Stratified rough sets and vagueness

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Abstract. The relationship between less detailed and more detailed versions of data is one of the major issues in processing geographic information. Fundamental to much work in model-oriented generalization, also called semantic generalization, is the notion of an equivalence relation. Given an equivalence relation on a set, the techniques of rough set theory can be applied to give generalized descriptions of subsets of the original set. The notion of equivalence relation, or partition, has recently been significantly extended by the introduction of the notion of a granular partition. A granular partition provides what may be thought of as a hierarchical family of partial equivalence relations. In this paper we show how the mechanisms for making rough descriptions with respect to an equivalence relation can be extended to give rough descriptions with respect to a granular partition. In order to do this, we also show how some of the theory of granular partitions can be reformulated; this clarifies the connections between equivalence relations and granular partitions. With the help of this correspondence we then can show how the notion of hierarchical systems of partial equivalence classes relates to partitions of partial sets, i.e., partitions of sets in which not all members are known. This gives us new insight into the relationships between roughness and vagueness.

1 Introduction

In processing geographic information, handling multiple levels of detail is of considerable practical importance. This is true both of cartographic generalization [MLW95], where the geometric presentation of the data is a major factor, and also of 'model-oriented generalization' in the sense of [M^+95]. In model-oriented generalization, the relevant attributes of the data are not geometric, but might for example be thematic classifications. In such a case the generalization might replace several distinct specific classifications with one more general one. As, say, in the process of ignoring the distinction between different kinds of road (motorways, major roads, minor roads, etc) and reducing to the single concept 'road'. A conceptually similar kind of generalization can be performed on raster data when deliberately reducing the resolution. In this case a number of pixels, which might be given a number of different colours could be replaced by a single pixel bearing just one colour.

An alternative terminology is used in Jones [Jon97, p271] where *semantic generalization* is described as being "... concerned with the meaning and function of a map and it depends on being able to identify hierarchical structure in the geographical information." This hierarchical structure has been used in making formal theories of the process of semantic generalization. The most obvious is a thematic classification given as a tree, but a richer notion of hierarchical structure is found, for example, in the studies of a lattice of resolution by Worboys [Wor98a,Wor98b].

In investigating the theory of semantic generalization we find the notion of equivalence relation, or partition, is a fundamental ingredient. In collapsing multiple kinds of road to a single one, we are imposing an equivalence relation on the available themes and putting the various kinds of road into the same *equivalence class*. To this equivalence class we give the label 'road'. In the example of raster data, the equivalence relation groups together the pixels at the more detailed level which become a single pixel at the coarser level of detail. This example may also exhibit a second equivalence relation which taking the labels of the more detailed pixels amalgamates them to a single equivalence class which is used to label the single pixel at the coarser level.

The basic way in which an equivalence relation is used may be summarized as follows. An equivalence relation groups together entities which are in some sense similar. Each collection of 'similar' entities forms new a single entity, called an equivalence class. A subset of the original set of entities can be given a rough description by specifying the extent to which each of the equivalence classes lies within the subset. In the most basic approach, this extent can be one of the three: wholly, partly, and not at all. Within geographic information, the use of equivalence relations has been explored in the context of rough sets [BS01], and the extension of equivalence relations on sets to the analogous structure on graphs has also been considered [Ste99]. A formal theory of partitions of space was provided by Erwig and Schneider [ES97].

An equivalence relation allows us to model the passage from one level of detail to another, but does not, on its own, model the considerably more than two levels of detail which are needed in practice. To deal with several levels of detail, a new concept has been proposed: the granular partitions of Bittner and Smith [BS03a]. A granular partition can be seen as an extension of the concept of equivalence relation, and it is the purpose of this paper to examine how the rough descriptions of the theory of rough sets can be extended from ordinary equivalence relations to the multi-level world of granular partitions.

The paper is structured as follows. To generalize the use of partitions in the study of roughness to granular partitions it is useful to present the theory of granular partitions in a new way (section 3 below), and to prepare for that we review the key notions of roughness (section 2 below). In section 4 we introduce systems of hierarchically ordered stratified rough sets. The ordering hereby corresponds to the degree of roughness of the underlying equivalence classes. In section 5 we generalize the notion of stratified rough sets by considering partial equivalence classes or equivalence classes in partial sets [MMO90]. In section 6 the notion of rough set is generalized in order to take into account vagueness. Conclusions are presented in section 7.

In places the paper is rather technical. This apparent complexity seems unavoidable and arises from the interaction between the granular partitions and the rough set concepts. This interaction produces a more intricate theory than is found in either of the two ingredients separately. Despite the technicality, the topic is, as explained above, one of considerable importance and we have provided examples in the paper which are designed to illustrate the main concepts.

2 Labelled partitions and rough sets

In this section we introduce the notions of K-labelled partitions and rough sets. We show that maps are an important class of K-labelled partitions and that rough sets can be used in order to approximate objects with indeterminate boundaries.

2.1 Labelled partitions

A partition here is understood in the standard mathematical sense: the subdivision of a set into jointly exhaustive and pairwise disjoint subsets via a corresponding equivalence relation. Partitions of a set, X, are often identified with functions of the form $f : X \to K$ which are surjective (that is where for every $k \in K$, there is some $x \in X$ for which $fx = k$).

Given such a function $f : X \to K$, we obtain a partition of X into subsets of the form $[x]_k = \{x \in X \mid fx = k\}$ where $k \in K$. The same partition however can arise from different functions. Consider, for example the subdivision of a part of the plane into subsets indicated by the 12 squares in Figure 1(i). In Figure 1(ii) and (iii) we have two different labelled versions of the same partition: $f_1 : X \to \{1, 2, \ldots, 11, 12\}$, and $f_2 : X \to \{a, b, \ldots, k, l\}$. Two functions $f : X \to K$ and $f' : X \to K'$ give rise to the same labelled partition if and only if there is a bijection $\varphi : K \to K'$ such that $\varphi f = f'$.

A surjective function $f : X \to K$ thus corresponds to something more than a partition of X: it is a partition of X together with a labelling (by the elements of K) of the blocks of the partition. It is useful to use the terms *blocks* and *cells* so that blocks are subsets of the partitioned set X, whereas cells are labels for these blocks. It may be helpful to imagine that the cells are labelled boxes or locations which are used to house the elements of X. The distinction between cells and blocks is then the distinction between a location and the contents of that location. To emphasize the importance of the labelling, we make the following definition.

