GEOMETRIC EVOLUTION PROBLEMS AND ACTION-MEASURES

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1. Introduction

Geometric evolution problems are connected to many interesting phenomena, such as ice melting, metal solidification, explosions, damage mechanics. Any such problem numbers among the unknowns a geometric object. The canonical example of a geometric evolution problem is the mean curvature flow of a surface. A more complex situation arises in the study of brittle crack propagation. The state of a brittle body is described by a pair displacement-crack, therefore the crack propagation problem has two unknowns. We have to suppose that, at any moment, the displacement has no discontinuities away from the crack. Moreover, the displacement is connected with the crack by the boundary conditions: these contain conditions such as unilateral contact of the lips of the crack.

In most of the studies the fracture propagation is not recognized to have a geometrical nature. It is the purpose of this paper to formulate a general geometric evolution problem based on the notion of action-measure, introduced here. For particular choices of the action-measure we obtain formulations of the mean curvature flow or the brittle fracture propagation problems.

2. Action measures and viscosity solutions

$(L, \leq, \tau)$ is a sequential topological ordered set (or t.o.s.) if $(L, \leq)$ is an ordered set and for any sequence $(\beta_h)_h$ in L, converging to some $\beta \in L$, if there exists $\alpha \in L$ such that $\beta_h \leq \alpha$ for any h, then $\beta \leq \alpha$.

Let us consider $F : X \to L$, where X is a topological space and L is a sequential t.o.s. A minimal element of F is any $x \in X$ such that for any $y \in X$, if $F(y) \leq F(x)$ then $F(y) = F(x)$. Remark however that, due to the

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lack of total ordering, a minimal element may not be a minimizer, i.e. even if $x \in X$ is a minimal element of F, it is not true that $F(x) \leq F(y)$ for any $y \in X$.

A particular case of t.o.s. is any space of measures. An action measure is a function defined over a topological space with values in a space of measures. The direct method in the calculus of variations can be reformulated in this frame. In particular, if the space of measures is a topological dual of a space of functions then the direct method can be written in a particular form. We leave to the reader the formulation of the general direct method and the reformulation of the theorem in this case.

Action measures are related to (first order) viscosity solutions (see [4], [5], [6]) . Indeed, take a function

$$H : R^n \times R^n \to R$$

C^1 in the first argument and positive one-homogeneous in the second. (Weaker assumptions may be taken.) Consider now L, the polar of H,

$$L(x,p) \ = \ \sup \{ \langle p, q \rangle - H(x,q) \, : \, q \in R^n \}$$

For any fixed $T > 0$ we define the set

$$A_T \ = \ \big\{ c : \overline{\Omega} \times [0,T] \to \overline{\Omega} \, : \, c(x,\cdot) \in C^1([0,T]) \ \forall x \in \Omega,$$
$$c(\cdot, 0) = \ id \ \ , \ c(x,T) \in \partial\Omega \ \forall x \in \Omega \}$$

and the function $F : A_T \to \mathcal{M}(\Omega)$

$$F(c)(B) = \int_B g(x,T) \ \mathrm{d}x + \int_B \int_0^T L(c(x,t), \dot{c}(x,t)) \ \mathrm{d}t \ \mathrm{d}x \quad .$$

Here g is a positive function defined on $\partial\Omega$. This action-measure has minimal elements. Moreover it has minimizing elements. Let c_0 be any one of them. Then

$$F(c_0)(B) = \int_B u(x) \ \mathrm{d}x \ \ \forall B \in \mathbf{B}(\Omega) \tag{1}$$

where u is the viscosity solution of the problem

$$H(u, \nabla u) = 0 \ \ , \ \ u = g \text{ on } \partial\Omega \quad . \tag{2}$$

Notice that in this setting of the problem (2) the primary unknown is the map c_0. The viscosity solution of (2), that is u, is the Lebesgue density of the measure $F(c_0)$.

Any function $c \in A_T$ can be identified with a path of deformations of Ω by $c(\cdot, t) \ \mapsto c_t(\cdot) : \overline{\Omega} \to \overline{\Omega}$. This fact make us formulate the following general problem:

Consider a space M of curves $t \mapsto \phi_t : \Omega \rightarrow \Omega$ and an action measure $A : M \rightarrow Meas(\Omega)$, where $Meas(\Omega)$ is a space of scalar measures over Ω. Find and describe, under suitable conditions over M and A, the minimal elements of the action measure A.

