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MONTE CARLO ANALYSIS OF RESISTIVE NETWORKS WITHOUT APRIORI PROBABILITY DISTRIBUTIONS

B. Ross Barmish and Houssain Kettani

ECE Department
University of Wisconsin
Madison, WI 53706

ABSTRACT

In this paper, we formulate and solve a new type of Monte Carlo problem for a resistive network. Given lower and upper bounds on the value of each resistor but no probability distribution, we consider the problem of finding the expected value for a designated gain. In view of the fact that no apriori probability distributions for the uncertain resistors are assumed, a certain type "distributional robustness" is sought. To this end, a new paradigm from the robustness literature is particularized to circuits and results are provided in this context. Some of the performance bounds obtained via this new approach differ considerably from those which result from a more conventional Monte Carlo simulation.

1. INTRODUCTION AND FORMULATION

In this paper, we consider a planar network N consisting of an input voltage source V_{in}, an output voltage V_{out} across a designated resistor $R_{out} = R_n$ and uncertain resistors $R \doteq (R_1, R_2, \ldots, R_n)$ as depicted in Figure 1. Although we work with only a single independent source which is a voltage, it is noted the results in this paper are readily modified to accommodate current sources and a variety of multiple source networks.

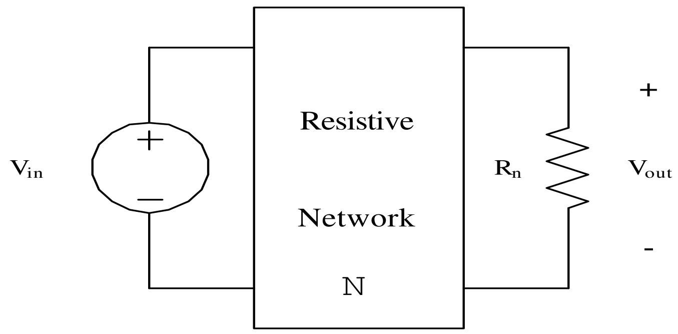

Figure 1: Network Configuration

1.1. Uncertain Resistors

To describe the uncertainty, for each resistor, we write

$$R_i \doteq R_{i,0} + \Delta R_i$$

with *nominal manufacturing value* $R_{i,0} > 0$ and *uncertainty* ΔR_i with given bounds

$$\Delta R_i \leq r_i; \quad r_i \geq 0$$

for $i = 1, 2, \ldots, n$. Writing

$$R_i \in \mathcal{R}_i \doteq [R_{i,0} - r_i, R_{i,0} + r_i],$$

the box of *admissible resistor uncertainty* is given by

$$\mathcal{R} \doteq \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n.$$

To complete the formal description of the basic circuit model, we also include the condition that only positive resistances are feasible. That is, $R_{i,0} > r_i$ for $i = 1, 2, \ldots, n$.

1.2. Uncertain Gain

For the system above, we focus attention on the uncertain gain

$$g(R) \doteq \frac{V_{out}}{V_{in}}.$$

With each uncertain resistor R_i assumed to be an independent random variable with probability density function $f_i(R_i)$, the joint probability density functions is

$$f(R) \doteq f(R_1, R_2, \ldots, R_n) = f_1(R_1) f_2(R_2) \cdots f_n(R_n)$$

and the multi-dimensional integral for the *expected gain* is given by

$$\mathcal{E}(g(R)) = \int_{\mathcal{R}} f(R) g(R) dR.$$

1.3. Monte Carlo Considerations

A classical Monte Carlo simulation for the network N above involves assuming apriori probability density functions $f_i(R_i)$; e.g., see [1]-[4]. In contrast, this paper addresses the case when there is no such apriori information available. Without knowledge of the probability density functions $f_i(R_i)$, we show, in the sense of *distributional robustness* to be described, that it is still possible to assess the expected gain performance. Unlike many classical Monte Carlo approaches to problems without apriori probability distributions, in this paper, we do not simply impose some "reasonable" distribution, such as normal or uniform, on the R_i. Our contention, consistent with other caveats found in the literature, is that the imposition of such an ad hoc probability distribution can lead to an unduly optimistic assessment of performance; e.g., see [1], [4] and [5]. Instead, we provide a new method for probabilistic assessment which we believe leads to more realistic estimates of performance in the absence of such apriori information.