Definition 1. *Let K and X be sets. Then a K-**labelled partition of** X is a surjective function from X to K.*

An important class of K-labelled partitions are maps (in the cartographic rather than the mathematical sense). Consider the left part of figure 2 which shows a part of the United States. The labelling function f here maps every point of the United States to names of federal states (Montana, Wyoming, Idaho, etc.).

2.2 Rough sets

Given a labelled partition of X (i.e. a surjective function $f : X \to K$ for some K) we obtain rough descriptions of the subsets of X in terms of the extent to which the cells

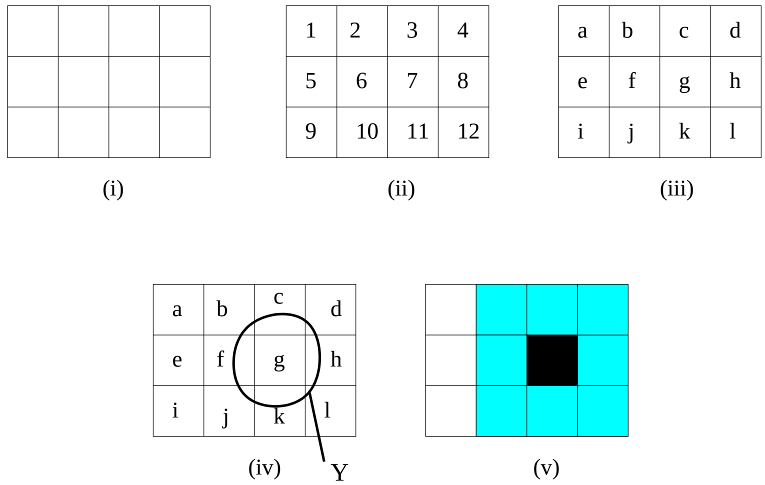


Fig. 1. A partition of a subset of a plane (i) with two different labellings (ii) and (iii) and a subset Y (iv) and its egg-yolk representation (v).

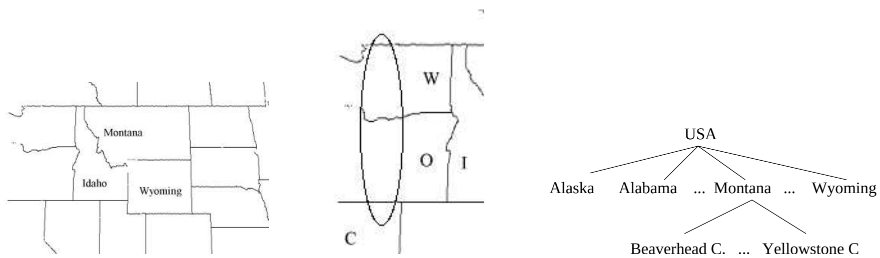


Fig. 2. A k-labelled partition (left); A rough approximation wrt. a k-labelled partition (middle); A stratified labelled partition (right).

are occupied by the subset. So, for any $Y \subseteq X$ we obtain a function $Y \boxplus f : K \to \{\mathsf{T}, \mathsf{B}, \mathsf{F}\}$. This function, read as 'Y is coarsened by f' is defined as follows.

$$(Y \boxplus f)\ k = \begin{cases} \mathsf{T} & \text{if } \forall x \in f^{-1}k\ x \in Y \\ \mathsf{B} & \text{if } \exists x, x' \in f^{-1}k\ x \in Y \text{ and } x' \notin Y \\ \mathsf{F} & \text{if } \forall x \in f^{-1}k\ x \notin Y \end{cases}$$

where f^{-1} is the inverse image of f, so that $f^{-1}k$ means $\{x \in X \mid fx = k\}$. The notation $\mathsf{T}, \mathsf{B}, \mathsf{F}$ is chosen as these three values are the concepts *True*, *Both*, and *False*. This is because if $(Y \boxplus f)\ k = \mathsf{T}$ then k is definitely in Y; if the value is F, then k is definitely not in Y; if the value is B, then k is both in and not in Y. The structure resulting from a coarsening operation is a *rough set* as defined in [Paw82].

The intuition is that the value of $(Y \boxplus f)\ k$ is T, B, or F according as the cell k is completely, partially or not at all occupied by elements of the subset Y. Consider 1 (iv).

Again, let X be the set of points of the part of the plane and let $Y \subset X$ a subset. The rough set approximation of Y with respect to the labelled partition f_2 is given below.

$k \in K$	a	b	c	d	e	f	g	h	i	j	k	l
$(Y \boxplus f_2)\ k$	F	B	B	B	F	B	T	B	F	B	B	B

The rough subset $Y \boxplus f$ can be represented by a pair of ordinary subsets of K: $\langle (Y \boxplus f)^{-1}\{\mathsf{T}\}, (Y \boxplus f)^{-1}\{\mathsf{T,B}\}\rangle$, leading to the usual 'egg-yolk' pictures (Figure 1 (v)). Here $(Y \boxplus f)^{-1}\{\mathsf{T}\}$ is the set $\{x \in (f^{-1}k) \mid (Y \boxplus f)\ k = \mathsf{T}\}$ and marked by the black square. Correspondingly $(Y \boxplus f)^{-1}\{\mathsf{T,B}\}$ is the set $\{x \in (f^{-1}k) \mid (Y \boxplus f)\ k \in \{\mathsf{T}, \mathsf{B}\}\}$ and corresponds to the union of the black and grey squares in the figure.

In the remainder we will use the phrases 'the rough set $Y \boxplus f$' and 'the (rough) approximation of Y with respect to the labelled partition f' synonymously.

Rough set approximations play an important role for the representation of objects with indeterminate boundaries [BS02], [BS03b]. Consider figure 2. In the middle we have a K labelled partition of the northwestern US and we have the Cascade mountains (CM), indicated by the ellipse, which cover parts of the states Washington (W), Oregon (O), and California (C). The rough set representation of the cascade mountains is $(CM \boxplus f_{USA})\ W = \mathsf{B}$, $(CM \boxplus f_{USA})\ O = \mathsf{B}$, $(CM \boxplus f_{USA})\ C = \mathsf{B}$, $(CM \boxplus f_{USA})\ I = \mathsf{F}$, etc. Rough set representations do not force us to draw crisp boundaries where no crisp boundaries exist.

3 Granular Partitions

Maps are often organized hierarchically. Consider the political subdivision of the US. Here we have counties which form states, which themselves form the US as a whole. This structure is visualized in the right part of figure 2. In this section we introduce the notion of K labelled stratified partition in order to take this hierarchical structure into account.

3.1 Cell Granulations

Above we considered only unstructured sets. Now we consider sets of cells upon which a *tree structure* has been defined.

