On the Average Case Behavior of the Multidimensional Assignment Problem[*]

Don A. Grundel[†]   Pavlo Krokhmal[‡]   Carlos A.S. Oliveira[§]

Panos M. Pardalos[¶]

May 18, 2004

This paper is dedicated to Professor T. Rockafellar in honor of his 70th birthday.

Abstract

The multidimensional assignment problem (MAP) is a combinatorial problem where elements of a variable number of sets must be matched, in order to find a minimum cost solution. The MAP has applications in a large number of areas, and is known to be NP-hard. We survey some of the recent work being done in the determination of the asymptotic value of optimal solutions to the MAP, when costs are drawn from a known distribution (e.g., exponential, uniform, or normal). Novel results, concerning the average number of local minima for random instances of the MAP for random distributions are discussed. We also present computational experiments with deterministic local and global search algorithms that illustrate the validity of our results.

1   Introduction

In the field of operations research, many problems require the computation of an optimal assignment among objects from different sets. Examples where this situation occurs abound in the literature, including optimal resource allocation [36], multitarget tracking [37], measurement of particle trajectories [38], and detection of tissue cells from a sequence of images [15].

A general way of formalizing such situations is given by the multidimensional assignment problem (MAP). In the MAP, elements from a variable number of sets can be mutually assigned. For each assignment a (from a set $\mathcal{A}$ of possible assignments), an associated cost c_a is given. A solution to the MAP is a complete assignment, i.e., each element of the first set is assigned to exactly one element in each of the other sets. The MAP asks for a complete assignment of elements with minimum cost.

A formal description of the MAP can be given in the following way. Let $A_1, \ldots, A_d$ be the d sets of elements (d is also known as the dimension of the problem), and let n_i be the number of elements in set A_i, i.e., $n_i = |A_i|$. Let $x_{i_1,\ldots,i_d}$ be a binary variable stating that element i_1 of set A_1 is assigned to element i_k of set A_k, for $k \in \{2, \ldots, d\}$. Also, let the cost of assignment $(i_1, \ldots, i_d)$ be

given by $c_{i_1,\ldots,i_d}$. Then, the MAP can be formulated as

$$\min \quad \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} c_{i_1\cdots i_d} x_{i_1\cdots i_d}$$

$$\text{s.t.} \quad \sum_{i_2=1}^{n_2} \cdots \sum_{i_d=1}^{n_d} x_{i_1\cdots i_d} = 1 \qquad \text{for all } i_1 = 1, 2, \ldots, n_1,$$

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_{k-1}=1}^{n_{k-1}} \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_d=1}^{n_d} x_{i_1\cdots i_d} \leq 1$$

$$\text{for all } k = 2, \ldots, d-1, \text{ and } i_k = 1, \ldots, n_k,$$

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_{d-1}=1}^{n_{d-1}} x_{i_1\cdots i_d} \leq 1 \qquad \text{for all } i_d = 1, 2, \ldots, n_d,$$

$$x_{i_1\cdots i_d} \in \{0, 1\} \qquad \text{for all } i_1, i_2, \ldots, i_d \in \{1, \ldots, n\},$$

$$n_1 \leq n_2 \leq \cdots n_d,$$

where d is the dimension of the MAP instance.

An equivalent formulation (when $n_1 = n_2 = \cdots = n_d = n$) states the MAP in terms of permutations $\delta_1, \ldots, \delta_{d-1}$ of numbers 1 to n. Using this notation, the MAP is equivalent to

$$\min_{\delta_1,\ldots,\delta_{d-1}\in\Pi^n} \sum_{i=1}^{n} c_{i,\delta_1(i),\ldots,\delta_{d-1}(i)},$$

where Π^n is the set of all permutations of $\{1, \ldots, n\}$. In this paper we assume $n_i = n$, for $i \in \{1, \ldots, d\}$.

1.1 Complexity Results

The MAP is known to be NP-hard. This follows from a reduction of the NP-complete problem 3-DIMENSIONAL MATCHING (3DM) [10]. The 3DM problem is to decide if there exists a stable matching of elements from three sets A_1, A_2, and A_3, where specific preferences are given for each triple of elements (a_1, a_2, a_3), $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$. That is, one wants to find a set of triples forming a match such that the preferences of all elements are satisfied. The 3DM problem can be easily reduced to the MAP with dimension $d = 3$ [27].

3

Some special cases of the MAP are, however, known to the solvable in polynomial time. For example, the most well known special case of the MAP (when $d = 2$) is the linear assignment problem (LAP). The LAP is a classical combinatorial optimization problem that can be solved in polynomial time by different algorithms, such as the maximum flow algorithm [28], and the Hungarian method [17].

In this paper we are interested in studying the average behavior of instances of the MAP, when assignment costs are drawn from random distributions such as the uniform, normal, or exponential distributions. By average behavior, we mean the value of some parameters, such as the expected cost of the optimal solution, and the number of local minima, when the instances have assignment costs drawn from a known distribution. We give in Section 2 a review of previous results related to the average behavior for diverse combinatorial optimization problems. In Section 3 we give a survey of our results related to the asymptotic behavior of the expected optimal value for MAP instances. In Section 4, we present new results on the average number of local minima for the MAP. Finally, in Section 5 we present computational experiments used to illustrate the results discussed in previous sections. We conclude in Section 6 with some remarks and future work.

2   Review of Previous Results

Studies on the average behavior of random combinatorial problems can be traced back to the work of Beardwood, Halton and Hammersley [4] on the traveling salesman problem (TSP). The TSP can be defined in the following way: let X_i, $i \in \{1, \ldots, n\}$, be independent random variables uniformly distributed on the unit square $[0, 1]^2$, and let L_n denote the length of the shortest closed path (on the usual Euclidean distance) which connects each element of $\{X_1, X_2, \ldots, X_n\}$. The classic result proved by [4] is

$$\lim_{n \to \infty} \frac{L_n}{\sqrt{n}} = \beta,$$

with probability one, for a finite constant β. This becomes significant, as addressed by Steele [39], because it is the key to Karp's algorithm [13] for solving the TSP. Karp uses a cellular dissection algorithm for approximating the solution of the TSP. His result may be summarized by saying that the optimal tour through n points is sharply predictable when n is large and the dissection method tends to give near-optimal solutions when n is large. This points to the idea of using asymptotic techniques to develop effective solution algorithms.