Our method is seen to be *distributionally robust* in the following sense: With resistor uncertainty as described above, the expected performance which we obtain is guaranteed for all probability distributions f in a given class $\mathcal{F}$. The definition of $\mathcal{F}$, initially provided in [5], is felt to be particularly apropos to circuits for which components are described by tolerances as in [9]. To this end, the definition of $\mathcal{F}$ is based on the intuitive notion that positive and negative deviations from the mean manufacturing value $R_{i,0}$ are equally likely and, that the larger the deviation in R_i from $R_{i,0}$, the less likely it is to occur. Instead of conducting a Monte Carlo simulation with an assumed probability distribution $f \in \mathcal{F}$ for the uncertain resistors, we solve a pair of variational problems (see Section 2) to obtain the maximum and minimum expected gains over $\mathcal{F}$. A solution $f^* \in \mathcal{F}$ to such a variational problem defines the probability distribution to be used in a subsequent Monte Carlo simulation. In other words, rather than assume an apriori probability distribution for simulation, we obtain an aposteriori distribution which is the solution of a maximum or minimum gain problem.

1.4. Admissible Probability Densities

It is assumed the R_i are an independent random variables supported in $\mathcal{R}_i$ with an unknown probability density function $f_i(R_i)$ which is symmetric about its mean $R_{i,0}$; i.e., it is assumed that positive and negative deviations away from $R_{i,0}$ as equally likely. It is also assumed that $f_i(R_i)$ is non-increasing in $|R_i - R_{i,0}|$; i.e., large deviations from the mean $R_{i,0}$ are less likely than small deviations. It is noted that this formulation allows each R_i to have a different density function.We write $f \in \mathcal{F}$ to denote an *admissible joint density function* $f(R)$ over $\mathcal{R}$. Given any $f \in \mathcal{F}$, the resulting random vector of resistors is denoted as R^f. Two important special cases of interest are obtained when $f = u \in \mathcal{F}$ is the uniform distribution and $f = \delta$ is the impulse (Dirac) distribution centered on $R_{i,0}$.

2. DISTRIBUTIONAL ROBUSTNESS

Within this new framework, the objective of this paper is to provide a method to compute the sharpest possible bounds for the expected gain. These bounds, defined below, are seen to be "distributionally robust" with respect to $\mathcal{F}$.

2.1. Sharp Bounds for Expected Gain

In the absence of an apriori probability density function for the resistors, we define the *maximum expected gain*

$$g^+ \doteq \max_{f\in\mathcal{F}} \mathcal{E}(g(R^f))$$

and the *minimum expected gain*

$$g^- \doteq \min_{f\in\mathcal{F}} \mathcal{E}(g(R^f)).$$

In the next section, conditions under which each of these two extrema is attained are provided; i.e., the use of maximum instead of supremum and minimum instead of infimum above is justified.

3. MAIN RESULTS

In order to convey the main result, a definition is required; see [6] for further details.

3.1. Essential Resistors

A resistor R_k is said to be *essential* if the following condition holds: There does not exist admissible values of the $n-1$ remaining resistors $R_i^* \in \mathcal{R}_i$, $i \neq k$ making the gain $g(R)$ independent of $R_k \in \mathcal{R}_k$. If R_k is essential, it can readily be shown that the partial derivative $\partial g/\partial R_k$ is non-zero over $\mathcal{R}$. Hence, by an intermediate value argument, if the resistor R_k is essential, the partial derivative $\partial g/\partial R_k$ has one sign over $\mathcal{R}$. In view of this, let

$$s_k \doteq \text{sign}\left(\frac{\partial g}{\partial R_k}\right)$$

denote this invariant sign; i.e., s_k is constant over $\mathcal{R}$ having the value $s_k = -1$ or $s_k = 1$. This essentiality condition is discussed further in the context of the example in Section 5 and conditions for its verification are given in the full version of this paper [6]. In the theorem below, it is seen that for the case of an essential resistor R_i, both the maximum and minimum expected gains are attained with a probability density functions f_i^* which is either an impulse at $R_i = R_{i,0}$ or uniform over all of $\mathcal{R}_i$. In view of the fact that such distributions can be viewed as extreme within the class $\mathcal{F}$, we call the result below an *Extremality Theorem*.

