach may mean that a region is not connected; an alternative approach would be to make regions maximally large such that they are still connected. For the points atoms case this involves making a region a connected component of $M^{-1}(t)$ for some $t \in T$.

This process of defining regions from themes can be part of a more complex summarisation procedure. For example, suppose we wanted to create a printed world map of population density, given that we have the population density figures over small areas of the earth, which may be considered as an atoms-map. A natural way of doing so is as follows. First, a themes-coarsening (classification): values of population density is be grouped into ranges, where each range may be allocated a particular colour or shading; this generates a new themes-division and a new atoms-map by themes-coarsening. Regions can then be defined to be the maximal connected components generated by this atoms-map; this generates a regions-map as described above. Finally, a 'neatening operation' may be applied: for example, smoothing the edges of the regions, removing very small regions inside another region, and so on. Obviously, this final operation will be the hardest to model formally.

5 Formal semantics of maps

The aim of this semantics is to be able to represent precisely and formally the information expressed by a map (of the forms described in section 2). It does this by associating a map with a set of worlds, each one representing a different way that reality might be.[6] First the meaning of an atoms-map is

[6] Since atoms-maps and regions-maps are just functions, the problems of syntactic ambiguity of maps discussed in (Pratt, 93) do not apply. Note, however, that a regions-map needs to be associated with a summarisation operator to be assigned a formal meaning—see section 5.2.

defined. Regions-maps will then be interpreted in terms of atoms-maps.

5.1 Meaning of atoms-maps

An atoms-map $M : X \to T$ is intended to mean: *for all* $x \in X$, $M(x)$ *holds at location* x; we will define this more closely in terms of a set of 'worlds', each of which represents a different way that the world might be.

We assume a set of states S (not all of which will necessarily be possible). A state is intended as a full description (for our purposes) of the world at a particular location. States are mutually exclusive and S is exhaustive, i.e. at any location, exactly one state correctly describes the situation there. There will be different ways to define the states; the key property that is needed is that each theme-statement u either holds or doesn't hold in any given state s. We write $s \models u$ if it does hold. S is essentially a themes-division which is finer than any other themes-division we're interested in. One way of defining S is so that a state s is a set of theme-statements which contains a theme-element of every themes-division we're interested in, e.g. $s = \{$mean rainfall = 500mm, height = 200m, land cover = scrub, ...$\}$; in which case, $s \models u$ whenever $u \in s$. For theme-element t, we say that s satisfies t, also written $s \models t$, if $s \models u$ for each $u \in t$. Let T be a themes-division. Because of mutually exclusivity and exhaustivity of T, for any possible state s we will have exactly one element t of T with $s \models t$.

We assume a set of (possible) worlds W representing the different ways reality could be; each possible world $w \in W$ (usually abbreviated to just 'world') is a function from atoms set X to S. World w means that, for each location x, the state at x is $w(x)$: it fully describes the state of the world at each location. (Note that the set of worlds W is relative to a particular atoms set X.) World w satisfies atoms-map M, written $w \models M$, if and only if for all atoms $x \in X$, every theme-statement in $M(x)$ holds in state $w(x)$, i.e. for all $x \in X$, $w(x) \models M(x)$. Define $[M]$, the meaning of M, to be the set of worlds satisfying M, i.e. $\{w \in W : w \models M\}$. An atoms-map $M : X \to T$ is said to be consistent if it has a model, i.e. $[M] \neq \emptyset$, so there exists $w \in W$ with $w \models M$.

Let $M_1 : X \to T_1$ and $M_2 : X \to T_2$ be two atoms-maps. We can say that M_1 and M_2 are consistent, written $\text{cons}\{M_1, M_2\}$, if there's a world which satisfies both, i.e. there exists a world $w \in W$ with $w \models M_1$ and $w \models M_2$. Similarly, for set of atoms-maps $\{M_i : i \in I\}$, where for $i \in I$, $M_i : X \to T_i$, we can say that this set is consistent, written $\text{cons}\{M_i : i \in I\}$ if there's a world satisfying every M_i.

Worlds themes-coarsening

Any consistent atoms-map $M : X \to T$ can be considered as a themes-coarsening (see section 3) of a world, because the set of states S is a finer division of possibilities than T. Let $\tau_T : S \to T$ be a function satisfying: *if* $\tau_T(s) = t$ *then* $s \models t$; such a function always exists, because of the exhaustivity property of T; furthermore, if s is a *possible* state, then $\tau_T(s) = t$ if and only if $s \models t$, because of the mutual exclusivity of elements of T. We can then themes-coarsen any world $w \in W$ to an atoms-map $M' : X \to T$ defined by $M' = w \circ \tau_T$. We write $w \circ \tau_T$ as $w^{\downarrow T}$, so that $w^{\downarrow T}(x) = \tau_T(w(x))$. Then, for atoms-map $M : X \to T$, we have $w(x) \models M(x)$ if and only if $\tau_T(w(x)) = M(x)$, i.e., $w^{\downarrow T}(x) = M(x)$. Therefore world w satisfies atoms-map M if and only if $w^{\downarrow T} = M$. Consequently, M is consistent if and only if there exists $w \in W$ with $w^{\downarrow T} = M$. A set of atoms-maps $\{M_i : i \in I\}$ is consistent, written cons$\{M_i : i \in I\}$, if and only if there exists world w such that for each $i \in I$, $w^{\downarrow T_i} = M_i$, where $M_i : X \to T_i$.

The possible states

Some states may be impossible, for example, a state s satisfying both the theme-statements *total 2002 rainfall is 500mm* and *total 2002 rainfall is between 700mm and 1000mm*, is impossible. Among S are the set $\underline{S}$ of states that can actually happen, the set of states that appear in some world at some location, i.e. $\{w(x) : w \in W, x \in X\}$.