Other work in this area includes studies of the minimum spanning tree [9, 41], quadratic assignment problem (QAP) [6, 30, 32] and, most notably, studies of the linear assignment problem (LAP) [1, 8, 14, 19, 33, 26, 31, 42]. More general work can be found in [22], where an analysis of parameters for random graphs was performed by Lueker. A nice introduction to probability topics in combinatorial optimization is given in [40].

In the minimum spanning tree problem, consider an undirected graph $G = (V, E)$ defined by the set V of n nodes and a set E of m arcs, with a length c_{ij} associated with each arc $(i, j) \in E$. The problem is to find a spanning tree of G, called a minimum spanning tree (MST) that has the smallest total length L_{MST} of its constituent arcs [28]. If we let each arc length c_{ij} be an independent random variable drawn from the uniform distribution on $[0, 1]$, Frieze [9] showed that

$$E[L_{MST}] \to \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.202\cdots \quad \text{as} \quad n \to \infty.$$

This result was followed by [41], where the Tutte polynomial for a connected graph is used to develop an exact formula for the expected value of L_{MST} in a finite graph with uniformly distributed arc costs.

For the Steiner tree problem, which is a NP-hard variant of the MST, Bollobás et al. [5] proved that with high probability the weight of the Steiner tree is $(1 + O(1))(k - 1)(\log n - \log k)/n$ when $k = O(n)$ and $n \to \infty$, where n is the number of vertices in a complete graph with edge weights chosen as i.i.d. random variables, distributed as exponential with mean one. Here, k is number of vertices contained in the Steiner tree.

A famous result that some call the Burkard-Fincke condition relates to the QAP. The QAP was introduced by Koopmans and Beckmann [16] in 1957 as a model for the location of a set of indivisible economical activities. QAP applications, extensions and solution methods are well covered in [12]. The Burkard-Fincke condition taken from [6] is stated as follows.

Proposition 1. *The ratio between the objective function values of worst and optimal solutions is arbitrarily close to one, with probability tending to one as the size of the problem approaches infinity.*

Another way of describing this is by saying that for a large problem any permutation is close to optimal. According to [6], this condition applies to all problems in the class of combinatorial optimization problems with sum- and bottleneck objective functions. The linear ordering problem (LOP) [7] falls into this category

as well. Burkard and Fincke suggest that this result means that very simple heuristic algorithms can yield good solutions for very large problems.

Recent work by Aldous and Steele [2] provides part survey, part tutorial on objective methods for understanding asymptotic characteristics of combinatorial problems. They provide some concrete examples of the approach and point out unavoidable limitations.

One of the most explored problems in this area has been the linear assignment problem (LAP). In the LAP we are given a matrix $C^{n \times n}$ with coefficients c_{ij} and the objective is to find a minimum cost assignment, i.e., a permutation π of the numbers $\{1, \ldots, n\}$ minimizing the objective function $\sum_{i=1}^{n} c_{i\pi(i)}$.

Mézard and Parisi conjectured [23, 24] that the optimum solution z^*, for instances where costs c_{ij} are drawn from the exponential or uniform distributions, approaches the asymptotic value $\pi^2/6$ when n (the size of the instance) approaches infinity. Additional empirical evidence of the validity of this conjecture was given by Pardalos and Ramakrishnan [31], with experiments over several very large dense LAPs solved with an interior point algorithm.

The Mézard-Parisi conjecture has been further strengthened by Coppersmith and Sorkin [8], who claimed that the expected value of the optimum k-assignment, for a fixed matrix of size $n \times m$, is given by

$$\sum_{i,j \geq 0, \, i+j < k} \frac{1}{(m-i)(n-j)}.$$

Coppersmith and Sorkin also presented proofs of this claim for small values of n, m and k. The conjecture is consistent with [23], since it can be proved that for $m = n = k$ this is simply

$$\sum_{i=1}^{n} \frac{1}{i^2}.$$

Moreover,

$$\sum_{i=1}^{n} \frac{1}{i^2} \rightarrow \frac{\pi^2}{6} \qquad \text{as } n \rightarrow \infty.$$

Although the proof of such conjectures concerning the MAP has eluded researchers for many years, there has been continuous progress in the determination of upper and lower bounds. In [42], Walkup proved an upper bound of 3 on the asymptotic value of the objective function, when the problem size increases. This was improved later by Karp [14], who showed that the limit is at most 2. On

6

the other hand, Lazarus [19] proved a lower bound of $1 + 1/e \approx 1.3679$. More recently this result was improved by Olin [26] to the tighter value of 1.51.

Finally, the conjectures of Mézard and Parisi have been solved in recent papers by Linusson and Wästlund [21] and Nair *et al.* [25]. Their results have also confirmed the conjecture of Coppersmith and Sorkin.

Concerning the MAP, the study of asymptotic behavior when the size of the problem increases is still in its early stages. Recently, however, we have made some progress on the determination of asymptotic behavior for the expected value of optimum solutions for the MAP [11], and on the average number of local minima, given the distribution of costs. This work is discussed in the next two sections.

3   Expected Value of Optimum Solutions for the MAP

One of the main difficulties encountered when solving instances of the MAP is the large increase in size of instances, when the dimension parameter d increases. Instances I with dimension d and n elements per dimension have size $|I| = n^d$. This suggests that the exponential computational time is inherent to the MAP, and it is also a real reason to develop estimates of the optimal solution for the MAP without checking all assignment costs in the instance.

It turns out that an estimate of optimum objective value for the MAP is possible, when we consider instances whose assignment costs are drawn from a known probability distribution. In this section we present results for the cases in which costs are taken from the uniform, exponential and normal distributions. The results in this section were first presented in [11].

3.1   Preliminary Results

Initially, let us consider an enumeration method for MAP solution costs known as the *index tree representation*. This type of representation was introduced by Pierskalla [36], and consists of ordering the assignment costs according to the sequence of its indices. Each node in the index tree (with the exception of the root) represents an assignment $(i_1, \ldots, i_d)$. A path in the index tree from the root to any leaf node represents a feasible solution to the current instance.

The resulting tree is composed of n levels. At level j of the tree, all assignments with $i_1 = j$ are listed. For a node v at level j, let A_i, for $i \in \{1, \ldots, d\}$, be the set of values appearing in position i on any of the assignments in the path

from the root node to v. Then, v is parent of all nodes w in level $j + 1$ such that $w = (j + 1, k_2, \ldots, k_d)$ and $k_i \notin A_i$, for $i \in \{2, \ldots, d\}$.