Theorem 1 *Assume that all resistors in N are essential. For the case of maximizing $\mathcal{E}(g(R^f))$, define probability density function f^* with marginals f_i^* as follows: Set $f_i^* = u$ if $s_i = -1$ and $f_i^* = \delta$ if $s_i = 1$. Then,*

$$\mathcal{E}(g(R^{f^*})) = \max_{f\in\mathcal{F}} \mathcal{E}(g(R^f)) = g^+.$$

For the case of minimizing $\mathcal{E}(g(R^f))$, define probability density function f^* with marginals f_i^* as follows: Set $f_i^* = \delta$ if $s_i = -1$ and $f_i^* = u$ if $s_i = 1$. Then,

$$\mathcal{E}(g(R^{f^*})) = \min_{f\in\mathcal{F}} \mathcal{E}(g(R^f)) = g^-.$$

4. PROOF OF THEOREM 1

We prove the result given for the maximum g^+ while noting that a nearly identical proof can be used for the minimum g^-. Indeed, in view of existing results on probabilistic robustness, for example, see [5] and [7], the maximum of $\mathcal{E}(g(R^f))$ over $f \in \mathcal{F}$ is equal to the maximum over truncated uniform distributions. That is, letting

$$T \doteq [0, r_1] \times [0, r_2] \times \cdots \times [0, r_n],$$

we have

$$g^+ = \max_{f\in\mathcal{F}} \mathcal{E}(g(R^f)) = \max_{t\in T} \mathcal{E}(g(R^t))$$

where R^t is the random vector with probability density function which is uniform over the *truncation box*

$$\mathcal{R}^t \doteq \mathcal{R}^{t,1} \times \mathcal{R}^{t,2} \times \cdots \times \mathcal{R}^{t,n}$$

where

$$\mathcal{R}^{t,i} \doteq [R_{i,0} - t_1, R_{i,0} + t_1].$$

Hence,

$$g^+ = \max_{t\in T} \mathcal{E}(g(R^t)) = \max_{t\in T} \frac{1}{2^n t_1 t_2 \cdots t_n} \int_{\mathcal{R}^t} g(R) dR$$

with the understanding that if $t_i = 0$, the corresponding integral with $\frac{1}{2t_1}$ multiplier is calculated using an appropriate impulse distribution or L'Hopital's rule.

To complete the proof of the theorem, let $t^* \in T$ be the truncation corresponding to probability density functions f_i^* as prescribed in the theorem. That is, if $f_i^* = u$, then, $t_i^* = r_i$. Alternatively, if $f_i^* = \delta$, then, $t_i^* = 0$. In addition, let $t \in T$ denote any candidate truncation for the maximization of $\mathcal{E}(g(R^t))$. To show that t^* attains the maximum, it will be shown that we can replace components t_k of t with corresponding components t_k^* of t^*, one at a time, without increasing $\mathcal{E}(g(R^t))$. For example, with $n = 3$, such a sequential replacement corresponds to the sequence of inequalities

$$\begin{aligned}\mathcal{E}(g(R^{(t_1,t_2,t_3)})) &\leq \mathcal{E}(g(R^{(t_1^*,t_2,t_3)})) \\ &\leq \mathcal{E}(g(R^{(t_1^*,t_2^*,t_3)})) \\ &\leq \mathcal{E}(g(R^{(t_1^*,t_2^*,t_3^*)})).\end{aligned}$$

