Perfect Recovery and Sensitivity Analysis of Time Encoded Bandlimited Signals *

Aurel A. Lazar[1] László T. Tóth[2]

[1]Department of Electrical Engineering
Columbia University, New York, NY 10027
aurel@ee.columbia.edu

[2]Department of Telecommunications and Media Informatics
Budapest University of Technology and Economics
Budapest, Hungary H-1117, toth@tmit.bme.hu

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Abstract

A Time Encoding Machine is a real-time asynchronous mechanism for encoding amplitude information into an increasing time sequence. We investigate the operating characteristics of a machine consisting of a feedback loop containing an adder, a linear filter and a noninverting Schmitt trigger. We show that the amplitude information of a bandlimited signal can be perfectly recovered if the difference between any two consecutive values of the time sequence is bounded by the inverse of the Nyquist rate. We also show how to build a non-linear inverse Time Decoding Machine that perfectly recovers the amplitude information from the time sequence.

The relationship between the recovery algorithms for time encoding and irregular sampling is described in the language of adjoint operators. This relationship further establishes time encoding as alternative information representation modality for bandlimited signals.

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We analyze the sensitivity of the time encoding recovery algorithm and demonstrate how to construct a Time Decoding Machine that perfectly recovers the amplitude information from the time sequence and is trigger parameter *insensitive*.

We derive bounds on the error in signal recovery introduced by the quantization of the time sequence. We compare these with the recovery error introduced by the quantization of the amplitude of the bandlimited signal when irregular sampling is employed. Under Nyquist-type rate conditions, quantization of a bandlimited signal in the time and amplitude domains are shown to be largely equivalent methods of information representation.

1 Introduction

A fundamental question arising in information processing is how to represent a signal as a discrete sequence. The classical sampling theorem ([6], [10]) calls for representing a bandlimited signal based on its samples taken at or above the Nyquist rate.

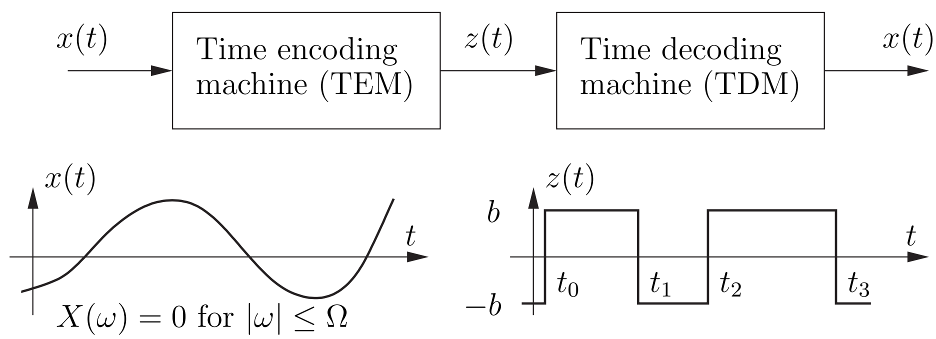

Figure 1: Time Encoding and Decoding.

A time encoding of a bandlimited function $x(t), t \in \mathbb{R}$, is a representation of $x(t)$ as a sequence of strictly increasing times $(t_k), k \in \mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers, respectively (see Figure 1). Alternately, the output of the encoder is a digital signal $z(t)$ that switches between two values $\pm b$ at times $t_k, k \in \mathbb{Z}$. Time encoding is an alternative to classical sampling and applications abound. In the field of neuroscience the representation of sensory information as a sequence of action potentials can be modeled as temporal encoding. The existence of a such a code was already postulated in [1]. Time encoding is also of great interest for the design and implementation of future analog to digital converters. Due to the ever decreasing size of integrated circuits and the attendant low voltage, high precision quantizers are more and more difficult to implement. These circuits provide increasing timing resolution, however, that a temporal code can take advantage of [9].

There are two natural requirements that a time encoding mechanism has to satisfy. The first is that the encoding should be implemented as a *real-time asynchronous* circuit. Secondly, the encoding mechanism should be *invertible*, that is the amplitude information can be recovered from the time sequence with arbitrary accuracy.

The encoding mechanism investigated here satisfies both of these conditions. We show that a Time Encoding Machine (TEM) consisting of a feedback loop that contains an adder, a linear filter and a noninverting Schmitt trigger has the required properties. We show that the amplitude information of a bandlimited signal can be perfectly recovered if the difference between any two consecutive values of the time sequence is bounded by the inverse of the Nyquist rate. We also show how to build a non-linear inverse Time Decoding Machine (TDM) (see Figure 1) that perfectly recovers the amplitude information from the time sequence. The relationship between the recovery algorithms for time encoding and irregular sampling is described in the language of adjoint operators.