In some situations, it may be reasonable to assume that what states are possible in a particular location does not depend on the location, in which case, if we define for atom x, S_x to be the set of states possible at x, so $S_x = \{w(x) : w \in W\}$, then for $x \in X$, we will have $S_x = \underline{S}$.

The set of possible states $\underline{S}$ can be used define implication between theme-elements. For theme-elements t and t', t implies t' if and only if for all $s \in \underline{S}$, $s \models t$ implies $s \models t'$, that is, whenever t holds, t' holds.

The possible worlds

Worlds are functions from X to S, but not all such functions are considered as possible descriptions of the world. What prevents a function from X to S being a world? Firstly, a world must in fact be a function from X to $\underline{S}$, since the state of a world at x must be a possible state. This limits W to being a subset of $F(X, \underline{S})$.

There are other ways that the real world is restricted. For example, certain variables such as temperature may be continuous. This condition can be viewed as saying that of the maps in $F(X, T)$ where T is some themes-division representing temperature, only those satisfying the continuity property are actually possible; for any other map $M \in F(X, T)$, $[M] = \emptyset$, hence restricting W. As a constraint on W this might be written as: $w^{\downarrow T}$ *is continuous*.

What we may know in practice about what is possible

We will not often have complete knowledge about W and about which maps are inconsistent. In practice, we will tend to have partial information about what combinations of theme-statements are possible, and about structural conditions. We will know that certain sets of theme-statements are impossible, thus implicitly restricting the set of possible states $\underline{S}$; we may also know that certain maps are impossible because of structural conditions such as continuity, e.g. a certain variable which is known (or assumed) to be continuous exhibits discontinuous behaviour in those maps. But our information about inconsistency will usually be incomplete; we will know that certain maps are inconsistent, but a map may be inconsistent without us knowing it.

Our information will also enable us to deduce that certain sets of maps are inconsistent: for example, $\{M_1, M_2\}$ where M_1 and M_2 are such that for some x, $M_1(x)$ and $M_2(x)$ are (known to be) incompatible theme-elements. So in practice we will have a relation $\text{cons}'\{M_1, M_2\}$, which approximates $\text{cons}\{M_1, M_2\}$, and is weaker than it, in the sense that $\text{cons}\{M_1, M_2\}$ implies $\text{cons}'\{M_1, M_2\}$ but not vice versa.

5.2 The meaning of regions-maps and other map structures

The previous section defined the formal semantics for an atoms-map M. This section extends the semantics to regions-maps by considering a regions-map as a summarisation of an unknown atoms-map.

The meaning of a set of atoms-maps. Let $\mathcal{M}$ be a set of atoms-maps, which is intended to represent that every $M \in \mathcal{M}$ is a correct representation of the world. A world then satisfies $\mathcal{M}$ if it satisfies every element of $\mathcal{M}$, i.e. $w \models \mathcal{M}$ if and only if $w \models M$ for all $M \in \mathcal{M}$, $\iff$ $w \in \bigcap_{M \in \mathcal{M}}[M]$. So we have $[\mathcal{M}]$, the set of worlds satisfying $\mathcal{M}$, is equal to $\bigcap_{M \in \mathcal{M}}[M]$.

The meaning of a disjunction of a set of atoms-maps. Let $\mathcal{N}$ be a set of atoms-maps. The disjunction of this set, $\bigvee \mathcal{N}$, is intended to represent that one of the maps tells us correct information about the world. Then world w satisfies $\bigvee \mathcal{N}$ if and only if it satisfies some element of $\mathcal{N}$, so that $[\bigvee \mathcal{N}] = \bigcup_{M \in \mathcal{N}}[M]$.

The meaning of a regions-map. Suppose regions-map $N : G \to T$ was produced from some unknown atoms-map $M : X \to T_M$, using a summarisation operator σ, so that $N = \sigma(M)$; suppose also that the unknown M gives us correct information. Write $\sigma^{-1}(N)$ for the set $\{M : \sigma(M) = N\}$. What N tells us is that some $M \in \sigma^{-1}(N)$ is a correct representation of the world, so N tells us the same information as $\bigvee \sigma^{-1}(N)$. The meaning of N is then defined to be the same as the meaning of $\bigvee \sigma^{-1}(N)$, so that $[N] = \bigcup_{M:\sigma(M)=N}[M]$. World w satisfies N if and only if w satisfies M for some atoms-map M that could have been summarised to give N. For the operators σ such that T_M does not depend on M, but is always equal to some T_0 (in particular the examples of operators discussed in this paper), this can be expressed neatly in terms of worlds themes-coarsening: $w \models N \iff \sigma(w^{\downarrow T_0}) = N$.

The meaning of a set of regions-maps. Suppose we are given a pair of regions-maps N_1 and N_2, produced using summarisation operators σ_1 and σ_2. Each of these tells us something about the true world; the meaning of this pair $\{N_1, N_2\}$ is then defined to be $[N_1] \cap [N_2]$, which is equal to

$$\left(\bigcup_{M_1:\sigma_1(M_1)=N_1} [M_1] \right) \cap \left(\bigcup_{M_2:\sigma_2(M_2)=N_2} [M_2] \right).$$

Similarly, the meaning $[\mathcal{N}]$ of a set of regions-maps is defined to be $\bigcap_{N \in \mathcal{N}}[N]$.

Map Structures. It is mathematically neater to define a general object covering all the above cases. We can recursively define a map structure to be either an atoms-map, a set of map structures, or a disjunction of a set of map structures. The meaning of a map structure can then be defined recursively from the meaning of an atoms-map. The meaning of a set of maps/map structures is the set of worlds satisfying all of them. The meaning of a set-disjunction of maps/map structures is the set of worlds satisfying at least one of them.