That is, by showing that

$$\mathcal{E}(g(R^t)) \leq \mathcal{E}(g(R^{(t_1,t_2,\ldots,t_{k-1},t_k^*,t_{k+1},\ldots,t_n)})),$$

holds for arbitrary k, we can replace components one at a time until we arrive at the desired result

$$\mathcal{E}(g(R^t)) \leq \mathcal{E}(g(R^{t^*})).$$

Indeed, without loss of generality, we take $k = 1$ and $s_1 = -1$ noting that the proof to follow is the virtually identical for $s_1 = 1$ and $k = 2, 3, \ldots, n$. Now, to separate out the dependence on R_1, we let $X \doteq R_1$ and $y \doteq (R_2, R_3, \ldots, R_n)$ and consider the conditional expectation

$$E(y, t_1) \doteq \frac{1}{2t_1}\int_{R_{1,0}-t_1}^{R_{1,0}+t_1} g(X, y)dX.$$

Claim: The inequality

$$E(y, t_1) \leq E(y, r_1)$$

holds for all admissible $y \in \mathcal{Y}$ where $\mathcal{Y}$ denotes the box of admissible resistor uncertainty for y.

To prove this claim, it is first noted that Kirchoff's Laws lead to a solution for the gain $g(X, y)$ which can be expressed in the linear fractional form as

$$g(x, y) = \frac{A(y)x + B(y)}{C(y)x + D(y)}$$

with $C(y), D(y) \geq 0$ for all admissible y and $C(y)X+D(y) > 0$ for all admissible (x, y); see [6] and [8]. In view of this form for $g(X, y)$, the conditional expectation under consideration is

$$\begin{aligned}E(y, t_1) &= \frac{1}{2t_1}\int_{R_{1,0}-t_1}^{R_{1,0}+t_1} \frac{AX + B}{CX + D}dX \\ &= \frac{1}{2t_1}\int_{-t_1}^{t_1} \frac{ax + b}{cx + d}dx \\ &= \frac{a}{c} + \frac{bc - da}{2t_1c^2}\log\frac{d + ct_1}{d - ct_1}\end{aligned}$$

where $a(y) \doteq A(y), b(y) \doteq A(y)R_{1,0}+B(y), c(y) \doteq C(y), d(y) \doteq C(y)R_{1,0} + D(y)$. Furthermore, we obtain partial derivative computed to be

$$\frac{\partial E}{\partial t_1} = (bc - ad)e(t_1)$$

where

$$e(t_1) \doteq \left(\frac{d}{ct_1(d - ct_1)(d + ct_1)} - \frac{1}{2c^2t_1^2}\log(\frac{d + ct_1}{d - ct_1})\right).$$

Next, we claim that the inequality $e(t_1) > 0$ holds. This claim is readily establishing by first noting that Kirchoff's Laws guarantee $c, d \geq 0$, $cx + d > 0$ and that the quotient

$$z \doteq \frac{d + ct}{d - ct} \geq 1.$$

Subsequently, positivity of $e(t_1)$ reduces to the simple problem of showing that

$$\frac{z}{2} - \frac{1}{2z} - \log z > 0$$

for all $z > 1$. Now, with $e(t_1) > 0$, in order to prove that $E(y, t_1)$ is maximized at $t_1 = 1$, we observe that

$$\frac{\partial g}{\partial x} = \frac{ad - bc}{(cx + d)^2}$$

has negative sign $s_1 = -1$ for all y. Hence, it follows that $ad - bc > 0$ and

$$\frac{\partial E}{\partial t_1} > 0.$$

Therefore, $E(t_1, y)$ is maximized at $t_1 = r_1$. The proof of the claim is now complete.

Finally, to complete the proof of the theorem, we now observe that it follows from the claim that

$$\begin{aligned}\mathcal{E}(g(R^t)) &= \frac{1}{2^{n-1}t_2t_3\cdots t_n}\int_{\mathcal{Y}} E(y, t_1)dy \\ &\leq \frac{1}{2^{n-1}t_2t_3\cdots t_n}\int_{\mathcal{Y}} E(y, r_1)dy \\ &= \mathcal{E}(g(R^{(t_1^*,t_2,\ldots,t_n)})).\end{aligned}$$

5. EXAMPLE

To illustrate the application of Extremality Theorem, we consider the ladder network shown in the Figure 2 and seek to find the maximum expected gain. Via a lengthy proof (see [6] for details), it can be verified that all resistors are essential. Moreover, for any inter-stage resistor R_{3k}, we have $s_{3k} = -1$ and for remaining resistors R_i, $i \neq 3k$, we have $s_i = 1$. Hence, in accordance with Theorem 1, the maximum expected gain is obtained by using an impulsive distribution for the inter-stage resistors and a uniform distribution for the remaining resistors.