We analyze the sensitivity of the recovery algorithm with respect to the precise knowledge of the parameters of the TEM. Based on a simple compensation principle we provide a perfect recovery algorithm that is insensitive with respect to these parameters.

Finally, we evaluate the error introduced by the quantization of the time sequence and derive bounds on the recovery error. We compare these with the recovery error introduced by quantization of the amplitude of an arbitrary bandlimited signal when irregular sampling is employed. Under Nyquist-type rate conditions, quantization of a bandlimited signal in the time and amplitude domain are shown to be largely equivalent methods of information representation.

This paper is organized as follows. In section 2 a method of mapping amplitude information into a time sequence is presented. An example of a TEM is given and its stability analyzed. In order to simplify the analysis an equivalent circuit that describes the key elements of the TEM is introduced. This circuit is used throughout the rest of the paper. Section 3 derives the recovery algorithm and presents the relationship between the irregular sampling and time encoding. Section 4 investigates the sensitivity of the recovery algorithm with respect to the parameters of the TEM. A Compensation Principle is derived and subsequently used to build a key parameter insensitive recovery algorithm. Finally, in section 5 the effects of quantization of a bandlimited signal in the time or amplitude domain are compared.

2 Time Encoding

The TEM investigated in this paper is depicted in Figure 2. The filter is assumed to be an integrator. Clearly the amplitude information at the input of the TEM is represented as a time sequence at its output.

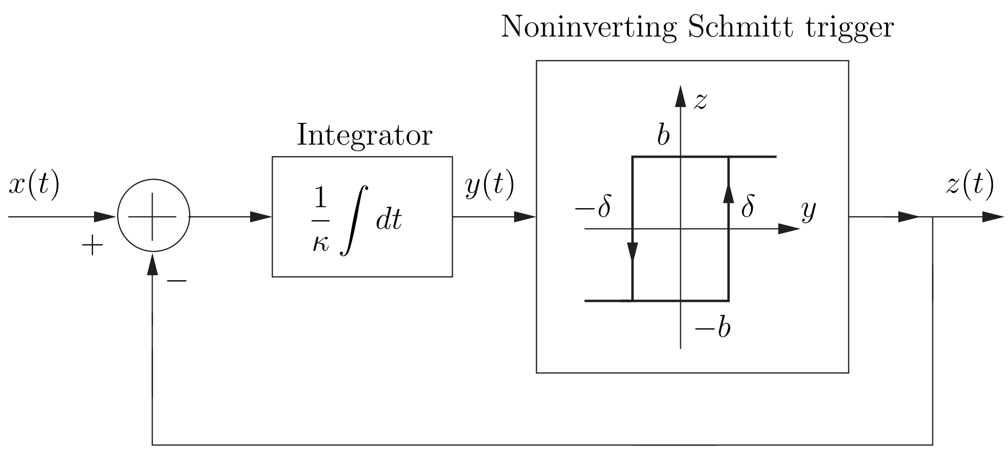

Figure 2: An Example of a Time Encoding Machine

The basic principle of operation of the TEM is very simple. The bounded input signal $x(t)$, $|x(t)| \le c < b$, is biased by a constant amount $+(-)b$ before being applied to the integrator. This bias guarantees that the integrator's output $y(t)$ is a positive (negative) increasing (decreasing) function of time. In steady state, there are two possible operating modes. In the first mode, the output of the TEM is in state $z(t) = -b$ and the input to the Schmitt trigger grows from $-\delta$ to δ. When the output of the integrator reaches the maximum value δ, a transition of the output $z(t)$ from $-b$ to $+b$ is triggered and the feedback becomes negative. In the second mode of operation, the TEM is in state $z(t) = b$ and the integrator output steadily decreases from

δ to $-\delta$. When the maximum negative value $-\delta$ is reached $z(t)$ will reverse to $-b$. Thus, while the transition times of the output $z(t)$ are non-uniformly spaced, the amplitude of the output signal remains constant. Therefore, a transition of the output from $-b$ to b or vice-versa takes place every time the integrator output reaches the triggering mark δ or $-\delta$ (called *quanta*). The time when this quanta is achieved depends on the signal as well as on the design parameters κ, δ and b. Hence, the TEM maps amplitude information into timing information. It achieves this by a *signal-dependent* sampling mechanism.