Definition 2. *A **cell tree** is a finite, partially ordered set of cells, $(K, \leq)$, which forms a tree. The partial order, $\leq$, is called the sub-cell relation, and the maximum element in this order will be the root of the tree. If a cell tree additionally satisfies the constraint that no node have just a single descendant then it is said to be **branching**.*

Consider figure 3 which shows a cell tree K with elements a, b, c, d, e, f, g, and h. Here the cell a is the root of the tree and we have $x \leq y$ if and only if the nodes x and y are connected by a line going upwards, or by a sequence of such lines.

The tree structure gives rise to a lattice (middle of figure 3), the elements of which are the cuts of the tree, defined as follows [RS95]:

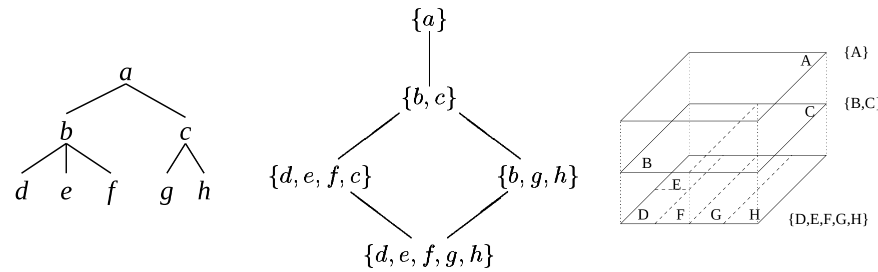


Fig. 3. A cell granulation (left), the corresponding cut lattice (middle), and the corresponding hierarchical subdivision of the point-set A (right).

Definition 3. *For any element k of a cell tree K, let $\mathsf{d}(k)$ denote the set of immediate descendants of k. A* ***cut*** *in K is a subset of K defined inductively as follows:*

1. *$\{r\}$ is a cut, where r is the root of the tree,*
2. *let C be a cut and $k \in C$ where $\mathsf{d}(k) \neq \varnothing$, then $(C - \{k\}) \cup \mathsf{d}(k)$ is a cut.*

It follows that the sets $\{a\}$, $\{b, c\}$, $\{d, e, f, c\}$, $\{b, g, h\}$, and $\{d, e, f, g, h\}$ are cuts in the tree K in the left part of figure 3.

Let C and D be cuts in the cell tree K. The cuts of a tree form a lattice ordered by $C \sqsubseteq D$ if for each $c \in C$ there is some $d \in D$ with $c \leq d$. This lattice will be referred to as the *cut lattice* of the cell tree K. The cut lattice of our example cell tree is shown in the middle of figure 3.

The cut lattice carries additional structure, which we will discuss now. Given cuts $C \sqsubseteq D$ there is a function $\ell : C \to D$ where $c \leq \ell c$. The facts (i) that C and D are cuts in a tree K and that (ii) $C \sqsubseteq D$ ensure that ℓ is a function. For example, in the cut lattice seen in the middle of figure 3, with cuts $C = \{d, e, f, c\}$ and $D = \{b, c\}$, the function ℓ is the following mapping: $d \mapsto b, e \mapsto b, f \mapsto b, c \mapsto c$. Note that when $C = D$ the function ℓ will be the identity.

We have seen that a set of cells structured as a tree gives rise to a lattice, the elements of which are sets of cells, and that these sets of sets are related by functions. All this structure can be derived from the tree, but it is often more convenient to deal with it directly than to always be thinking of it as generated from the tree. Thus we will refer to the lattice and associated structure as a **cell granulation**; it consists of:

1. A lattice, $(\mathcal{L}, \sqsubseteq)$, of levels of detail
2. for each level of detail $i \in \mathcal{L}$ there is a set of cells $\mathcal{L}_i$, and
3. for each pair i, j of levels of detail, where $i \sqsubseteq j$, there is a function $\ell_{ij} : \mathcal{L}_i \to \mathcal{L}_j$.

It should be noted that not every structure of the above form will be a cell granulation, as only lattices of a certain form can arise as lattices of cuts of trees. The cell granulation is derived from the cell tree, and will be denoted simply as $\mathcal{L}$ when there is no danger of confusing this with the underlying lattice. If it is necessary to emphasize the dependence on the tree K we can write $\mathcal{L}(K)$ rather than just $\mathcal{L}$. Cuts or *levels of detail* or *levels of granularity* will be referred to by their index i or by the corresponding set $\mathcal{L}_i$.

3.2 Stratified Labelled Partitions

Having described the granulation structure on the set of cells, we now see how these are used to construct stratified labelled partitions. Recall that in the ordinary case a partition of a set X labelled by a set of cells K is a surjective function from X to K. In the granular case, the role of K is taken by the cell granulation $\mathcal{L}(K)$ introduced in section 3.1 above, so it remains to explain what plays the role of the surjective function in the ordinary case.

Definition 4. *Let $\mathcal{L}$ be the cell granulation derived from a cell tree K, and let X be a set. Then a K-**labelled stratified partition** consists of for each $i \in \mathcal{L}$ a partial and surjective function $f_i : X \to \mathcal{L}_i$ such that whenever $i \sqsubseteq j$*

$$(\forall x \in X)\ (\ell_{ij}\, f_i\, x = f_j\, x)$$

whenever the left hand side of the equation is defined.

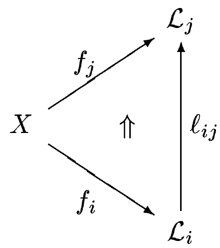
The introduction of partial functions here is significant, and is motivated by the theory of granular partitions. At a particular level of detail, we allow that the collection of cells, or labels, at our disposal may not cover all the entities to be classified. It should remembered that the definition of a partial function allows for the function to be undefined for some elements of its domain, but it does not exclude the possibility that the function is total. Thus, partiality corresponds to the *potential* for having unclassified entities, it does not mean that there have to be some things which are unclassified.