3. Evolution driven by diffeomorphisms

$Diff_0(\Omega)$ denotes the space of C^∞ diffeomorphisms of Ω with compact support, that is the set of all C^∞ functions $\phi : R^n \rightarrow R^n$ such that $\phi^{-1} \in C^\infty$ and *supp* $(\phi - id) \subset\subset \Omega$. It is well known that any vector-field $\eta \in C_0^\infty(\Omega, R^n)$ (i.e. with compact support in Ω) generates a one-parameter flow $t \mapsto \phi_t \in Diff_0(\Omega)$, solution of the problem: $\dot{\phi}_t \ = \ \eta.\phi_t \ \ , \ \ \phi_0 \ = \ id$, where the dot "." denotes function composition.

Consider a sufficiently regular set $B \subset \Omega$. Let ξ_B be the characteristic function of B. For any $\phi \in Diff(\Omega)$ we have the equality: $\xi_{\phi(B)} \ = \ \xi_B.\phi^{-1}$.

A geometric evolution of the set B is any curve $t \mapsto B(t)$, such that $B(0) \ = \ B$. A particular case of geometric evolution of B is when $B(t)$ is isotopically equivalent to B. Such an evolution (which we call isotopic) can be obtained by considering a curve $t \mapsto \phi_t \in Diff_0(\Omega)$, $\phi_0 = id$. Any such curve induces a geometric evolution of B by $B(t) = \phi_t(B)$. Therefore, this kind of geometric evolution of the set B is equivalent to a curve in $Diff_0(\Omega)$, with origin at id.

We can make weaker assumptions upon the geometric evolution of B. In this paper we shall introduce the notion of geometric evolution driven by diffeomorphisms. The advantage of this notion is that potentially complex evolutions of B are locally approximated by isotopic evolutions. We describe further what an evolution driven by diffeomorphisms is.

The regularity assumptions upon the initial set B are described first. $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. We shall suppose that B has k Hausdorff dimension. We suppose also that for any vector-field $\eta \in C_0^\infty(\Omega, R^n)$ there exists the derivative with respect to t of the function $t \mapsto \xi_{\phi_t(B)}\mathcal{H}^k$, where ϕ_t is the one-parameter flow generated by η. Moreover, this derivative is supposed to be absolutely continuous with respect to the measure $\mathcal{H}^{k-1}$.

An evolution of B driven by diffeomorphisms is a curve $t \mapsto B(t)$, $B(0) = B$, such that:

- i) $d/dt \, \xi_{B(t)}\mathcal{H}^k$ is absolutely continuous with respect to $\mathcal{H}^{k-1}$. The support of this measure is denoted by $\partial^* B(t)$ and is called the border of $B(t)$.
- ii) there is a curve $t \mapsto \eta(t) \in C_0^\infty(\Omega, R^n)$ such that for almost any t we have the inequality of measures:

$$\frac{d}{dt}\xi_{B(t)}\mathcal{H}^k \ \leq \ \frac{d}{ds}\xi_{B(t)\cdot\phi^{-1}_{s,\eta(t)}}\mathcal{H}^k$$

where $s \mapsto \phi_{s,\eta(t)}$ is the one-parameter flow generated by $\eta(t)$ and the derivative with respect to s is made for $s = 0$.

iii) the function $t \mapsto d/ds\ \xi_{B(t)} . \phi^{-1}_{s,\eta(t)} \mathcal{H}^k(\Omega)$ is measurable.

iv) for any $t < t'$ we have $B(t) \subset B(t')$.

Let us denote by $Bar^+(t, Q)$ the set of all $\eta \in C_0^\infty(\Omega, R^n)$ with compact support in $Q \subset \Omega$ which satisfy: $d/dt\ \xi_{B(t)}\mathcal{H}^k \ \leq\ d/ds\ \xi_{B(t)} . \phi^{-1}_{s,\eta}\mathcal{H}^k$. Obviously, the set $Bar^+(t, Q)$ depends on the evolution $t \mapsto B(t)$.