Discussion. As observed earlier, this formal semantics is relative to a particular atoms set X. Sometimes there can be different reasonable choices of atoms set (over a given domain D) in framing a problem: these will have different associated semantics; might these then give different answers? Spatial atoms are intended to be the finest spatial units that are of interest; if there are a set Δ of reasonable choices for X, we could generate the unique coarsest atoms set which is finer than each element of Δ, and use this. Alternatively, one could often just use a points atoms set, which is a finer division of space than any other atoms set over a domain D. Very often, however, the choice of atoms set will not make a difference to the queries of interest; see section 7.2 for an example of this.

6 Queries

A dataset implicitly contains information about the world (reality); to get at that information we need to query the dataset. A query asks something about the way the world is, and expects a particular type of answer; for example, the query *What is the average height above sea level (in metres) in a region R?* expects a number as an answer. So fundamentally a query can be viewed as a function from the set of worlds W to a set of answers A. We call such a function an *underlying query*. Another way of viewing this is that a query divides up the set of worlds into sets corresponding to different answers.

A special case is boolean queries, where the set of worlds is divided into two. If $A = \{\text{yes}, \text{no}\}$ then the query is asking if the world has some particular property; the set corresponding to 'yes' is the set of worlds which have that property. Such a query can be considered as a proposition: it is either true (with answer 'yes') or false (with answer 'no') in any particular world.

The set of worlds will usually be a very large and unwieldy mathematical object, so it will not usually be practical to explicitly define underlying queries, or deal directly with worlds. The language used to express a query will depend on the type of dataset it is expecting; in our case it will often expect a particular type of map, because of referring to a particular theme; for example, the query '*What was the 2002 yearly rainfall at* x_0' could be answered using an atoms-map $M : X \to T$ giving 2002 yearly rainfall values. The kind of query we are interested in can be expressed, for some theme-division T and set A, as a function $q : F(X,T) \to A$. We call this an *atoms-map query*. This defines an underlying query $\underline{q} : W \to A$ by $\underline{q}(w) = q(M)$ where $w \models M$. Atoms-map query q can be viewed as a simpler representation of underlying query $\underline{q}$.

6.1 Underlying queries

An underlying query $\underline{q}$ is a function from the set of worlds W to a set A. If w is the world then the answer to the query is $\underline{q}(w)$. However, we will usually not know precisely which world w is the true world. $\underline{q}$ can be extended to a function on the set of subsets of W in the obvious way: for $Z \subseteq W$, $\underline{q}(Z) = \{\underline{q}(w) : w \in Z\}$. If we know that the true world is in Z then the answer to the query must be in $\underline{q}(Z)$.

Suppose we know that atoms-map $M : X \to T$ is an accurate description of the world; this is saying that the true world w is in $[M]$. We can define $\underline{q}(M)$ to be $\underline{q}([M])$. The set $\underline{q}(M)$ gives the possible correct answers to the query given that M is accurate.

Suppose instead we know that regions-map $N : G \to T$ is accurate, where $N = \sigma(M)$ for summarisation operator σ and some unknown atoms-map M. We can define $\underline{q}(N)$ to be $\underline{q}([N])$. The set of worlds $[N]$ equals $\bigcup_{\sigma(M)=N}[M]$, so $\underline{q}(N) = \bigcup_{\sigma(M)=N} \underline{q}(M)$. We can extend this for any map structure $\mathcal{N}$: define $\underline{q}(\mathcal{N})$ to be $\underline{q}([\mathcal{N}])$.

6.2 Atoms-maps queries

A query will often relate to a certain themes-division T (e.g. rainfall values, land cover type): it is expecting a particular type of map.

Define an atoms-map query to be a function from $F(X,T)$ to A, for some T and A. We can ask the query of any map, not just the type of map it's intended for: but we may not get a precise answer, as the map may not carry enough information to determine a unique answer. Let $q : F(X,T) \to A$ be a query, and $N : X \to U$ be an atoms-map. $\hat{q}(N)$ is intended to be the set of possible answers to the query given that N is an accurate description of the world. We look at all maps $M : X \to T$ which are consistent with N, and define $\hat{q}(N)$ to be $\{q(M) : \text{cons}\{M, N\}\}$. In practice, as discussed in 5.1, we may well only have relation cons′ which approximates cons, so we may approximate $\hat{q}(N)$ by $\{q(M) : \text{cons}'\{M, N\}\}$ which contains $\hat{q}(N)$.

We can say that N answers the query q precisely if $\hat{q}(N)$ is a singleton: i.e. there's only one answer to the query that is consistent with the map N.

The meaning of an atoms-map query $q : F(X, T) \to A$ is understood to be the same as the underlying query $\underline{q}$ defined by $\underline{q}(w) = q(w^{\downarrow T})$, i.e. $\underline{q}(w) = q(M)$ where M is the atoms-map such that $[M] \ni w$. We then have $\hat{q}(N) = \underline{q}(N)$, since $\underline{q}(N) = \underline{q}([N]) = \{\underline{q}(w) : w \models N\} = \{q(M) : \exists w : w \models M, N\} = \{q(M) : \text{cons}\{M, N\}\} = \hat{q}(N)$.

The function $\hat{q}$ can be extended to regions-maps. Suppose we now know that regions-map $N : G \to U$ is accurate, where $N = \sigma(M')$ for summarisation operator σ and unknown atoms-map M'. We have $\underline{q}(N) = \{\underline{q}(w) : w \models N\} = \{q(M) : \exists w : w \models M, N\} = \{q(M) : \text{cons}\{M, N\}\}$. So we can define $\hat{q}(N) = \{q(M) : \text{cons}\{M, N\}\}$. Then $\hat{q}(N)$ is the set of possible answers to the query given that N is accurate.

Similarly, $\hat{q}$ can be extended to any map structure $\mathcal{N}$, by $\hat{q}(\mathcal{N}) = \{q(M) : \text{cons}\{M, \mathcal{N}\}\}$, where $\text{cons}\{M, \mathcal{N}\}$ if and only if $[M] \cap [\mathcal{N}] \neq \emptyset$. Again we have $\hat{q}(\mathcal{N}) = \underline{q}(\mathcal{N})$.