Consider the right part of figure 3 which shows the subdivision of the point set A in subsets which form partitions of A at different levels of detail. (In this example we use capital letters to denote sets and corresponding non-capital letters for their labels.) At the top level we have the set A as a whole. At the intermediate level we have a partition of A formed by the subsets B and C. At the finest level we have a partition of A formed by the subsets D, E, F, G, and H. Also, the subsets D, E and F form a partition of the set B and the subsets G and H form a partition of C. For every partition of the set A into subsets there is now a corresponding labelled partition:

$$\begin{array}{ll} f_1 : A \to \{a\}, & f_2 : A \to \{b, c\}, \\ f_3 : A \to \{d, e, f, c\}, & f_4 : A \to \{b, g, h\}, \\ f_5 : A \to \{d, e, f, g, h\}. & \end{array} \quad (1)$$

One can see that every co-domain of the labelled partitions $f_1 \ldots f_5$ corresponds to a cut in the cell granulation K formed by the cells $\{a, b, c, d, e, f, g, h\}$ depicted in the left part of figure 3. Now consider the labelled partitions f_4 and f_2 and assume $x \in G$. It then follows that $f_4\ x = g$. Since g is a subcell of c we have $\ell_{42}\ g = c$. On the other hand, since $x \in G$ and $G \subset C$ we also have $x \in C$. Consequently we have $f_2\ x = c$ and hence $f_2\ x = (\ell_{42}\ f_4)\ x$.

Definition 4 can be neatly summarized by a diagram in the ordered category of sets and partial functions:



3.3 Granular partitions

We shall now establish the correspondence between the notion of a granular partition introduced by [BS03a] and the notion of stratified labelled partitions introduced above. Basic components of a granular partition are a cell tree K, a corresponding set X, and mappings between them. However a granular partition does not have multiple surjective functions from X to cuts in K but rather a single order-preserving mapping, π, from K into the powerset of X. This notion of granular partition is very general. In this subsection we will establish the equivalence of labelled stratified partitions and a class of specific, particularly well-formed granular partitions:

Definition 5. *Let $(K, \leq)$ be a cell tree, X be a set, $\mathcal{P}_+ X$ denote the set of non-empty subsets of X, and let $\pi : K \to \mathcal{P}_+ X$ be a function such that for all $k_1, k_2 \in K$,*

$$\begin{array}{l} (i)\ \ k_1 \leq k_2 \Leftrightarrow (\pi\ k_1) \subseteq (\pi\ k_2), \\ (ii)\ \pi k_1 \cap \pi k_2 \neq \varnothing \Rightarrow (k_1 \leq k_2\ \ or\ \ k_2 \leq k_1). \end{array}$$

The triple $\Pi = ((K, \leq), X, \pi)$ is then called a strict mereological monotonic granular partition. Condition (i) expresses the constraint that π be an order-isomorphism.

This particular class of granular partitions is such that the mapping π preserves the tree-structure of K, which is equivalent to saying that the subsets of X singled out by π have a tree structure (with respect to the subset relation) which is isomorphic to that of the cell tree K.

Consider the left and right part of figure 3. A granular partition then is a triple $\Pi = ((K, \leq), A, \pi)$ such that $(K, \leq)$ is as depicted in the left part of the figure and π is defined as follows: $\pi\ a = A$, $\pi\ b = B$, $\ldots$, $\pi\ h = H$, where capital letters refer to sets in the right part of the figure.

Given a cell granulation, we can define $\pi : K \to \mathcal{P}_+ X$ by $\pi k = \{x \in X \mid \lambda_i x = k$ for some $i\}$. The following result shows that this construction provides a strict mereologically monotonic granular partition provided that the cell tree is branching (no node having just a single descendant).

Theorem 1. *If the cell granulation $(\mathcal{L}(K), X, \lambda_1, \ldots \lambda_n)$ with $\lambda_i : X \to \mathcal{L}_i$ is a $\mathcal{K}$ branching labelled stratified partition then $\Pi = ((K, \leq), X, \pi)$ is a strict mereologically monotonic granular partition.*

Proof First we show that if $\pi k_1 \cap \pi k_2$ is non-empty then either $k_1 \leq k_2$ or $k_2 \leq k_2$. If k_1 and k_2 are unrelated in the order then some cut, say q, must contain both of them. But then $\lambda_q x = k_1$ and $\lambda_q x = k_2$ contradicting the unrelatedness of k_1 and k_2.

Next we tackle one half of the first condition for a strict mereologically monotonic granular partition. Suppose that $k_1 \leq k_2$ and let $\lambda_i x = k_1$. Then there must be $j \geq i$ with $k_2 \in \mathcal{L}_j$, and $\ell_{ij} k_1 = k_2$. Hence $\lambda_j x = k_2$, and so $x \in \pi k_2$.

Finally, we have to show that if $\pi k_1 \subseteq \pi k_2$ then $k_1 \leq k_2$. As $\pi k_1 \cap \pi k_2$ is non-empty then either $k_1 \leq k_2$ or $k_2 \leq k_1$. If $k_2 \leq k_1$ then we have $\pi k_1 = \pi k_2$. The possibility that $k_2 < k_1$ can be excluded. For k_1 must have another descendant besides k_2, say k', at level i where $k_2, k' \in \mathcal{L}_i$. Now, as λ_i is surjective, there are distinct x, x' where $\lambda_i x = k_2$ and $\lambda_i x' = k'$. But $\lambda_j x' = k_1$ for some j, as k' is a descendant of k_1, and so $\pi k_1 \neq \pi k_2$. Hence, having ruled out $k_2 < k_1$, we conclude $k_1 \leq k_2$.

□

We note that if the original cell tree is not necessarily branching, then we can only prove that π is an order homomorphism (i.e. $k_1 \leq k_2 \Rightarrow \pi k_1 \subseteq \pi k_2$).

In the opposite direction, we can start with a strict mereologically monotonic granular partition and construct a K-labelled stratified partition. For each cut i, $\lambda_i x$ is defined if there is $k \in \mathcal{L}_i$ with $x \in \pi k$. In this case, $\lambda_i x = k$. That this construction has the appropriate properties is established in the following result.

Theorem 2. *If $\Pi = ((K, \leq), X, \pi)$ is a strict mereologically monotonic granular partition then the cell granulation $(\mathcal{L}(K), X, \lambda_1, \ldots \lambda_n)$ with $\lambda_i : X \to \mathcal{L}_i$ is a K labelled stratified partition.*

Proof The λ_i are well defined, for if $x \in \pi k_1 \cap \pi k_2$ we have $k_1 = k_2$ as $k_1 < k_2$ is impossible for distinct elements of the same cut. The λ_i are clearly surjective. It remains to show that if $i \leq j$ and $\lambda_i x$ is defined, then $\lambda_j x$ is defined and $\ell_{ij} \lambda_i x = \lambda_j x$. If $\lambda_i x = k \in \mathcal{L}_i$ then we can find $k' \in \mathcal{L}_j$ with $k \leq k'$, thus $\pi k \subseteq \pi k'$ and $x \in \pi k'$. As $k \leq k'$ we get $\ell_{ij} k = k'$ and so $\ell_{ij} \lambda_i x = \lambda_j x$. □

It follows that the notions *strict mereologically monotonic granular partition* and $\mathcal{K}$ *labelled stratified partition* are equivalent. In the remainder we focus onto the latter.