Given an instance of the MAP, let n be the number of elements in each set, and d the number of sets (dimension of the instance). Let $\mathcal{A}$ be the set of feasible solutions to the problem instance. We define z^* as the optimum value of the MAP instance. We are interested in computing the value of $E(z^*)$, given instances with random distributed costs.

The main tool for our study is the following proposition, which gives a lower bound on the number of feasible solutions for a random subset of elements in $\mathcal{A}$. This proposition is based on the technique known as *probabilistic method* [3]. In this technique, some combinatorial configuration is shown to exist with certainty, given the probability distribution of all possible configurations. We investigate the distribution of nodes in an index tree representation, and show that among a specified number of such nodes there is at least one feasible solution.

Proposition 2 ([11]). *Using an index tree to represent the cost coefficients of the MAP, randomly select α different nodes from each level of the tree and combine these nodes from each level into set $\mathcal{L}$. Then, $\mathcal{L}$ is expected to produce at least one feasible solution to the MAP when*

$$\alpha = \lceil n^{d-1}/n!^{\,(d-1)/n} \rceil \quad \text{and} \quad |\mathcal{L}| = n\alpha. \tag{1}$$

Recall that if $X_1, X_2, \ldots, X_k$ are k independent identically distributed variables and $X_{(i)}$ is the i-th smallest of these, then $X_{(i)}$ is called the i-th *order statistic* for the set $\{X_1, X_2, \ldots, X_k\}$. In the rest of this section, we will consider bounds for the value of the α-th order statistic of i.i.d. variables drawn from a random distribution. This value will be used to derive an upper bound on the cost of the optimal solution for random instances, when n or d increases.

Notice that in the following we assume $\alpha = n^{d-1}/n!^{\frac{d-1}{n}}$ (without the ceiling operation). The importance of determining the expected value of the α-th order statistic is described in the next proposition.

Proposition 3 ([11]). *Let $z_u^* = nE[X_{(\alpha)}]$, where $E[X_{(\alpha)}]$ is the expected value of the αth order statistic for each level of the index tree representation of the MAP. Then, z_u^* is an upper bound to the mean optimal solution cost of an instance of a MAP with independent identically distributed cost coefficients.*

3.2 Exponential and Uniform Distributions

The propositions presented above can be used to derive some upper bounds on the asymptotic optimum value for instances with assignment costs taken from the exponential or uniform distributions. This is shown in the next theorems.

Theorem 1 ([11]). *Given a d-dimensional MAP with n elements in each dimension, if the n^d cost coefficients are independent exponentially distributed random variables with mean $\lambda > 0$, then $z^* \to 0$ as $n \to \infty$ or $d \to \infty$.*

Theorem 2 ([11]). *Given a d-dimensional MAP with n elements in each dimension, if the n^d cost coefficients are independent uniformly distributed random variables in $[0, 1]$, then $z^* \to 0$ as $n \to \infty$ or $d \to \infty$.*

Theorem 3 ([11]). *Given a d-dimensional MAP with n elements in each dimension, for some fixed n, if the n^d cost coefficients are independent, uniformly distributed random variables in $[a, b]$, then $z^* \to na$ as $d \to \infty$.*

These theorems say that for uniform and exponential distributed assignment costs, the value of the optimal solutions for the MAP approaches zero when n or d increases. It is not a surprise that this happens when d increases, since then the number of assignments also increases exponentially. However, Theorems 1, and 2 imply this behavior when n increases.

There are also practical implications of these theorems for algorithms that work on data that can be shown to follow the random distributions above. For example, if n or d is large, it is possible to dismiss assignments with large costs, knowing that with high probability they will not appear on the optimal solution.

3.3 Normal Distribution

Asymptotic results similar to the ones presented above can be shown, for the case where cost values are taken from a normal distribution. This can be done using the same technique employed in the previous section to bound the cost of the optimal solution for normal distributed random MAPs. However, in this case a simpler bound is given by the median order statistics (see [11]).

Theorem 4 ([11]). *Given a d-dimensional MAP, for a fixed d, with n elements in each dimension, if the n^d cost coefficients are independent standard normal random variables, then $z^* \to -\infty$ as $n \to \infty$.*

Theorem 5 ([11]). *Given a d-dimensional MAP with n elements in each dimension, for a fixed n, if the n^d cost coefficients are independent standard normal random variables, then $z^* \to -\infty$ as $d \to \infty$.*

4 Expected Number of Local Minima

Another important characteristic of a combinatorial optimization problem is given by its number of local minima. Since a problem's global optimum must be found among the set $\mathcal{M}$ of its local minima, the size M of $\mathcal{M}$ is one of the indicators of how difficult it is to find the optimum solution for a combinatorial optimization problem. Clearly, the larger the number of local minima, the harder it is to find a global solution (unless most local solutions are also optimal).

Another important issue is related to the complexity of finding local minima in optimization problems. Algorithms for finding a local optimum, known as *local search algorithms*, typically employ a steepest descent strategy, based on greedy decisions. The computational complexity of such algorithms typically depends on the number of local minima in the instance.

In this section we discuss some types of local search techniques for the MAP, and study the distribution of the number M of local minima for random instances, when using a 2-exchange neighborhood.

4.1 Definitions

To study local search algorithms, a first step is to establish the concept of neighborhood of a feasible solution. Let s be a feasible solution, $\mathcal{S}$ the set of all feasible solutions, and $g(s) : \mathcal{S} \to 2^{\mathcal{S}}$ the set of solutions that can be found starting from s and applying a single rule of transformation γ. The set $g(s)$ is called a *local neighborhood* of s (more information about local search and related concepts can be found in [28]).

Depending on the transformation γ, different local neighborhoods can be derived. A very general type of neighborhood, applicable practically to all combinatorial optimization problems is the *k-exchange neighborhood*. It consists of taking a subset B of elements appearing in the current solution s, such that $|B| = k$, finding a new subset $B' \neq B$, such that $|B'| = k$, of the elements not already in the current solution, and making

$$s \leftarrow (s \setminus B) \cup B'.$$

For a concrete example, the n-exchange neighborhood for the MAP can be implemented by selecting any one of the d sets, taking the current permutation of n elements in this set (B), and substituting by a new permutation of n elements (B'). In this case, the number J of neighbors for a solution is clearly equal to $d(n! - 1)$.