Example

We are interested in knowing the value of mean temperature over the year 2002 for a particular small areal atom x_0 in atoms set X. Furthermore we would like to know the atom in X which has highest mean temperature (we assume this is unique).

Both queries are concerned with an unknown function $M : X \to T$ where each element of T represents a precise numerical value in degrees Centigrade, and $M(x) = t$ means that the mean temperature over 2002 at x is $t°C$.

The first query can be written as a function $q_1 : F(X, T) \to T$ where $q_1(M) = M(x_0)$. The second query can be written as a function $q_2 : F(X, T) \to X$ where $q_2(M)$ is the atom x with highest value of $M(x)$.

Now suppose the information we have is a function $N : X \to T'$ which is a degraded form of M; here T' is a set of disjoint ranges of temperature, including e.g. a range $[0°C, 10°C)$, which together cover all possible temperatures, and $N = \tau(M)$ is the natural themes-coarsening; it is generated from $\tau : T \to T'$ given by: $\tau(t)$ is the range of temperature containing value t.

$\hat{q}_1(N)$ is then the set of possible answers to the query given that N is correct. $\hat{q}_1(N) = \{q_1(M) : \text{cons}\{M, N\}\}$. M and N are consistent if and only if $N = \tau(M)$, so $\hat{q}_1(N)$ equals $\{M(x_0) : \tau(M) = N\} = \{t \in T : \tau(t) = N(x_0)\}$ which is just the set of points in the interval $N(x_0)$. Similarly, $\hat{q}_2(N) = \{q_2(M) : \tau(M) = N\}$ which is the set of atoms x such that $N(x) = t'$, where t' is the highest interval range attained by N over X.

6.3 Regions-maps queries

A query may be couched in terms of a region or set of regions, and implicitly refer to a particular spatial coarsening. These can be interpreted as atoms-maps queries. For example, a query may ask the mean value of an attribute over a region g. If G is a regions set containing g, then q can be written as a function from $F(G, T)$ to set of answers A, where each regions-map $F(G, T)$ is understood to have been produced from an atoms-map in $F(X, T)$ by mean-coarsening. q then corresponds to an atoms-map query $q' : F(X, T) \to A$ given by $q'(M) = q(\mu_G(M))$, and has underlying query $\underline{q} : W \to A$ given by $\underline{q}(w) = q(\mu_G(w^{\downarrow T}))$.

6.4 Answering queries in terms of consequence relation between propositions

By a proposition we mean something that, in any world, is either unambiguously true (it holds) or is unambiguously false. An atoms-map M can be considered as a proposition in this sense because in any particular world w, either M is satisfied or not. For proposition p write $w \models p$ if p holds in world w, and for set of propositions J write $w \models J$ if for all $p \in J$, $w \models p$, i.e., every element of J holds in w. Let q be an (underlying, atoms-map or regions-map) query with answer set A. For any $a \in A$, the statement $q = a$, meaning that the answer to query q is equal to a, can be considered as a proposition: it is true or false in any particular world; similarly if $B \subseteq A$, a statement $q \in B$, meaning that the answer to the query is in B.

There is a natural consequence relation on propositions: $p \models p'$ if and only if $w \models p$ implies $w \models p'$, i.e. p' holds whenever p holds. Similarly, for set J of propositions, we say $J \models p'$ if $w \models p'$ holds for all w such that $w \models J$ holds. This consequence relation $\models$ between propositions is reflexive, transitive and (hence) monotonic. Answering queries can be expressed in terms of this consequence relation. In particular, if M is consistent we have $M \models (q = a)$ if and only if $\hat{q}(M) = \{a\}$. We also have $M \models (q \in B)$ if and only if $\hat{q}(M) \subseteq B$. Therefore $\hat{q}(M)$ is the (unique) smallest set B such that $M \models (q \in B)$.

The monotonicity property of the consequence relation implies that anything deducible from either map M_1 or map M_2 is deducible from $\{M_1, M_2\}$.

7 Integration of maps

This section looks at how integration works for some pairs of maps of the forms discussed above. The aim is to produce mathematical representations for the integrated dataset which can be used to answer particular types of query. The first case we consider is the combination of atoms-maps; here we can construct an atoms-map that is semantically equivalent to a pair (or set) of atoms-maps. This differs from the other cases we will look at: then a pair of maps is not semantically equivalent to any single map, but a disjunctive set of maps.

For combining the information given by regions-maps there are lots of possibilities, according to how the regions-map was produced, even among the various spatial coarsenings we have discussed. Two cases are examined: combining a mean-coarsened map with a theme-generated map, and combining two mean-coarsened maps. Other possibilities can be approached in a similar fashion.

7.1 Combination of atoms-maps

Suppose we have two atoms-maps on the same set of atoms X. $M_1 : X \to T_1$ and $M_2 : X \to T_2$. We assume that both give accurate information about the world. What can we deduce from the pair of maps? M_1 tells us that the set of theme-statements $M_1(x)$ holds at x. M_2 tells us that $M_2(x)$ holds at x. Together they tell us precisely that at x both $M_1(x)$ and $M_2(x)$ hold, so $M_1(x) \cup M_2(x)$ holds.

Define the combined themes-division $T_1 \otimes T_2$ to be $\{t_1 \cup t_2 : t_1 \in T_1, t_2 \in T_2\}$. The theme-elements in $T_1 \otimes T_2$ are mutually exclusive and exhaustive. Not all elements in this combination will necessarily be possible. A theme-element $t_1 \cup t_2$ is possible if and only if there exists some state s in $\underline{S}$, the set of possible states, with $s \models u$ for all theme-statements $u \in t_1 \cup t_2$.