4 Stratified Rough Sets

As mentioned in section 2 above, an ordinary labelled partition $f : X \to K$ provides for each $Y \subseteq X$ a rough set $Y \boxplus f$. What happens to this process when we have a stratified labelled partition? In order to answer this question we now extend the notion of stratified rough set introduced by [Yao99].

Let $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ a stratified labelled granular partition with a *total* surjective function of the form $\lambda_i : X \to \mathcal{L}_i$ for each level of detail $\mathcal{L}_1, \ldots, \mathcal{L}_n$ in $(\mathcal{L}, \sqsubseteq)$. [Yao99] then defines a stratified rough set as a sequence of rough sets $(Y \boxplus \lambda_1), \ldots (Y \boxplus \lambda_n)$ as follows. Let $(Y \boxplus \lambda_i)^{-1}\{\mathsf{T}\}$ be the 'egg', and $(Y \boxplus \lambda_i)^{-1}\{\mathsf{T},\mathsf{F}\}$ be the union of 'egg' and 'yolk' in the corresponding egg-yolk representation of Y at the level of detail formed by $\mathcal{L}_i$ (remember Figure 1 (v)). Then whenever $i \sqsubseteq j$ the 'egg' at level i is a subset of the 'egg' of level j which itself is a subset of Y, which is in turn a subset of the union of 'egg' and 'yolk' at level i and so on:

$$(i)\ \ (Y \boxplus \lambda_i)^{-1}\{\mathsf{T}\} \subseteq (Y \boxplus \lambda_j)^{-1}\{\mathsf{T}\} \subseteq Y,$$
$$(ii)\ Y \subseteq (Y \boxplus \lambda_i)^{-1}\{\mathsf{T,B}\} \subseteq (Y \boxplus \lambda_j)^{-1}\{\mathsf{T,B}\}$$

Let $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ as defined above and let $\omega_{ij}^{p} = \{(Y \boxplus \lambda_i)\ k \mid k \in (\ell_{ij}^{-1}\ p)\}$ be the set of approximation values under $(T \boxplus \lambda_i)$ with respect to the subcells $(\ell_{ij}^{-1}\ p) \subseteq L_i$ of the cell $p \in \mathcal{L}_j$. We then define a stratified rough set as follows:

Definition 6. *A stratified rough set is a family of rough sets $(Y \boxplus \lambda_1), \ldots (Y \boxplus \lambda_n)$, such that whenever $i \sqsubseteq j$ then there exists a mapping $\alpha_{ij} : \{T, B, F\} \rightarrow \{T, B, F\}$ such that the following holds:*

$$(\alpha_{ij}\ (Y \boxplus \lambda_i))\ k = ((Y \boxplus \lambda_j)\ \ell_{ij})\ k$$

with

$$(\alpha_{ij}\ (Y \boxplus \lambda_i))\ k = \begin{cases} T & \text{iff } \omega_{ij}^{(\ell_{ij}\ k)} = \{T\} \\ F & \text{iff } \omega_{ij}^{(\ell_{ij}\ k)} = \{F\} \\ B & \text{otherwise} \end{cases}$$

Correspondingly we can draw the commutative diagram in figure 4.

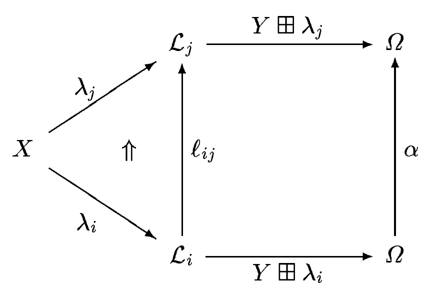


Fig. 4. Rough sets at different levels of granularity in a *cumulative* granular stratified partition.

Consider figure 5 which corresponds to the labelled stratified partition shown in figure 3 with λ_i corresponding to f_i in equation (1). In figure 5 (i) we have six subsets of the set A five of which form a partition and one (T) which lies skew to this partition. Figures 5 (ii – iv) show stratified rough sets representations of T at different levels of detail. Here the gray color of the set E in figure 5 (ii) indicates that $(T \boxplus \lambda_5)\ e = \mathsf{B}$. Similarly the gray color of the set B in figure 5 (iii) indicates that $(T \boxplus \lambda_2)\ b = \mathsf{B}$. The white color of the set G in figure 5 (ii) indicates that $(T \boxplus \lambda_5)\ g = \mathsf{F}$.

Let ω_{ij}^{p} be as defined above. In figure 5 (ii) we have $\omega_{54}^{b} = \{\mathsf{B}, \mathsf{F}\}$ and $\omega_{21}^{a} = \{\mathsf{B}, \mathsf{F}\}$, and hence $(\alpha_{54}\ (T \boxplus \lambda_4))\ b = \mathsf{B}$ and $(\alpha_{21}\ (T \boxplus \lambda_4))\ a = \mathsf{B}$.

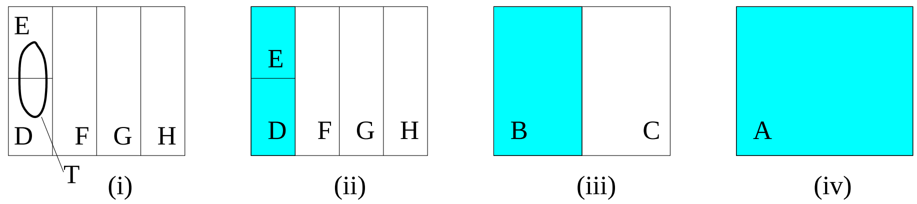


Fig. 5. A stratified rough set representations of Y at different levels of detail.

5 Rough sets in non-cumulative labelled stratified partitions

An important assumption in the previous section was that in the stratified labelled partition $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ the λ_i are *total* surjective function. Consider the K labelled partition depicted as a map of the United States in the left part of figure 2. That the labelling function is total here means that there is no 'white space' or no undiscovered land in the space covered by this map. In this case we also say that the underlying granular partition is *cumulative*.