In the next subsection we study the expected number of local minima for the MAP, for the case of a 2-exchange neighborhood. We consider initially the situation when $n = 2$, followed by the case where assignment costs are taken from a normal distribution.

4.2   The 2-Exchange Neighborhood

A very common type of neighborhood is the so-called *2-exchange neighborhood*. It has been used for numerous problems, and its classical reference is the paper by Lin and Kernighan [20]. A possible implementation for the MAP consists of three steps:

- selecting a value k from 1 to d,

- selecting a pair of values $i, j \in A_k$, and

- interchanging the values i and j in the assignments where they appear.

Thus, this neighborhood is defined as all 2-element exchange permutations [29] of the feasible solution. Using this definition, one can easily compute the size J of the 2-exchange neighborhood as $J = d\binom{n}{2}$. If $\hat{x}$ is a feasible assignment set of the MAP and $z_{\hat{x}}$ is the corresponding solution, then $\hat{x}$ is a discrete local minimum if and only if $z_{\hat{x}} \leq z_{\hat{y}}$ for all $\hat{y}$ in the neighborhood of $\hat{x}$. As an example, consider the following feasible solution to a MAP instance with $d = 3$, $n = 3$: $\{111, 222, 333\}$. There are nine neighbors of this solution. An example of such a neighbor is $\{111, 322, 233\}$. The solution $\{111, 222, 333\}$ is a local minimum if its solution cost is less than or equal to all nine of its neighbors' solution costs.

4.2.1   The case $n = 2$

In the special case of a MAP where $n = 2$, $d \geq 3$, and cost elements are i.i.d. random variables from some cumulative distribution F, we can establish a closed form expression for the expected number of local minima. This is based on the following proposition.

Proposition 4. *In an instance of the MAP with $n = 2$ and with cost coefficients that are i.i.d. random variables with continuous distribution F, all feasible solutions are i.i.d. random variables with distribution F_2, where F_2 represents the convolution operation $F * F$.*

Proof. Let I be an instance of MAP with $n = 2$. Each feasible solution for I is an assignment $a_1 = c_{1,\delta_1(1),...,\delta_{d-1}(1)}$, $a_2 = c_{2,\delta_1(2),...,\delta_{d-1}(2)}$, with cost $z = a_1 + a_2$. The important feature of such assignments is that for each fixed entry $c_{1,\delta_1(1),...,\delta_{d-1}(1)}$, there is just one remaining possibility, namely $c_{2,\delta_1(2),...,\delta_{d-1}(2)}$, since each dimension has only two elements. This implies that different assignments cannot share elements in the cost vector, and therefore different assignments have independent costs z. Now, a_1 and a_2 are independent variables from F. Thus $z = a_1 + a_2$ is a random variable with distribution F_2. □ □

We are now ready to prove the following theorem.

Theorem 6. *Let M be the number of local minima for an instance of the MAP with cost coefficients that are i.i.d. continuous random variables. If the instance has $n = 2$ and $d \geq 3$, then $E[M] = 2^{d-1}/(d + 1)$.*

Proof. Let N be the number of feasible solutions to the MAP where $N = n!^{d-1} = 2^{d-1}$. Using indicator variables, let

$$Y_k = \begin{cases} 1, & \text{if } k\text{-th solution, } k \in \{1, \ldots, N\}, \text{ is a local minimum;} \\ 0, & \text{otherwise.} \end{cases}$$

Now, M can be written as the sum of indicator variables such that

$$M = \sum_{k=1}^{N} Y_k.$$

Using the properties of expectation we can show that

$$E[M] = \sum_{k=1}^{N} E[Y_k] = \sum_{k=1}^{N} P(Y_k = 1), \tag{2}$$

where $P(Y_k = 1)$ is simply the probability that the cost of the k-th feasible solution is less than or equal to any of its neighbors. As the k-th solution has $J = d\binom{n}{2} = d$ neighbors, and using the fact that the current solution and all its neighbors have the same distribution, then we establish that $P(Y_k = 1) = 1/(d + 1)$. After making some simple substitutions into (2) we have

$$E[M] = 2^{d-1}/(d + 1)$$

□ □

4.2.2 The case of normal distributed costs

We now discuss the more general case where n can be greater than 2. We assume however, in this section, that assignment costs are drawn from a normal distribution. This is necessary in order to simplify the formulas used for calculating the average number of local minima.

Let $z_s = f(s)$ represent the objective cost of the current solution s to the MAP instance. If we allow the n^d cost coefficients of the MAP to be independent standard normal random variables, then the random variable z_s is normally distributed because it is the sum of n standard normal variables. In the 2-exchange neighborhood, a neighbor differs from the current solution by replacing two cost coefficients in the solution. Therefore, the effect of the transformation γ (the 2-exchange operation) on s can be summarized as adding two standard normal variables and subtracting two other standard normal variables.

As an example of neighborhood, consider the current solution, $s = \{111, 222, 333\}$ for a MAP with $d = 3$ and $n = 3$. As mentioned above, there are $J = dn(n-1)/2 = 9$ neighbors of this solution. The difference of costs for each of the neighbors of s, represented as $s^1, \ldots, s^9$ can be calculated as follows:

$$z_s - z_{s^1} = c_{111} + c_{222} - c_{112} - c_{221},$$
$$z_s - z_{s^2} = c_{111} + c_{222} - c_{121} - c_{212},$$
$$z_s - z_{s^3} = c_{111} + c_{222} - c_{211} - c_{122},$$
$$z_s - z_{s^4} = c_{111} + c_{333} - c_{113} - c_{331},$$
$$z_s - z_{s^5} = c_{111} + c_{333} - c_{131} - c_{313},$$
$$z_s - z_{s^6} = c_{111} + c_{333} - c_{311} - c_{133},$$
$$z_s - z_{s^7} = c_{222} + c_{333} - c_{223} - c_{332},$$
$$z_s - z_{s^8} = c_{222} + c_{333} - c_{232} - c_{323},$$
$$z_s - z_{s^9} = c_{222} + c_{333} - c_{322} - c_{233}.$$

Let us denote by X_1 to X_J the random variables representing the difference in cost between the current solution and its J neighbors. Given the variables $X_1, \ldots, X_J$, where $J = dn(n-1)/2$, and each $X_i \sim N(0, \sigma)$, for $i \in \{1, \ldots, J\}$, if we want to compute

$$\Pr(X_1 \leq x_1, \ldots, X_M \leq x_M) = \Phi(X),$$

then we can use the formula for the multivariate normal distribution, applied to

vector $X = (X_1, \ldots, X_J)$ (from [35])

$$\Phi(x) = \frac{\exp(\frac{-1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu))}{\sqrt{(2\pi)^n |\Sigma|}},$$

where μ is the vector of means of the random variables, and Σ is the matrix of covariances.