Define the combined atoms-map $M_1 \otimes M_2 : X \to T_1 \otimes T_2$ by, for $x \in X$, $(M_1 \otimes M_2)(x) = M_1(x) \cup M_2(x)$. The operation $\otimes$ is commutative and associative, and the combined map $M_1 \otimes M_2$ conveys exactly the same information as the pair of maps M_1 and M_2. We can extend this combination to arbitrary sets of atoms-maps. Let $\{T_i : i \in I\}$ be a set of themes-divisions. Define themes-division $\bigotimes_{i \in I} T_i$ to be the set of all unions $\bigcup_{i \in I} t_i$, for all different combinations of choices of elements t_i of each T_i, so that $\bigotimes_{i \in I} T_i = \{\bigcup_{i \in I} t_i : \forall i \in I, t_i \in T_i\}$. This extends the definition of a pair of themes-divisions: $\bigotimes \{T_1, T_2\} = T_1 \otimes T_2$, and $\bigotimes \{T_1, T_2, \ldots, T_k\} = T_1 \otimes T_2 \otimes \cdots \otimes T_k$.

Let, for each $i \in I$, $M_i : X \to T_i$ be an atoms-map. Then define $M = \bigotimes_{i \in I} M_i$ from atoms set X to themes-division $\bigotimes_{i \in I} T_i$ by $M(x) = \bigcup_{i \in I} M_i(x)$. This extends the definition for a pair of maps: $\bigotimes \{M_1, M_2\} = M_1 \otimes M_2$, and $\bigotimes \{M_1, M_2, \ldots, M_k\} = M_1 \otimes M_2 \otimes \cdots \otimes M_k$.

Note that we don't need to know anything about the set of worlds in order to combine.

The operation $\otimes$ is closely related to the operations $\otimes$ described in (Worboys and Duckham, 2002) (a difference is that $\otimes$ as defined in (Worboys and Duckham, 2002) is strictly speaking not associative).

The semantic equivalence of the combined map. $[M_1]$ is the set of worlds w such that $w \models M_1$, i.e. the set of possible worlds given that M_1 is accurate. If we know that both M_1 and M_2 are accurate then we know that the true world w is in $[M_1] \cap [M_2]$, which happens if and only if for all x in X, $w(x)$ satisfies both $M_1(x)$ and $M_2(x)$, which happens if and only if for all x, $w(x)$ satisfies $M_1(x) \cup M_2(x)$. This means that $[M_1] \cap [M_2] = [M_1 \otimes M_2]$, so that $M_1 \otimes M_2$ conveys exactly the same information as the pair of maps M_1, M_2. Similarly for $w \in W$, w satisfies $\bigotimes_{i \in I} M_i$ if and only if for all $x \in X$, $w(x) \models \bigcup_{i \in I} M_i(x)$, which happens if and only if, for all $i \in I$, $w(x) \models M_i(x)$, i.e. for all $i \in I$, w satisfies M_i. This shows that $[\bigotimes_{i \in I} M_i] = [\{M_i : i \in I\}]$.

This means that consistency of a (finite or infinite) set of atoms-maps can be reduced to consistency of a single atoms-map. Let $\mathcal{M}$ be a set of atoms-maps $\{M_i : i \in I\}$, where $M_i : X \to T_i$. Then $\mathcal{M}$ is consistent, cons$\mathcal{M}$, if and only if the atoms-map $M = \bigotimes_{i \in I} M_i$ is consistent. In particular, the pair $\{M_1, M_2\}$ is consistent if and only if the atoms-map $M_1 \otimes M_2$ is consistent.

7.2 Combining a mean-coarsened map with a theme-generated map

This section considers the integration of a mean-coarsened map (see 4.1) and a theme-generated map (see 4.3). The integration of maps M and P' in the introductory example is a special case of this.

Let $M_1 : G \to T$ and $M_2 : H \to T'$ be regions-maps over areal atoms set X, where G and H have the same spatial extent, i.e. $\bigcup_{g \in G} g = \bigcup_{h \in H} h$. (The case where G and H have different spatial extents can be managed in a similar way.) Both atoms-maps are summarisations of an underlying atoms-map $M : X \to T$ which gives, for each atom x, the 2002 rainfall in millimetres at x; mathematically, we consider T just as $[0, \infty)$. M_1 gives the rainfall values averaged over each region $g \in G$, so is a mean-coarsening of M. T' is a division of the values of rainfall into intervals, for example, $[0, 100)$, $[100, 200)$, $[200, 300)$, $[300, 350)$, etc.; (the method we give applies to any division into intervals). $M_2(h) = t'$ means that the rainfall value *for every atom in* h is in the interval t'. M_2 can thus be considered as a theme-generated regions-map generated from a themes-coarsening of M. We make no *a priori* assumptions on the form of M, so that any function $X \to T$ is considered initially possible.

We would like to know if these two maps are consistent, and also what they tell us about mean rainfall values over $h \in H$; M_2 gives us upper and lower bounds for this: it tells us that the mean rainfall value over h is in the interval $M_2(h)$; we would like to know if learning the information given by M_1 enables us to deduce stronger information about those mean rainfall values.

What M_1 tells us about M: We know $M_1 = \mu_G(M)$ where μ_G is the mean-coarsening operator to regions set G, so that for all $g \in G$, $\sum_{x \subseteq g} \frac{|x|}{|g|} M(x) = M_1(g)$.

What M_2 tells us about M: For all $x \subseteq h$, $M(x) \in M_2(h)$, i.e. $M_2^L(h) \leq M(x) < M_2^U(h)$, writing the interval $M_2(h)$ as $[M_2^L(h), M_2^U(h))$.

We are assuming that we know M_1, M_2 and the area of each atom, so the only unknown values in these equations and inequalities are the values of $M(x)$ for each $x \in X$.