Definition 7. *Let $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ be a labelled stratified partition. The level of granularity $\mathcal{L}_i$ is cumulative if and only if the function λ_i is total. The partition as a whole is cumulative if each level of granularity is cumulative.*

However there *are* maps with 'white space', unexplored territories, or not well understood domains. In order to take this into account we now generalize the notion of stratified rough sets by giving up this constraint of cumulativeness and allow the λ_i to be partial surjective functions. What results corresponds to what Mislove calls murky sets [MMO90] in the theory of partial sets and to what Bittner and Smith call non-cumulative granular partitions [BS03a]. In Mislove's terminology we now consider stratified rough sets in labelled partitions of murky sets. Roughly, murky sets are sets which are such that we do not know all of their members. In the terminology of Bittner and Smith we consider rough approximations with respect to non-cumulative granular partitions [BS03b].

If the underlying labelling functions λ_i are total surjective functions, then the rough set representations at a coarser levels of detail can be derived from a rough set represented at finer level of detail. Consider levels of detail $i \sqsubseteq j$. Given a rough set $(Y \boxplus \lambda_i)$ we can determine the rough set $(Y \boxplus \lambda_j)$ in the way described in definition 6. In general, however, we cannot assume that the underlying labelling functions are total because this assumes complete knowledge about the underlying set which may not be available.

Under circumstances where the labelling functions λ_i are not total it will be impossible to define a unique generalization mapping α_{ij} in the way shown in figure 4. Moreover a multitude of generalization mappings, each yielding one possible generalization of the rough set at hand will be needed. The example shown in figures 6 and 7 will help to explain this.

In figure 6 (i) we see the set Q with 12 elements, each of them labelled by a natural number. Five of these form the subset $R \subset Q$ and six of them form the subset $S \subset Q$

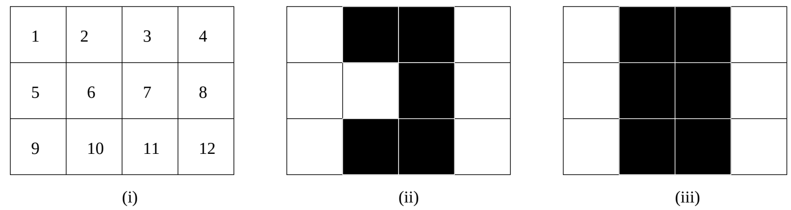


Fig. 6. A set Q, with 12 elements (i), a 5 element subset $R \subset Q$ (ii), and a 6 element subset $S \subset Q$ (iii).

$(\lambda_i\ x) =$	values of λ_i and x
a	$(\lambda_1\ 1) \ldots (\lambda_1\ 12)$
b	$(\lambda_2\ 1)$, $(\lambda_2\ 5)$, $(\lambda_2\ 9)$
	$(\lambda_4\ 1)$, $(\lambda_4\ 5)$, $(\lambda_4\ 9)$
c	$(\lambda_2\ 2)$, $(\lambda_2\ 3)$, $(\lambda_2\ 6)$, $(\lambda_2\ 7)$, $(\lambda_2\ 10)$, $(\lambda_2\ 11)$
g	$(\lambda_4\ 2)$, $(\lambda_4\ 3)$
h	$(\lambda_4\ 10)$, $(\lambda_4\ 11)$

Table 1. The mappings $\lambda_1, \lambda_2, \lambda_4$ with $\mathcal{L}_1 = \{a\}$, $\mathcal{L}_2 = \{b, c\}$, and $\mathcal{L}_4 = \{b, g, h\}$.

(figure 6 (ii) and (iii)). The set Q can be given a stratified labelled partition, using the cell tree and the granularity lattice shown in the left and middle of figure 3, and the mappings λ_i given in table 1. The table is read as follows: (row 1) the mapping λ_1 maps all elements of Q onto the label a; (row 2) λ_2 maps the elements 1, 5, and 9 onto b, and so does λ_4. The other rows follow the same pattern. The mappings targeting the granularity levels $\{d, e, f, c\}$ and $\{d, e, f, g, h\}$ are omitted here.

Table 1 tells us that the mapping λ_1 is surjective and total. The other mappings are surjective but partial. No λ_i with $i > 1$ maps the elements $4, 8, 12 \in Q$ to any cell in their target domain $\mathcal{L}_i$. Moreover λ_4 in addition also fails to map the elements $6, 7 \in Q$.

Consider figure 7: (i) depicts the rough set representation of R *and* S for the level of granularity $\mathcal{L}_4 = \{b, g, h\}$; (ii) depicts the rough set representation of R for the level of granularity $\mathcal{L}_2 = \{b, c\}$, and (iii) depicts the rough set representation of S for $\mathcal{L}_2$. We have $\mathcal{L}_4 \sqsubseteq \mathcal{L}_2$. The color grey indicates the approximation value B as in $(R \boxplus \lambda_2)\ c = \mathsf{B}$, black indicates the approximation value T as in $(R \boxplus \lambda_4)\ g = \mathsf{T}$, and the diagonal line pattern represents the approximation value F as in $(R \boxplus \lambda_4)\ b = \mathsf{F}$. The white spaces in the figures 7 (i–iv) indicates the partial character of the mappings λ_2 and λ_4.

Two significant features appear in this example:

1. At the level of granularity $\mathcal{L}_4$ we cannot distinguish between the sets R and S – both are are represented by the rough set depicted in figure 7 (i).
2. The rough approximation of R with respect to $\mathcal{L}_2$ cannot be derived from that at the finer level of detail $\mathcal{L}_4$ using a generalization mapping α_{24} as defined in definition 6 – applying the generalization mapping defined in 6 to the rough set $(R \boxplus \lambda_4)$

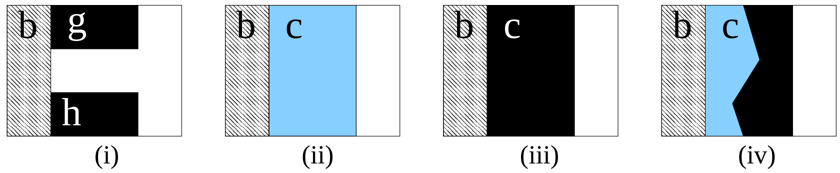


Fig. 7. Rough set representations of R and S at the levels of granularity λ_4 and λ_2.

yields the rough set depicted in figure 7 (iii) and not the one depicted in (ii) as one would want.

This is due to the fact that the function λ_4 is to a larger degree partial than λ_2: g and h do not make up the whole of c. From the rough set representation of R and S at the finer level of detail $\mathcal{L}_4$ alone we are unable to determine whether the part of Q labelled c is wholly or only partly covered by R and by S respectively. Consequently, given tha partial character of the mapping λ_4 and the rough set depicted in figure 7 (i) the two rough sets depicted in the figures 7 (ii) and (iii) equally good candidates for being the result of performing a generalization on (i). This is indicated in figure 7 (iv).