In our case, this distribution becomes simpler, because the means are equal to zero, and therefore μ is the $\vec{0}$ vector. The formula then reduces to

$$\Phi(x) = \frac{\exp(\frac{-1}{2}x^T \Sigma^{-1} x)}{\sqrt{(2\pi)^n |\Sigma|}}.$$

Now, we want to calculate the probability of the current solution being a local optimum, which can be described as $\Pr(X_1 < 0, \ldots, X_J < 0)$. This is

$$\Phi(0) = \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} \frac{\exp(\frac{-1}{2}x^T \Sigma^{-1} x)}{\sqrt{(2\pi)^n |\Sigma|}}.$$

In order to do this, we need to study the composition of the covariance matrix for the MAP, using the 2-exchange neighborhood.

4.2.3 Computing Covariances

The entries in the covariance matrix Σ are defined by the relationship (the transformation γ) between the costs of a solution s and a neighbor $s' \in g(s)$. In the present case, this relationship is simplified, since the difference between s and s' is itself a normal random variable. To analyze the value of this difference, we use the following simple lemma.

Lemma 1. *If X, Y, and Z are normal distributed random variables with mean 0 and standard deviation σ, then the covariance of $X + Y$ and $X + Z$ is equal to σ^2.*

Proof. We know that for $A, B \sim N(0, \sigma)$, the following is true:

$$Cov(A, B) = E(AB) - E(A)E(B) = E(AB),$$

since $E(A)$ and $E(B)$ are zero. Then, if $X, Y, Z \sim N(0, \sigma)$,

$$Cov(X + Y, X + Z) = E(X^2 + XZ + XY + YZ) = E(X^2),$$

because X, Y, and Z are independent, so e.g., $E(XY) = EX\,EY = 0$. But as $X \sim N(0, 1)$, $E(X^2) = \sigma^2 + \mu^2 = \sigma^2$. $\square$ $\square$

14

We can use the lemma above to compute all entries in the covariance matrix. Initially, in the main diagonal of the matrix, all entries are the variance of the variables $X_1, \ldots, X_J$. It is known that the variance of the sum or difference of normal random variables is given by the sum of the variance of its terms. Thus, given the selected assignments a and b and the new assignments a' and b', for any $X_k = c_a + c_b - (c_{a'} + c_{b'})$, $k \in \{1, \ldots, J\}$, we have $Var(X_k) = 4$.

Now, to compute the remaining entries of the matrix Σ of covariances for the 2-exchange local search, we consider the cases in which different exchanges in a permutation can occur. For entries where the same two assignment positions i, j are selected (there are only $d - 1$ of these assignment for each neighbor, because they vary only the dimension where the exchange is made), the covariance is given by $Cov(A - B, A - C)$, where A is the shared part of the assignment and B and C are the sum of two $N(0, 1)$ random variables. According to the previous lemma, $Cov(A - B, A - C) = \sigma_A^2 = 2$.

There are also more $2d(J - 2)$ entries in each row of Σ that share only one entry (since there are d dimensions, 2 options for the first position and $J - 2$ options for the second position to be exchanged). In this case, using the lemma above we have a covariance of $Cov(X, Y) = 1$.

Finally, each row of the matrix Σ has $J - 2d(J-2) - d$ remaining entries where no cost is shared, since the positions exchanged in the assignment are completely different. These entries have covariance equal to zero.

Example 1. *Continuing with the example where $n = 3$ and $d = 3$, by using the procedure described above we can compute the covariance matrix as the following:*

$$
\Sigma^{9 \times 9} =
\begin{bmatrix}
4 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 4 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 4 & 2 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 4 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 2 & 4 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 4 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 4 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4
\end{bmatrix}
$$

5 Computational Experiments

In the previous section, we discussed some results related to the expected value of the optimal solution and the expected number of local minima for an instance of the MAP. One of the questions that remain after this discussion, however, is about how fast the convergence of these values is, when the size of the instances increase.

In this section we present results of computational experiments performed to verify the convergence of some asymptotic results discussed in the previous sections. We used a set of randomly generated instances of the MAP, where the costs were drawn from random distributions such as the exponential, uniform, and normal. These values were generated in the following way.

Values from the uniform distribution ($U[0, 1]$) were generated using the standard `rand` function form the C library (in the Windows platform). The next distribution used was the exponential with mean one, being determined by $X \sim \text{EXP}(1) = -\ln U$, where $U \sim U[0, 1]$. Finally, the third distribution used was the standard normal, $N(0, 1)$, with values $X \sim N(0, 1)$ determined by the polar method [18] as shown in Algorithm 1.

Algorithm 1: Generate a r.v. $X \sim N(0, 1)$ by the polar method.

    Output: a random variable $X \sim N(0, 1)$
    $W \leftarrow 2$
    while $W > 1$ **do**
        Generate U_1 and U_2, for $U_1, U_2 \sim U[0, 1]$; $V_1 \leftarrow 2U_1 - 1$
        $V_2 \leftarrow 2U_2 - 1$
        $W \leftarrow V_1^2 + V_2^2$
    end
    $X = V_1 \sqrt{-2(\ln W)/W}$.

5.1 Expected Value of Optimal Solutions

When considering the expected value of optimal solutions, it would be interesting to determine how fast these values converge to the limit, determined on Section 3.

In order to find this rate of convergence in practice, we performed a number of experiments with random instances, generated as described above. For each of the random distributions considered, we run an exact algorithm for the MAP,

applied to varying parameters d and n. The algorithm employed for solving exactly the MAP is a branch-and-bound, which uses some ideas from the index tree representation discussed in Section 3, and is fully described in [34].