Let $\mathcal{M}$ be the set of all atoms-maps M satisfying these conditions for M_1

and M_2. Consistency of the maps is equivalent to $\mathcal{M}$ being non-empty. The interval of possible values for the mean rainfall for $h \in H$ is the set of values $\{\mu_H(M)(h) : M \in \mathcal{M}\}$, where μ_H is the mean-coarsening operator to regions set H, and $\mu_H(M)(h) = \sum_{x \subseteq h} \frac{|x|}{|h|} M(x)$. The equations and inequalities are linear, and the number of variables is finite (the number of atoms), so a linear programming approach can be used to test for consistency and find the intervals of possible values for $h \in H$. However, there is a much more efficient way to solve the problem.

Define regions set GH to be $\{g \cap h : g \in G, h \in H, g \cap h \neq \emptyset\}$. Let $\mathcal{N}$ be the set of regions-maps $N : GH \to T$ satisfying

(i) for each $g \in G$, $\sum_{h \in H} \frac{|g \cap h|}{|g|} N(g \cap h) = M_1(g)$ (where we define $N(\emptyset) = 0$); and

(ii) for all $g \cap h \in GH$, $N(g \cap h) \in M_2(h)$, i.e. $M_2^L(h) \leq N(g \cap h) < M_2^U(h)$.

It can be shown[7] that $\mathcal{N} = \mu_{GH}(\mathcal{M})$, where μ_{GH} is the mean-coarsening to GH; in other words, for all $M \in \mathcal{M}$, $\mu_{GH}(M)$ satisfies conditions (i) and (ii), and, furthermore, for any N satisfying (i) and (ii) there exists $M \in \mathcal{M}$ with $N = \mu_{GH}(M)$. We then have M_1 and M_2 are consistent $\iff \mathcal{M} \neq \emptyset \iff \mathcal{N} \neq \emptyset \iff$ (i) and (ii) have a solution N.

(i) and (ii) have a simple structure that makes checking if they have a solution very easy; the constraints are clearer if we change the notation a little: for $N \in \mathcal{N}$, $g \in G$, $h \in H$, define $N'(g,h) = |g \cap h| \, N(g \cap h)$, define $l_{g,h} = |g \cap h| \, M_2^L(h)$ and $u_{g,h} = |g \cap h| \, M_2^U(h)$. Also define $m_g = |g| M_1(g)$. The values $N'(g,h)$ are unknowns, and we know the values $l_{g,h}$, $u_{g,h}$ and m_g. The constraints can now be written as:

for all $g \in G$, $\sum_{h \in H} N'(g,h) = m_g$, and,

for all $g \in G$, $h \in H$, if $g \cap h = \emptyset$, $N'(g,h) = 0$, and if $g \cap h \neq \emptyset$, $l_{g,h} \leq N'(g,h) < u_{g,h}$.

Considering $G \times H$ as a table, we have upper and lower bounds on each element of the table, and we also know the sums of each row. It is then clear that the constraints are consistent if and only if for each $g \in G$, $\sum_{h \in H} l_{g,h} \leq$

[7] This also shows that the results do not depend on the particular atoms set X (given that X is such that G and H are regions sets on X).

$m_g < \sum_{h\in H} u_{g,h}$, i.e. if and only if

$$\sum_{h\in H} \frac{|g\cap h|}{|g|} M_2^L(h) \leq M_1(g) < \sum_{h\in H} \frac{|g\cap h|}{|g|} M_2^U(h).$$

Suppose now that this condition holds, so that the two maps are consistent. We would like to know the mean value of rainfall over a particular $h_0 \in H$. One way of expressing this query is as a function $q : F(H,T) \to [0,\infty)$, where elements of $F(H,T)$ are understood as mean-coarsenings of atoms-maps. For $K : H \to T$ the value of $q(K)$ is $K(h_0)$. We would like to know what M_1 and M_2 tell us about the answer to q; this is given by $\hat{q}(\{M_1, M_2\})$ which is the set of possible values $K(h_0)$ consistent with both M_1 and M_2, which equals $\{\mu_H(M)(h_0) : M \in \mathcal{M}\}$; write this as $\mu_H(\mathcal{M})(h_0)$. By the transitivity property of mean-coarsening, $\mu_H(\mathcal{M}) = \mu_H^{GH}(\mu_{GH}(\mathcal{M})) = \mu_H^{GH}(\mathcal{N})$. The set $\mu_H^{GH}(\mathcal{N})(h_0)$ can be written as $\{\frac{1}{|h_0|}\sum_{g\in G} N'(g,h_0) : N' \in \mathcal{N}'\}$, where $\mathcal{N}'$ is the set of all N' satisfying the constraints. So the possible values are proportional to the set of possible sums of column h_0 in the table $G \times H$. Each element in the column has independent constraints, hence the set of possible values is the set of all values $\frac{1}{|h_0|}\sum_{g\in G} n(g,h_0)$ where $n(g,h_0)$ is a possible value for $N'(g,h_0)$. Write $L_{g,h_0} = \max(l_{g,h_0}, m_g - \sum_{h\neq h_0} u_{g,h})$, and $U_{g,h_0} = \min(u_{g,h_0}, m_g - \sum_{h\neq h_0} l_{g,h})$. The infimum of possible values for $N'(g,h_0)$ is L_{g,h_0}, and the supremum of possible values is U_{g,h_0}. Therefore the infimum and supremum possible values for the mean rainfall over h_0 are $\frac{1}{|h_0|}\sum_{g\in G} L_{g,h_0}$ and $\frac{1}{|h_0|}\sum_{g\in G} U_{g,h_0}$, respectively; any value between these two is a possible value for the mean rainfall over h_0.