It follows that we need to extend the notion of generalization mapping α_{ij} which was set out in definition 6 in order to take into account the non-cumulative character of the underlying labelled stratified partition. Let $Y \boxplus \lambda_i$ be a rough set based on a non-cumulative granularity-level $\mathcal{L}_i$, let p be a cell belonging to granularity level $\mathcal{L}_j$ and let $i \sqsubseteq j$. We then need to distinguish three cases:

1. If we have $\{(Y \boxplus \lambda_i)\ k \mid k \in (\ell_{ij}^{-1}\ p)\} = \{\mathsf{T}\}$ then there might be elements of the underlying set X which are not labelled at granularity-level $\mathcal{L}_i$ which may or may not belong Y. Therefore we need to have two generalization mappings α_{ij}^{l} and α_{ij}^{l+1} such that $(\alpha_{ij}^{l}(Y \boxplus \lambda_i))\ p = \mathsf{T}$ and $(\alpha_{ij}^{l+1}(Y \boxplus \lambda_i))\ p = \mathsf{B}$.
2. If we have $\{(Y \boxplus \lambda_i)\ k \mid k \in (\ell_{ij}^{-1}\ p)\} = \{\mathsf{F}\}$ then, again, there might be elements of X which are not labelled at granularity-level $\mathcal{L}_i$ which may or may not belong Y. Therefore we need two generalization mappings α_{ij}^{l} and α_{ij}^{l+1} such that $(\alpha_{ij}^{l}(Y \boxplus \lambda_i))\ p = \mathsf{F}$ and $(\alpha_{ij}^{l+1}(Y \boxplus \lambda_i))\ p = \mathsf{B}$.
3. If we have $\mathsf{B} \in \{(Y \boxplus \lambda_i)\ k \mid k \in (\ell_{ij}^{-1}\ p)\}$ then we can apply definition 6.

Now compare the generalization from a cumulative level of granularity with generalization from from a non-cumulative level of granularity. In a cumulative level of granularity there is a unique generalization function doing the transformation job. When we generalize from a non-cumulative level of granularity $\mathcal{L}_i$ to a level of granularity $\mathcal{L}_j$ with a single cell then there may be *two* generalization functions: α_{ij}^{1} and α_{ij}^{2}. This case is represented in figure 8: The generalization mappings α_{ij}^{1} and α_{ij}^{2} satisfy the equations in the left of the figure. A corresponding diagram representation is given in the right part of the figure.

The more cells the target level of granularity $\mathcal{L}_j$ has two cells the more generalization functions need to be added. This reflects the phenomenon of *vagueness* which is caused by the non-cumulativeness of the underlying stratified partition.

$$(\alpha^1_{ij}\ (Y \boxplus \lambda_i))\ k = ((Y \boxplus_1 \lambda_j)\ \ell_{ij})\ k$$
$$(\alpha^2_{ij}\ (Y \boxplus \lambda_i))\ k = ((Y \boxplus_2 \lambda_j)\ \ell_{ij})\ k$$

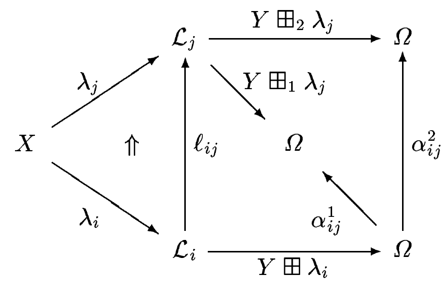


Fig. 8. The multiplicity of possible generalizations in non-cumulative labelled stratified partitions.

6 Rough sets and vagueness

In the previous section we dealt with the problem of vagueness by adding more and more generalization transformations – each yielding one possible rough set at the targeted level of granularity. An alternative way of dealing with the problem of vagueness is to introduce the notion of *vague rough set* and to provide an unique generalization transformation between vague rough sets. The idea hereby is to considerer sets of approximation values rather than sets of possible approximations.

6.1 Vague rough sets

Let $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ be a labelled non-cumulative granular partition with $\lambda_i : X \to \mathcal{L}_i$. In order to represent vagueness we consider the following subsets:

$$\tilde{\Omega} = \{\{\mathsf{F}\}, \{\mathsf{B}\}, \{\mathsf{F}\}, \{\mathsf{T}, \mathsf{B}\}, \{\mathsf{B}, \mathsf{F}\}, \{\mathsf{T}, \mathsf{B}, \mathsf{F}\}\}$$

The ordering of $\tilde{\Omega}$ corresponding to the subset relation is given in the diagram in figure 9.

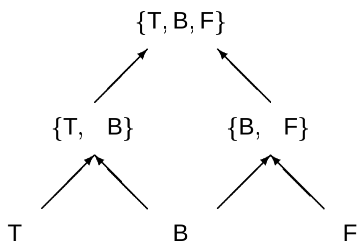
Given a subset $Y \subseteq X$ we define a *vague* rough set as a mapping of signature $(Y \boxtimes \lambda_i) : K \to \tilde{\Omega}$ (notice the difference between $\boxplus$ and $\boxtimes$). The value of $(Y \boxtimes \lambda_i)\ k$ is interpreted as a disjunction of possible relations between the subsets Y and $(\lambda_i^{-1}\ k)$. For example, the value of $(Y \boxtimes \lambda_i)\ k$ is $\{\mathsf{B}, \mathsf{F}\}$ if either Y contains some but not all of elements of $(\lambda_i^{-1}\ k)$ or if there is no overlap between Y and $(\lambda_i^{-1}\ k)$ at all. Under this interpretation the ordering in the diagram in figure 9 represents an increasing degree of vagueness.

Let $\mathcal{L}_i$ be a non-cumulative level of granularity. The rough set $X \boxplus \lambda_i$ is a *crisping* of the vague rough set $Y \boxtimes \lambda_i$ if and only if for every cell k the label $(X \boxplus \lambda_i)\ k$ is one of the disjuncts in $(Y \boxtimes \lambda_i)\ k$:[1]

$$\mathsf{CR}\ (X \boxplus \lambda_i)(Y \boxtimes \lambda_i) \equiv \forall k \in \mathcal{L}_i : (X \boxplus \lambda_i)\ k \in (Y \boxtimes \lambda_i)\ k$$

[1] In cumulative granular partitions crisping is more complicated. See [Bit03] for details.

Consider figure 7(iv). Let $N \subset Q$ be a set of which we know only the vague rough set representation corresponding to the figure: $(N \boxtimes \lambda_2)\ b = \{\mathsf{F}\}$ and $(N \boxtimes \lambda_2)\ c = \{\mathsf{T}, \mathsf{B}\}$. Crispings of $N \boxtimes \lambda_2$ then are $R \boxplus \lambda_2$ and $S \boxplus \lambda_2$ as depicted in 7(ii) and 7(iii).