In the next two sections we discuss the conditions for stability of the TEM and introduce an equivalent circuit that describes the key elements of the TEM.

2.1 Stability

In Figure 2, κ, δ, b are strictly positive real numbers and $x = x(t)$ is a Lebesgues measurable function that models the input signal to the TEM for all $t,\ t \in \mathbb{R}$. The output of the integrator in a small neighborhood of t_0, $t > t_0$ is given by:

$$y(t) = y(t_0) + \frac{1}{\kappa}\int_{t_0}^{t}[x(u) - z(u)]du. \tag{1}$$

Note that $y = y(t)$ is a continous increasing (decreasing) function whenever the value of the feedback is positive (negative). Here, $z : \mathbb{R} \to \{-b, b\}$ for all $t, t \in \mathbb{R}$, is the function corresponding to the output of the TEM in Figure 2. z switches between two values $+b$ and $-b$ at a set of trigger times (t_k), for all $k,\ k \in \mathbb{Z}$, and $z(t_0) = -b$ by convention.

Remark 1 Informally, the information of the input $x(t)$ is carried by the signal amplitude whereas the information of the output signal $z(t)$ is carried by the trigger times. A fundamental question, therefore, is whether the TEM encodes information loss-free. Loss-free encoding means that $x(t)$ can be perfectly recovered from $z(t)$.

In what follows $1_{[t_k,t_{k+1})}(t)$, $t \in \mathbb{R}$, denotes a pulse of length $t_{k+1} - t_k$ and unit amplitude.

Lemma 1 (Stability) *For all input signals $x = x(t),\ t \in \mathbb{R}$, with $|x(t)| \leq c < b$ the Time Encoding Machine is stable, i.e., $|y(t)| \leq \delta$, for all $t,\ t \in \mathbb{R}$.*

The output z is given by $z(t) = (-1)^{k+1}\ b\ 1_{[t_k,t_{k+1})}(t)$, $t \in \mathbb{R}$, where the set of trigger times (t_k), $k \in \mathbb{Z}$, is generated by the recursive equation

$$\int_{t_k}^{t_{k+1}} x(u)du = (-1)^k[-b(t_{k+1} - t_k) + 2\kappa\delta]. \tag{2}$$

for all k, $k \in \mathbb{Z}$.

Proof: Due to the operating characteristic of the Schmitt trigger, y reaches the value δ if the feedback is b or the value $-\delta$ if the feedback is $-b$ for any arbitrary initial value of the integrator. Therefore, without loss of generality we can assume that for some initial condition $t = t_0$ we have $(y, z) = (-\delta, -b)$ and the TEM is described in a small neighborhood of t_0, $t > t_0$, by:

$$-\delta + \frac{1}{\kappa}\int_{t_0}^{t}[x(u) + b]du = \delta. \tag{3}$$

Since the left hand side is a continuously increasig function, there exists a time $t = t_1$, $t_0 < t_1$, such that the equation above holds. Similarly starting with $(y, z) = (\delta, b)$ at time t_1 the equation:

$$\delta + \frac{1}{\kappa}\int_{t_1}^{t}[x(u) - b]du = -\delta, \tag{4}$$

is satisfied for some $t = t_2$, $t_1 < t_2$. Thus, the strictly increasing sequence (t_k), $k \in \mathbb{Z}$, defined by the equations (2) uniquely describes the (output) function $z = z(t)$, for all t, $t \in \mathbb{R}$, and $|y| \leq \delta$ by construction.

2.2 An Equivalent Circuit

By dividing with b on both sides of equation (2) we obtain:

$$\int_{t_k}^{t_{k+1}} \frac{x(u)}{b}du = (-1)^k[-(t_{k+1} - t_k) + \frac{2\kappa\delta}{b}]. \tag{5}$$

for all k, $k \in \mathbb{Z}$. Therefore, the increasing time sequence (t_k), $k \in \mathbb{Z}$, can be generated by an equivalent circuit with integration constant $\kappa = 1$ and a Schmidt trigger with parameters $\kappa\delta/b$ and 1 (see Figure 3).