The results found after running a large number of iterations of the algorithm are summarized in Figures 1 to 4. These figures show how the costs of optimal solutions for the MAP present a fast rate of convergence, even for small values of n and d. Figure 1 shows a comparison of convergence between the cases where $d = 3$ and $d = 5$. The curves show that the costs follow a fast decreasing tendency, approaching the value zero when n increases. Also, it can be observed that when the dimension increases to 5, the rate of convergence is still greater, with the objective value being nearly equal to zero for $n = 12$. The figure also shows the convergence of standard deviation. Again, the trend of decreasing standard deviation occurs faster when the dimension increases from 3 to 5.

Figure 2 displays a similar outcome when the assignment costs are normal distributed variables. However, in this case the objective costs goes to minus infinity. Here, for the smaller dimension $d = 3$ the convergence of the standard deviation is not so pronounced, but it appears more clearly when $d = 5$.

Figure 3 is a summary of these observations, with the mean optimal cost being plotted for a varying number of elements n and dimensions d. Finally, Figure 4 shows a direct comparison of convergence for different values of d when n increases. Notice that some values could not be computed for large combinations of dimension–number of elements, due to the high computational complexity involved.

5.2   Expected Number of Local Minima

In order to evaluate some of our results concerning the expected number of local minima, tests have been performed to determine this number for some random generated instances. In this case, however, a numerical verification of the results is much more difficult from the computational point of view. To determine the number of local minima one needs to completely enumerate all solutions, which is a daunting task even for small problems.

Instead of completely enumerating solutions, we decided to run a local search algorithm to probe for local minima in the given instances. The procedure consists of starting at a random solution and execute iterations of the local search algorithm until a local minimum is found. After a large number of iterations, we take the ratio between the number of solutions explored and the number of local minima found. As we increase the number of iterations, this value should approach the
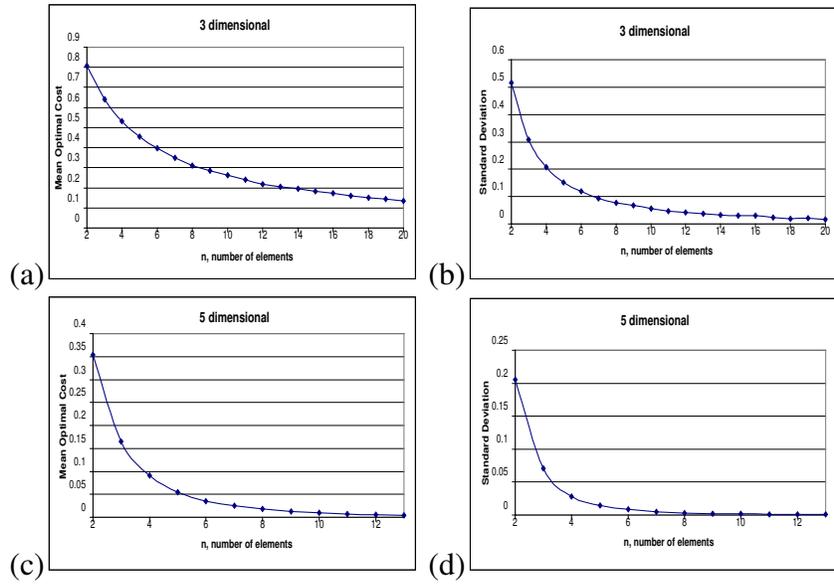
17

Figure 1: Convergence of optimal solution value for MAP instances with $d = 3$ (a and b), and $d = 5$ (c and d), and costs drawn from EXP(1). The solution values are show in (a) and (c), while (b) and (d) indicate the convergence of the standard deviation.
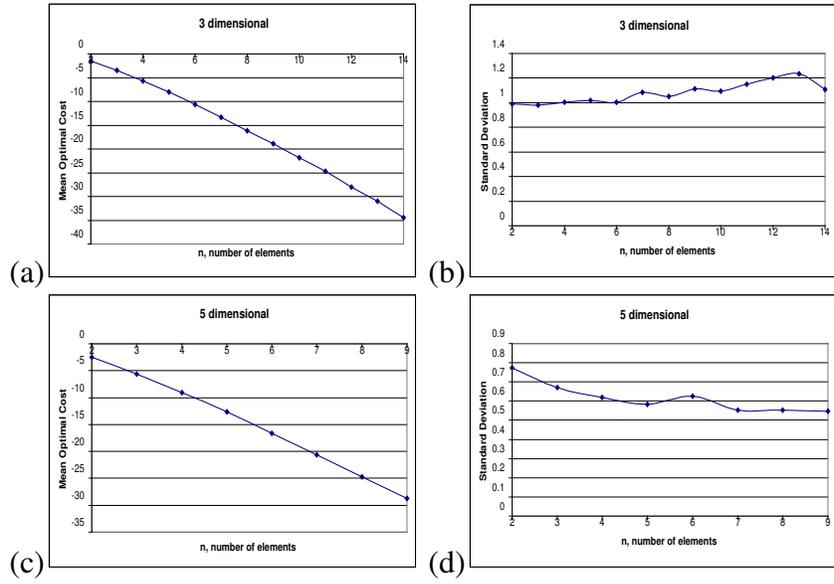
18

Figure 2: Convergence of optimal solution value for MAP instances with $d = 3$ (a and b), and $d = 5$ (c and d), and costs drawn from Normal(0,1). The solution values are show in (a) and (c), while (b) and (d) indicate the convergence of the standard deviation.
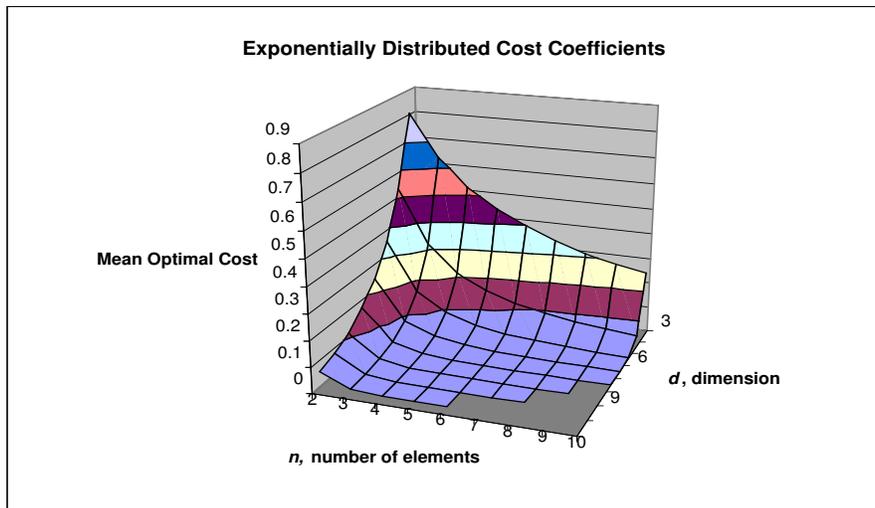