	$\bigcap \omega_{ij}^{\ell_{ij}\ k}$	$(\tilde{\alpha}(Y \boxtimes \lambda_i))\ k$
1	{B}	{B}
2	{F}	{F, B}
3	{T}	{T, B}
4	{B, T}	{B, T}
5	{F, B}	{F, B}
6	{F, B, T}	{F, B, T}
7	{}	{B}

Fig. 9. Representing vagueness as sets of labels.

6.2 Generalization of vague rough sets

We now discuss generalization transformations of vague rough sets of the form $(Y \boxtimes \lambda_i)$ from granularity level $\mathcal{L}_i$ to $\mathcal{L}_j$ with $i \sqsubseteq j$. Let $\omega_{ij}^{p} = \{(Y \boxtimes \lambda_i)\ k \mid k \in \ell_{ij}^{-1}\ p\} \subset \mathcal{P}\,\tilde{\Omega}$ be the set containing the sets of approximation values under $(Y \boxtimes \lambda_i)$ with respect to the cells $(\ell_{ij}^{-1}\ p) \subseteq \mathcal{L}_i$.

Consider table 2 and assume sets $O, P, U \in Q$ of which we only know their vague rough set representation with respect to the granularity level $\mathcal{L}_4$ as given in columns δb, δg, and δh of the table. In column ω_{42}^{c} we have the subset of $\mathcal{P}\,\tilde{\Omega}$ corresponding vague rough set in column δ with respect to the cells $g, h \in (\ell_{42}^{-1}\ c)$.

δ	$\delta\ b$	$\delta\ g$	$\delta\ h$	ω_{42}^{c}	$\bigcap \omega_{42}^{c}$	$\tilde{\alpha}(\delta\ c)$
$O \boxtimes \lambda_4$	{B}	{T, B}	{F, B}	{{T, B}, {F, B}}	{B}	{B}
$P \boxtimes \lambda_4$	{B}	{T, B, F}	{F, B}	{{T, B, F}, {F, B}}	{B, F}	{B, F}
$U \boxtimes \lambda_4$	{B}	{T}	{F}	{{T}, {F}}	{}	{B}

Table 2. Examples for the generalization of vague rough sets from granularity level $\mathcal{L}_4$ to $\mathcal{L}_2$.

We define the generalization mapping $\tilde{\alpha}_{ij} : \mathcal{P}\,\tilde{\Omega} \to \mathcal{P}\,\tilde{\Omega}$ which transforms vague rough sets from granularity level $\mathcal{L}_i$ to granularity level $\mathcal{L}_j$ with $i \sqsubseteq j$ by reading the table in figure 9 row-wise as follows (using ω be a shorthand for $\omega_{ij}^{\ell_{ij}\ k}$):

Row1: if $\bigcap \omega = \{\mathsf{B}\}$ then $(\tilde{\alpha}\ (Y \boxtimes \lambda_i))\ k = \{\mathsf{B}\}$
...
Row7: if $\bigcap \omega = \{\}$ then $(\tilde{\alpha}(Y \boxtimes \lambda_i))\ k = \{\mathsf{B}\}$

Consider table 2. In the last two columns we see the values $\bigcap \omega_{42}^{c}$ and $\tilde{\alpha}(\delta\ c)$ according to the table in figure 9 for the corresponding rough sets in column δ.

Definition 8. *A stratified vague rough set is a family of vague rough sets* $(Y \boxtimes \lambda_1), \ldots (Y \boxtimes \lambda_n)$*, such that whenever* $i \sqsubseteq j$ *then there exists a mapping* $\tilde{\alpha}_{ij} : \mathcal{P}\{\mathsf{T},\mathsf{B},\mathsf{F}\} \to \mathcal{P}\{\mathsf{T},\mathsf{B},\mathsf{F}\}$ *as defined in table 9 such that the following holds:*

$$(\tilde{\alpha}_{ij}\ (Y \boxtimes \lambda_i))\ k = ((Y \boxtimes \lambda_j)\ \ell_{ij})\ k.$$

One then can verify that if $((K, \leq), (\mathcal{L}, \sqsubseteq), \ell, \lambda_1, \ldots \lambda_n)$ is a labelled non-cumulative granular partition and $(Y \boxtimes \lambda_i)$ is a vague rough set then $\tilde{\alpha}(Y \boxtimes \lambda_i)$ is the result of applying the generalization mapping $\tilde{\alpha}_{ij}$ to $(Y \boxtimes \lambda_i)$ if and only if every crisping of $\tilde{\alpha}(Y \boxtimes \lambda_i)$ is the result of a crisp generalization α, of a crisping of $(Y \boxtimes \lambda_i)$.

7 Conclusions and Further Work

In this paper we have shown how the technique of making rough descriptions of a subset with respect to an equivalence relation can be extended to descriptions with respect to a granular partition. The work has also revealed a new way of looking at a granular partition as a generalization of an equivalence relation. In this generalization, a set of names of equivalence classes is replaced by a tree structure and certain subsets of the tree are extracted to form labels for equivalence classes. In this way we obtain a hierarchy of equivalence classes. This is relevant to Spatial Information Theory because (a) most spatial representations, in particular maps, are granular partitions, (b) those representations are often hierarchical [PM97]; and (c) because approximations with respect to sets of equivalence classes are important in order to deal with vagueness and indeterminacy inherent in many geographic phenomena.

This identification of the way in which the equivalence classes at the various levels of detail relate to each other is an important contribution. It enables us to understand the relationship of granular partitions to the stratified map spaces of Stell and Worboys [SW98]. The stratified map space concept is applicable to problems involving level of detail in temporal data, as for example in the work of Hornsby and Egenhofer [HE99]. The extension to rough descriptions using granular partitions for temporal data is one area for further work which we intend to pursue.

Another area for further work is to extend the results of this paper to richer structures than sets. In particular, graphs represent a significant challenge, and have clear connections with practical issues in spatial information theory. To carry out the extension to graphs would entail replacing the set which is subjected to the family of equivalence relations in a granular partition, by a graph. This would require identification of the appropriate generalization of equivalence relations for the richer context. A number of possibilities for such a generalization have been discussed in the literature [Ste99], and it is possible that more than one could be made to work with granular partitions. If the work were extended in this way, we would expect it to yield new techniques for the rough description of networks, such as those of roads, railways etc.

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