We have the following result: for almost any t there is a positive vector-field $v(t)$, with support on $\partial^* B(t)$, called the normal velocity field, such that for any $\eta \in Bar^+(t, \Omega)$ we have

$$\frac{d}{dt}\xi_{B(t)}\mathcal{H}^k \ \leq\ v(t)\mathcal{H}^{k-1} \ \leq\ \frac{d}{ds}\xi_{B(t)} . \phi^{-1}_{s,\eta}\mathcal{H}^k \ .$$

4. A general geometric evolution problem

Consider now a set $C \subset \mathcal{P}(\Omega)$, which contains only regular closed sets $B \subset \Omega$ and let M be a family of evolutions of an initial set $B_0 \in C$ driven by diffeomorphisms, such that for any t and any curve $t \mapsto B(t) \quad \in M$ we have $B(t) \in C$. Let us consider also a functional $E : C \rightarrow R$, such that $E(B) \geq E(B')$ if $B \subset B'$. E is smooth in the following sense: for any $B \in C$ and any one-parameter flow $t \mapsto \phi_{t,\eta}$ the function $t \mapsto E(\phi_{t,\eta}(B))$ is derivable in $t = 0$. This derivative will be denoted by $dE(B, \eta)$. Given a geometric evolution $t \mapsto B(t) \ \in M$, for any borelian set $Q \in \mathbf{B}(\Omega)$, the variation of E at $B(t) \in C$, inside Q is defined by the formula:

$$dE(B(t))(Q) \ = \ sup \ \left\{ dE(B(t), \eta) : \exists\ \lambda > 0,\ \lambda\eta \in Bar^+(t, Q) \ , \right.$$
$$\left. d(\partial^* B, \phi_{1,\eta}(\partial^* B)) \ \leq \ 1 \ \right\} \quad .$$

Under suitable assumptions $-dE(B(t))$ is a positive measure.

We introduce now the action-measure defined for any geometric evolution $t \mapsto B(t) \quad \in M$ by the expression:

$$A(t \mapsto B(t))(Q) \ = \ \int_0^T \int_{\partial^* B(t) \cap Q} \mathbf{v}(t) \ \mathrm{d}\mathcal{H}^{k-1} \ \mathrm{d}t \quad + \ \int_0^T dE(B(t))(Q) \ \mathrm{d}t \quad .$$

Notice that the first term of A can be written as the variation of $\mathcal{H}^k(B(t))$ from 0 to T. Remark also that we can consider functions $E = E(B, t)$, such that $E(B, t) \geq E(B, t')$ if $B \subset B'$.

Example 1. Mean curvature flow. (see [1]) Let us take $k = n$ in the regularity assumptions, that is B_0 n-dimensional, and $E(B) \ = \ - \ \mathcal{H}^{n-1}(\partial^* B)$. Then any minimal element of the action measure A defined above is a supersolution of the mean curvature flow problem, that is for almost any t and almost any $x \in \partial^* B(t)$ we have $v(t) \geq k(x,t)$, where $k(x,t)$ is the mean curvature of $\partial^* B(t)$ in x (with the convention of positive curvature for spheres).

Example 2. Brittle crack propagation. By a crack set in Ω we mean a closed, finite rectifiable set B. Ω represents the reference configuration and $\mathbf{u} : \overline{\Omega} \rightarrow R^n$ is the deformation of a hyper-elastic body. The free energy density is $w(\nabla \mathbf{u})$; in the case of infinitesimal deformations $\mathbf{u}$ represents the displacement of the body and w is a quadratic function of the symmetric gradient of $\mathbf{u}$.

A path $t \mapsto \mathbf{v}(t)$ of deformations (or displacements) is given on $\partial\Omega$. The evolution of the body is supposed to be quasi-static. An initial crack set B_0 is present in the body. We are interested in the propagation of this crack under the path of imposed deformations. We introduce for this the following functional, defined for any crack set B and any moment t:

$$E(B,t) \ = \ inf \ \left\{ \int_\Omega w(\nabla(\mathbf{u}) \ \mathrm{d}x \, : \, \mathbf{u} \in C^1(\overline{\Omega} \setminus B) \ , \ \mathbf{u} = \mathbf{v}(t) \text{ on } \partial\Omega \setminus B \right\} \quad .$$

Our principle of brittle crack propagation states that the evolution of the initial crack B_0 is a minimal element of the action-measure:

$$A(t \mapsto B(t))(Q) \ = \ G \ \mathcal{H}^{n-1}(B(T) \cap Q) \ + \ \int_0^T dE(B(t),t)(Q) \ \mathrm{d}t \quad .$$

The physical meaning of this principle is: choose the crack propagation $t \mapsto B(t)$ such that the energy consumed by the body in order to produce in Q the crack growth $t \mapsto B(t) \cap Q$ is less than the energy released in Q due only to crack propagation.