Figure 3: General convergence of optimal solution value for the MAP.
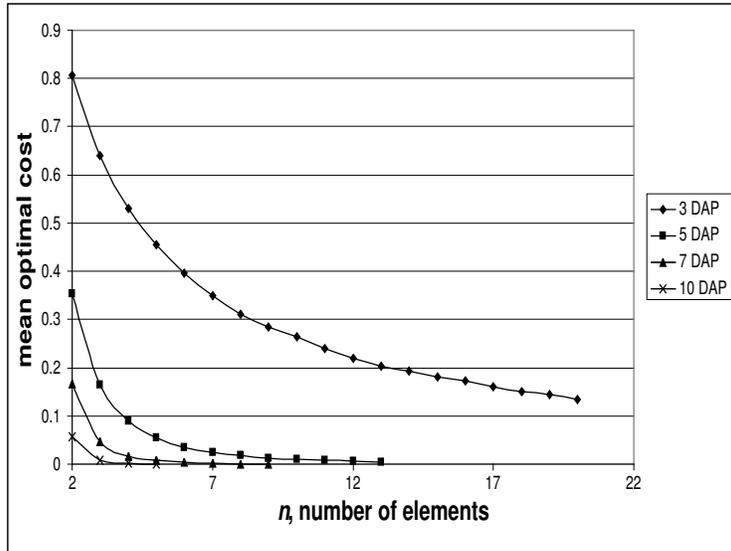
19

Figure 4: Comparison of convergence of optimal solution value for different values of d.

true ratio of solutions to local minima.

Some computational results of our search procedure are given in Tables 1–4. Table 1 gives the average number of local minima for the 2-exchange neighborhood. Here, the assignment costs are distributed according to a Normal(0,1) distribution, as discussed in Section 4. We note that these values agree with the formulas derived in that section, when numerical methods are used to compute the number of local minima. Table 2 gives a summary of ratios between the number of local minima and the number of feasible solutions, when applying a 2-exchange neighborhood. Similar results are shown in Tables 3 and 4, when a 3-exchange neighborhood is used instead.

6 Concluding Remarks

In this paper, some results obtained by the authors on the average case behavior of the MAP are presented. Initially, a quick review of the existing literature related to asymptotic features of combinatorial optimization problems is shown.

Then, we survey some of the recent results obtained by the authors on the

$n \backslash d$	3	4	5	6
2	1.00003	1.59812	2.66971	4.56438
3	1.75544	7.05994	30.7258	164.13
4	4.6201	62.2495	952.73	22383.7
5	16.2187	938.5	65048.8	4.75E+06
6	77.6563	23589.3	7.90E+06	3.48E+09

Table 1: Average number of local minima for 2-exchange neighborhood. Costs are distributed as Normal(0,1).

$n \backslash d$	3	4	5	6
2	0.250008	0.199765	0.166857	0.142637
3	0.048762	0.032685	0.023708	0.021107
4	0.008021	0.004503	0.002872	0.002811
5	0.001126	0.000543	0.000314	0.000191
6	0.00015	6.32E-05	2.94E-05	1.8E-05

Table 2: Ratio of local minima to number of feasible solutions for a 2-exchange neighborhood. Costs are distributed as Normal(0,1).

$n \backslash d$	3	4	5	6
3	1.52222	6.20525	26.617	122.597
4	3.22272	43.3244	672.444	11179
5	9.216	516.326	34767.4	2.65E+06
6	30.793	8709.12	3.06E+06	–

Table 3: Average number of local minima for 3-exchange neighborhood. Costs are distributed as Normal(0,1).

$n \backslash d$	3	4	5	6
3	0.042284	0.028728	0.020538	0.015766
4	0.005595	0.003134	0.002027	0.001404
5	0.00064	0.000299	0.000168	0.000106
6	5.94E-05	2.33E-05	1.14E-05	–

Table 4: Ratio of local minima to number of feasible solutions for a 3-exchange neighborhood. Costs are distributed as Normal(0,1).

average case behavior of optimal solution costs for the MAP. The conclusion is that, when the size of the MAP instances approach ∞, if the assignment costs are exponential or uniform$(0, 1)$ random variables, then the optimum value approaches zero. The optimum value approaches $-\infty$ when the assignment costs are normal$(0, 1)$ random variables.

We presented novel results on the asymptotic number of local minima occurring in random MAP instances, when $n = 2$, and when costs are taken from the normal distribution. We described techniques to calculate the number of local minima, which include computing the covariance matrix for the random variables representing the costs of 2-exchanges in a local search algorithm.

These studies provide an insight into the problem intrinsic computational difficulty and can be used to understand the performance of certain algorithms in practice for different problem data. Future work on this area include, for example, the determination of improved methods for computing local minima for the MAP. The methods discussed in this paper could also be extended to related combinatorial problems such as the linear ordering problem.

References

[1] D. Aldous. Asymptotics in the random assignment problem. *Probability Theory and Related Fields*, 93:507–534, 1992.

[2] D. Aldous and J.M. Steele. The objective method: Probabilistic combinatorial optimization and local weak convergence. In H. Kesten, editor, *Discrete and Combinatorial Probability*. Springer-Verlag, 2003.

[3] N. Alon and J. Spencer. *The Probabilistic Method*. Interscience Series in Discrete Mathematics and Optimization. John Wiley, second edition, 2000.

[4] J. Beardwood, J.H. Halton, and J.M. Hammersley. The shortest path through many points. *Proc. Cambridge Philosophical Society*, 55:299–327, 1959.

[5] B. Bollobás, D. Gamarnik, O. Riordan, and B. Sudakov. On the value of a random minimum weight steiner tree. To appear in Combinatorica, 2002.

[6] R. Burkard and U. Fincke. Probabilistic asymptotic properties of some combinatorial optimization problems. *Discrete Appl. Math.*, 12, 1985.

[7] B.H. Chiarini, W. Chaovalitwongse, and P. Pardalos. A new algorithm for the triangulation of input-output tables in economics. In P.M. Pardalos, Athanasios Migdalas, and George Baourakis, editors, *Supply Chain and Finance*. World Scientific Publishers, 2004.