The ideas above are now illustrated for a three stage network with nominal values $R_{1,0} = R_{4,0} = R_{5,0} = R_{7,0} = R_{8,0} = 1$, $R_{2,0} = 2$, $R_{3,0} = 3$, $R_{6,0} = 5$ and $R_{9,0} = 7$. To provide a case where a classical Monte Carlo simulation yields an expected gain which differ dramatically from the one obtained via the methods in this paper, we assume uncertainty bounds equal to 80% of the nominal for the inter-stage resistors and 10% for the remaining resistors. As prescribed by Theorem 1, we carry out a Monte Carlo simulation using an impulsive distribution for R_3, R_6 and R_9 and a uniform distribution for the remaining R_i. With $100,000$ samples, we obtained the estimate

$$\mathcal{E}(g(R^{f^*})) \approx 0.1864.$$

Next, a classical Monte Carlo simulation using the uniform distribution for all resistors was carried out. This time, an estimate

$$\mathcal{E}(g(R^u)) \approx 0.1554.$$

was obtained. In conclusion, it is apparent that the classical expected gain is less than the distributionally robust expected gain by about 20%.

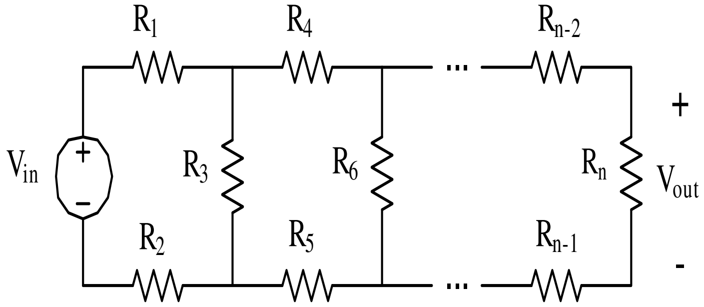

Figure 2: Ladder Network

6. FURTHER RESEARCH

To conclude this paper, we mention three directions for further research. First, it would be of interest to consider the extent to which the results in this paper can be generalized to address non-planar networks and networks with multiple and dependent sources. Regarding the non-planar case, a fundamental technical issue arises because the gain can no longer be expressed as a quotient of multilinear functions in the R_i. Second it would be of interest to consider more general versions of the resistor problem involving other elements such as capacitors and inductors. Third, it would be of interest to conduct further research involving the essentiality condition; see Section 3.1. To motivate such work, consider the circuit in Figure 3.

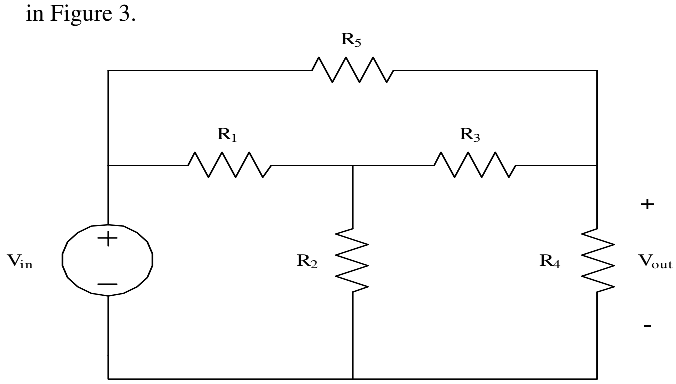

Figure 3: Violation of Essentiality Condition

For this circuit, if one considers the effect of resistor R_3 on the gain $g(R)$, for appropriately chosen nominal circuit values and uncertainty bounds, it can readily be shown that the essentiality condition is violated. However, since extensive computation for many combinations of probability density functions for the R_i has continually resulted in the expected gain being maximized with either uniform or impulsive distribution, the obvious conjecture to consider is that the Extremality Theorem holds without requiring that the resistors be essential.

7. ACKNOWLEDGEMENTS

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