In the particular case of infinitesimal deformations if we take the curve $t \mapsto B_0(t) = B_0$ we see that $A(B_0(\cdot))(Q) = 0$ for any Q, therefore $A(B(\cdot))$ is a negative measure. Therefore, in this case, a generalization of Griffith criterion holds.

In [2], [3] we have proposed a minimizing movement model of brittle crack propagation in infinitesimal deformations ([3], definitions 4.1 and 5.1). The model is presented here in a condensed form. Let us consider the set M of all $(\mathbf{u}, K)$ such that $K \subset \overline{\Omega}$ is a crack set, $\mathbf{u} \in \mathbf{u} \in C^1(\overline{\Omega} \setminus K, R^n)$ and for $\mathcal{H}^{n-1}$-almost any $x \in K$ there exist the normal $\mathbf{n}(x)$ at K in x and $\mathbf{u}^+(x)$, $\mathbf{u}^-(x)$.

We define the functions

$$J : M \times M \rightarrow R \ ,$$

$$J\left((\mathbf{u},K),(\mathbf{v},L)\right) \ = \ \int_\Omega w(\nabla \mathbf{v}) \ \mathrm{d}x \ + \ G\mathcal{H}^{n-1}(L \setminus K) \quad ,$$

$$\Psi : [0,\infty) \times M \rightarrow \{0, +\infty\} \ ,$$

$$\Psi(\lambda, (\mathbf{v}, K)) \ = \ \begin{cases} 0 & \text{if } \mathbf{v} = \mathbf{u}_0(\lambda) \text{ on } \partial\Omega \setminus K \\ +\infty & \text{otherwise} \end{cases} \ .$$

We consider the initial data $(\mathbf{u}_0, K) \in M$ such that $\mathbf{u}_0 \ = \ \mathbf{u}(\mathbf{u}_0(0), K)$. For any $s \geq 1$ we define the sequences

$$k \in N \ \mapsto \mathbf{u}^s(k) \ , \ L^s(k) \ , \ K^s(k) \ \ ,$$

$(\mathbf{u}^s(k), L^s(k)) \in M$ and $(\mathbf{u}^s(k), K^s(k)) \in M$, recursively:

i) $(\mathbf{u}^s, K^s)(0) \ = \ (\mathbf{u}_0, K)$, $L^s(0) \ = \ K$,

ii) for any $k \in N$ $(\mathbf{u}^s, L^s)(k+1) \in M$ minimizes the functional

$$(\mathbf{v}, L) \in M \ \mapsto \ J\left(((\mathbf{u}^s, K^s)(k), (\mathbf{v}, L)\right) \ + \ \Psi((k+1)/s, (\mathbf{v}, L))$$

over M. $K^s(k+1)$ is defined by the formula:

$$K^s(k+1) \ = \ K^s(k) \cup L^s(k+1) \ \ .$$

$(\mathbf{u}, L, K) : [0, +\infty) \rightarrow M$ is an energy minimizing movement associated to J with the constraint Ψ and initial data $(\mathbf{u}_0, K)$ if there is a diverging sequence (s_i) such that for any $t > 0$ we have: $\mathbf{u}^{s_i}([s_i t]) \ \rightarrow \ \mathbf{u}(t)$ in $L^2(\Omega, R^n)$. $L(t)$ is called the active crack at the moment t and

$$K(t) \ = \ \cup_{s \in [0,t]} S(s)$$

is the total damaged region at the same moment.

We have the following result which connects these two models of brittle crack propagation presented here.

Theorem. *Let us consider an energy minimizing brittle crack propagation* $t \mapsto (\mathbf{u}_t, \mathbf{S}(t), K(t))$. *Suppose that* $t \mapsto K(t)$ *is driven by diffeomorphisms. Then the curve* $t \mapsto K(t)$ *is a minimal element of the action-measure* A *defined above, in the case of infinitesimal deformations.*

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