[8] D. Coppersmith and G. Sorkin. Constructive bounds and exact expectations for the random assignment problem. *Random Structures and Algorithms*, 15:133–144, 1999.

[9] A.M. Frieze. On the value of a random minimum spanning tree problem. *Discrete Appl. Math.*, 10:47–56, 1985.

[10] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. W.H. Freeman and Company, 1979.

[11] D.A. Grundel, C.A.S. Oliveira, and P.M. Pardalos. Asymptotic properties of random multidimensional assignment problems. *Journal of Optimization Theory and Applications*, 122(3), 2004.

[12] R. Horst, P.M. Pardalos, and N.V. Thoai. *Introduction to Global Optimization*. Kluwer Academic Publishers, Dordrecht, 2nd edition, 2000.

[13] R.M. Karp. The probabilistic analysis of some combinatorial search algorithms. In J.F. Taub, editor, *Algorithms and Complexity: New Directions and Recent Results*, pages 1–19. Academic Press, New York, 1976.

[14] R.M. Karp. An upper bound on the expected cost of an optimal assignment. In D. Johnson et al., editors, *Discrete Algorithms and Complexity: Proceedings of the Japan-US Joint Seminar*, pages 1–4. Academic Press, New York, 1987.

[15] T. Kirubarajan, Y. Bar-Shalom, and K.R. Pattipati. Multiassignment for tracking a large number of overlapping objects. *IEEE Trans. Aerospace and Electronic Systems*, 37(1):2–21, 2001.

[16] T. C. Koopmans and M. J. Berkmann. Assignment problems and the location of economic activities. *Econometrica*, 25:53–76, 1957.

[17] H.W. Kuhn. The hungarian method for the assignment and transportation problems. *Naval Research Logistics Quarterly*, 2:83–97, 1955.

[18] A. M. Law and W. D. Kelton. *Simulation Modeling and Analysis*. McGraw-Hill, Inc., New York, 2nd edition, 1991.

[19] A.J. Lazarus. The assignment problem with uniform $(0, 1)$ cost matrix. Master's thesis, Dept. of Mathematics, Princeton University, 1979.

[20] S. Lin and B.W. Kernighan. An effective heuristic algorithm for the traveling salesman problem. *Operations Research*, 21:498–516, 1973.

[21] S. Linusson and J. Wästlund. A proof of Parisi's conjecture on the random assignment problem. Available at `http://www.mai.liu.se/~jowas/`, 2003.

[22] G.S. Lueker. Optimization problems on graphs with independent random edge weights. *SIAM Journal of Computing*, 10(2):338–351, 1981.

[23] M. Mézard and G. Parisi. Replicas and optimization. *J Phys Letters*, 46:771–778, 1985.

[24] M. Mézard and G. Parisi. On the solution of the random link matching problems. *J Phys Letters*, 48:1451–1459, 1987.

[25] C. Nair, B. Prabhakar, and M. Sharma. A proof of the conjecture due to parisi for the finite random assignment problem. Available at `http://www.stanford.edu/~balaji/rap.html`, 2003.

[26] B. Olin. *Asymptotic properties of random assignment problems*. PhD thesis, Kungl Tekniska Högskolan, Stockholm, Sweden, 1992.

[27] C.A.S. Oliveira and P.M. Pardalos. Randomized parallel algorithms for the multidimensional assignment problem. *Applied Numerical Mathematics*, 49:117–133, 2004.

[28] C.H. Papadimitriou and K. Steiglitz. *Combinatorial Optimization: Algorithms & Complexity*. Dover Publications, Inc., 1998.

[29] P.M. Pardalos, R.A. Murphey, and L. Pitsoulis. A greed randomized adaptive search procedure for the multitarget multisensor tracking problem. In P.M. Pardalos and Ding-Zhu Du, editors, *Network design: connectivity and facility location*, volume 40 of *DIMACS Series on Discrete Mathematics and Theoretical Computer Science*, pages 277–302. American Mathematical Society, 1997.

[30] P.M. Pardalos and L. Pitsoulis, editors. *Nonlinear Assignment: Problems, Algorithms and Applications*. Kluwer, Dordrecht, 2000.

[31] P.M. Pardalos and K.G. Ramakrishnan. On the expected optimal value of random assignment problems: Experimental results and open questions. *Computational Optimization and Applications*, 2:261–271, 1993.

[32] P.M. Pardalos and H. Wolkowicz, editors. *Quadratic Assignment and Related Problems*, volume 16 of *DIMACS Series*. American Mathematical Society, 1994.

[33] G. Parisi. A conjecture on random bipartite matching. on Physics e-Print archive, `http://xxx.lanl.gov/ps/cond-mat/9801176`, January 1998.

[34] E. Pasiliao. *Algorithms for Multidimensional Assignment Problems*. PhD thesis, Dept. of Industrial and Systems Engineering, University of Florida, 2003.

[35] J.K. Patel and B.C. Read. *Handbook of the Normal Distribution*. Dekker, New York, 1982.

[36] W.P. Pierskalla. The multidimensional assignment problem. *Operations Research*, 16:422–431, 1968.

[37] A. B. Poore, N. Rijavec, M. Liggins, and V. Vannicola. Data association problems posed as multidimensional assignment problems: problem formulation. In O. E. Drummond, editor, *Signal and Data Processing of Small Targets*, pages 552–561. SPIE, Bellingham, WA, 1993.

[38] J. Pusztaszeri, P.E. Rensing, and T.M. Liebling. Tracking elementary particles near their primary vertex: A combinatorial approach. *Journal of Global Optimization*, 9(1):41–64, 1996.

[39] J.M. Steele. Complete convergence of short paths and Karp's algorithm for the TSP. *Mathematics of Operations Research*, 6(3):374–378, 1981.

[40] J.M. Steele. *Probability Theory and Combinatorial Optimization*. CBMS-NSF regional conference series in applied mathematics. SIAM, 1997.

[41] J.M. Steele. Minimal spanning trees for graphs with random edge lengths. In B. Chauvin, D. Gardy Ph. Flajolet, and A. Mokkadem, editors, *Mathematics and Computer Science II: Algorithms, Trees, Combinatorics and Probabilities*. Birkhuser, Boston, 2002.

[42] D.W. Walkup. On the expected value of a random assignment problem. *SIAM Journal on Computing*, 8:440–442, 1979.