7.3 Combining two mean-coarsened maps

Consider now the case where we are given two regions-maps $M_1 : G \to T$ and $M_2 : H \to T$, with G and H being regions sets over areal atoms set X with the same spatial extent. As before, they are both summarisations of an underlying atoms-map $M : X \to T$ which gives, for each atom x, the 2002 rainfall in millimetres for x; M_1 gives the mean rainfall over the regions $g \in G$, so is a mean-coarsening of M. Here M_2 also gives the mean rainfall over regions $h \in H$, so is also a mean-coarsening of M. Again we make no *a priori* assumptions on the form of M, so that any function $X \to T$ is considered initially possible.

As before, M_1 and M_2 give us constraints on M; let $\mathcal{M}$ be the set of M satisfying these constraints, that is, the set of all M which mean-coarsened to G give M_1 and when mean-coarsened to H give M_2.

Let $\mathcal{N}$ be $\{\mu_{GH}(M) : \mu_G(M) = M_1, \mu_H(M) = M_2\}$, the result of mean-coarsening each M in $\mathcal{M}$ to $GH = \{g \cap h : g \in G, h \in H, g \cap h \neq \emptyset\}$. It can be shown that the function $N : GH \to T$ is in $\mathcal{N}$ if and only if satisfies $\mu_G^{GH}(N) = M_1$ and $\mu_H^{GH}(N) = M_2$.

In terms of the semantics, $w \models \bigvee \mathcal{M}$ if and only if $w \models M$ for some M such that $\mu_G(M) = M_1$ and $\mu_H(M) = M_2$. Since $w \models M \iff w^{\downarrow T} = M$, $w \models \bigvee \mathcal{M}$ if and only if $\mu_G(w^{\downarrow T}) = M_1$ and $\mu_H(w^{\downarrow T}) = M_2$, which is if and only if $w \models \{M_1, M_2\}$. We have $w \models \bigvee \mathcal{N}$ if and only if $w \models N$ for some N such that $\mu_G^{GH}(N) = M_1$ and $\mu_H^{GH}(N) = M_2$. $w \models N \iff \mu_{GH}(w^{\downarrow T}) = N$, so $w \models \bigvee \mathcal{N}$ if and only if $\mu_G^{GH}(\mu_{GH}(w^{\downarrow T})) = M_1$ and $\mu_H^{GH}(\mu_{GH}(w^{\downarrow T})) = M_2$, i.e. $\mu_G(w^{\downarrow T})) = M_1$ and $\mu_H(w^{\downarrow T}) = M_2$, because of the transitivity property. Hence $[\{M_1, M_2\}] = [\bigvee \mathcal{M}] = [\bigvee \mathcal{N}]$, and the disjunctive set of models $\mathcal{N}$ carries exactly the same information as the pair of maps $\{M_1, M_2\}$.

Regions-map $N : GH \to T$ is in $\mathcal{N}$ if and only if $\mu_G^{GH}(N) = M_1$ and $\mu_H^{GH}(N) = M_2$. Therefore N is in $\mathcal{N}$ if and only if it satisfies the two conditions:

(i) for each $g \in G$, $\sum_{h \in H} \frac{|g \cap h|}{|g|} N(g \cap h) = M_1(g)$; and

(ii) for each $h \in H$, $\sum_{g \in G} \frac{|g \cap h|}{|h|} N(g \cap h) = M_2(h)$.

(i) and (ii) are linear constraints on the $|GH|$ variables $N(e)$ for $e \in GH$. We can use standard techniques to test whether these have a solution, so whether M_1 and M_2 are consistent.

Similarly if we want to compute the bounds on mean rainfall values for a region $g_0 \in G$ we can consider the new variable N_{g_0} defined to be $\sum_{h \in H} \frac{|g_0 \cap h|}{|g_0|} N(g_0 \cap h)$ and compute the upper and lower bounds of this. Other queries can be answered in a similar fashion.

Mode-coarsened maps and proportional-coarsened maps also give rise to linear constraints, so combining e.g. a proportional-coarsened map on G and a proportional-coarsened map on H again leads to linear constraints on the values of $e \in GH$.

8 Discussion of representation of uncertain information

This paper has not so far dealt with uncertainty; it has been assumed that when we are given an atoms-map, it gives us accurate information, and that a regions-map is a summarisation of an accurate atoms-map. This is a simplification which will not always be reasonable; a regions-map will often be a summarisation of an atoms-map that represents noisy data; in particular, the atoms-map may be the result of interpolating data at a number of locations (and the data here may also be somewhat inaccurate). A representation of uncertainty can also be used to deal with cases where part of the summarisation process is too complex to fully model. This section briefly indicates how uncertainty may be represented within this framework.

Logical representation for uncertainty

Suppose we are given atoms-map $M_0 : X \to T$. Let $M : X \to T$ be the true atoms-map in $F(X, T)$, which tells us which theme-element $M(x) \in T$ actually holds at each atom $x \in X$; we want to be able to deduce information about M in order to answer queries. If M_0 can be reasonably assumed to be completely accurate then we can say that $M = M_0$. Otherwise M_0 will usually still give us useful information about M: it may restrict the plausible identity of M to a set $C_{M_0} \subseteq F(X, T)$. Often C_{M_0} will be all possible atoms-maps that are close (in some sense) to M_0. The information we are given is taken to be semantically equivalent to $\bigvee C_{M_0}$, so the possible worlds are all those with $w^{\downarrow T} \in C_{M_0}$.

As a very simple example, suppose we are given an areal atoms-map $M_0 : X \to T$ giving mean temperatures over the year 2002 for each atom. We may decide that each value of M_0 is accurate within $2°C$, leading to C_{M_0} being all $M : X \to T$ with for all $x \in X$, $|M(x) - M_0(x)| \leq 2$.

Suppose that instead of M_0 we are given regions-map $N : G \to T$, which was produced by summarisation operator σ from some noisy data $M_0 : X \to T$. The atoms-maps $M : X \to T$ consistent with this information are all those such that M is in C_{M_0} for some M_0 with $\sigma(M_0) = N$.

The approach to query-answering and integration described above can be applied also to this representation of uncertain information.

Graded representation of uncertainty

The previous representation of uncertain information doesn't allow grades of plausibility of atoms-maps: a map is either considered plausible or implausible. Often it can be natural to have degrees of plausiblity/implausibility.

Firstly there is *a priori* implausibility: some maps (and hence some worlds) are in themselves less plausible than others. For example, large changes in values of a particular attribute between neighbouring areal atoms, may be considered fairly implausible, though not impossible. In addition, we may be certain that some particular combinations of theme-statements are incompatible, but only fairly sure that other combinations are incompatible.

Secondly, there is the issue of implausibility of maps after we have been given a particular map M_0. We can also consider grades of how plausible maps are, given M_0. For example, it is natural to view maps closer to the input map as more plausible than those further away.

There are various uncertainty formalisms that can be used for representing and reasoning with this type of information. Perhaps the simplest is possibilistic logic (Dubois, Lang and Prade, 1994) which involves reasoning with constraints on an unknown possibility distribution, allocating totally ordered grades to worlds and hence maps.

Another class of formalisms which could be useful for reasoning with uncertain map data is default/non-monotonic logics e.g., (Reiter, 80). For example, suppose a parcel in a land cover map is labelled 'grassland', where the label is the result of a summarisation process involving some sort of averaging over pixels in the parcel. If a user is interested in a sub-region h of the parcel we might report a land cover class of 'grassland' for h, using a default inference. However, it should be made clear to the user that this is just a tentative conclusion, which may very well be incorrect (even if the land cover map itself is entirely correct). See also (Pratt, 93) for a discussion of the use of default reasoning in a formal semantics for maps.

9 Summary and extensions

It can be very important to describe as precisely as possible the meaning of a map or other spatial dataset, especially when integrating different sources of data; without this it will often not be possible to say what deductions can reliably be drawn from the integrated dataset. The meaning of a spatial dataset can sometimes come from considering how the dataset was produced: thinking in terms of all the ways the world could be that are consistent with this spatial dataset being produced. Because of this a precise description of the production method can be very important metadata. In particular, a map is intended to summarise information which is spatially-distributed; analysis of this summarisation operator can lead to an understanding of the meaning of the map.

This paper describes the beginnings of a formal framework for the meaning of certain kinds of thematic maps and similar spatial datasets, interpreting them in terms of a set of possible worlds, by considering the summarisation operator that generated the dataset. Two important basic types of summarisation operators are discussed: themes-coarsening—classifying the values of an attribute that is being mapped—and spatial coarsening, for example, averaging the values of an attribute over areas of the domain. Queries are formally defined, which allow one to express what can be deduced from a single spatial dataset, or from an integrated dataset.

There are many directions in which this work could be extended. Spatial datasets will not often be exactly interpretable in terms of such simple summarisation operators. Representation of uncertainty will usually be essential for taking into account noisy data and complex aspects of the summarisation (and interpolation) procedures. Logical, possibilistic and non-monotonic reasoning approaches were discussed briefly in section 8; other natural possibilities include likelihood-based and Dempster-Shafer (Evidential Reasoning) approaches. Other potential extensions include: exploration of other summarisation operators, analysis of interpolation techniques within the framework, and extension to temporal as well as spatial datasets. Furthermore, it could be valuable to develop this kind of approach for other types of maps, e.g., with representations of objects such as roads and rivers.

Acknowledgements. This work was supported by the REVIGIS project, IST-1999-14189. I am very grateful to my colleagues for helpful and interesting discussions on related topics, in particular, Lex Comber, Matt Duckham, Mike Worboys, Jenny Lingham, Mike Goodchild, Richard Wadsworth, Peter Fisher and Keith Mason.

References

DENT, B. D., 1996, *Cartography. Thematic Map Design.* Fourth Edition (Wm. C. Brown Publishers).

DUBOIS, D., LANG, J., PRADE, H., 1994, Possibilistic logic. In *Handbook of Logic in Artificial Intelligence and Logic Programming*, D. Gabbay, C. Hogger, J. Robinson (eds) (Oxford University Press), Vol. 3, pp439–513, .

FULLER, R. M., SMITH, G. M., SANDERSON, J. M., HILL, R. A., THOMSON, A. G., 2002, The UK Land Cover Map 2000: Construction of a parcel-based vector map from satellite images. *The Cartographic Journal* **39** No. 1, 15–25, June 2002. See also `http://www.ceh.ac.uk/data/lcm/LCM2000.shtm`

PETIT, C. C., LAMBIN, E. F., 2001, Integration of multi-source remote sensing data for land cover change detection, *International Journal of Geographical Information Science* **15**, no. 8, 785–803.

MARK, D. M., 1999, Spatial representation: a cognitive view. In *Geographical Information Systems Volume 1 Principles and Technical Issues* (second edition), Longley, P. A., Goodchild, M. F., Maguire, D.J., and Rhind, D.W. (eds), (John Wiley & Sons Inc.), pp81–89.

PRATT, I., 1993, Map semantics. In *Spatial Information Theory, A Theoretical Basis for GIS, COSIT '93*, Frank, A.U., Campari, I., (eds) (Springer-Verlag), pp77–91.

REITER, R., 1980, A logic for default reasoning, *Artificial Intelligence* **13** (Nos. 1, 2): 81–132.

SHAFER, G., 1976, *A Mathematical Theory of Evidence* (Princeton, NJ: Princeton University Press).

WEIBEL, R., 1997, Generalization of spatial data: principles and selected algorithms. Chapter 5 of *Algorithmic Foundations of Geographic Information Systems*, van Kreveld, M., Nievergelt, J., Roos, T.,n Widmayer, P., (eds) (Springer).

WORBOYS, M. F., DUCKHAM, M., 2002, Integrating spatio-thematic information, *GI Science 2002*, Boulder, Colorado, USA.