In what follows, without any loss of generality, a simple version of the TEM will be used. The input to the TEM is a bounded Lebesgues measurable

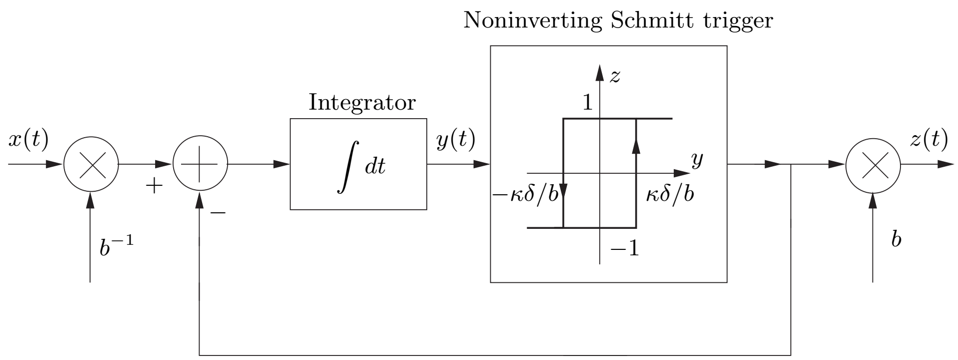

Figure 3: Equivalent Circuit Representation of the TEM

function $x(t)$ with $|x(t)| \leq c < 1$ for all $t, t \in \mathbb{R}$. The output of the TEM is function z taking two values $z : \mathbb{R} \to \{-1, 1\}$ for all $t, t \in \mathbb{R}$, with transition times (t_k), $k \in \mathbb{Z}$, generated by the recursive equations

$$\int_{t_k}^{t_{k+1}} x(u)du = (-1)^k[-(t_{k+1} - t_k) + \delta], \tag{6}$$

for all k, $k \in \mathbb{Z}$. These equations map the amplitude information of the signal $x(t)$, $t \in \mathbb{R}$, into the time sequence (t_k), $k \in \mathbb{Z}$. Equation (6) is an instantiation of the *t-transform*. In what follows the TEM consists of an integrator with integrator constant $\kappa = 1$ and Schmitt trigger with parameters $(\delta/2, 1)$.

Example 1 Assume that $x(t) = c$ (not necessarily positive), where c denotes a given DC level. For $c < -1$ or $c > 1$ the output of the integrator becomes unbounded, and thus the overall TEM becomes unstable. This might lead to information loss because the output $z(t)$ can not track the input $x(t)$. If $|c| < 1$, the TEM is stable and equation (6) reduces to the simple recursion

$$t_{k+1} = t_k + \frac{\delta}{1 + (-1)^k c},$$

and, therefore, $t_{k+1} > t_k$, $k \in \mathbb{Z}$. Both $y(t)$ and $z(t)$ are periodic signals with period T and

$$T = t_{k+2} - t_k = \frac{2\delta}{1 - c^2}. \tag{7}$$

for all k, $k \in \mathbb{Z}$. The integrator output, $y(t)$, and the overall output, $z(t)$, are as shown in Fig. 4.

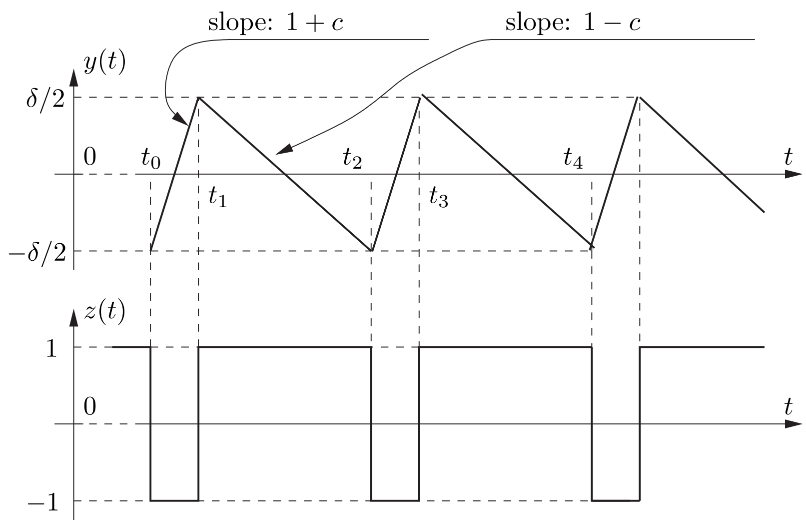

Figure 4: Time-domain illustration of the operation of the TEM with DC input.

The mean value (the 0-th order Fourier-series coefficient) of $z(t)$ amounts to:

$$\frac{(-1)^k[-(t_{k+1}-t_k)+(t_{k+2}-t_{k+1})]}{T}=c, \tag{8}$$

that is, the input and the output of the TEM have the same average value.

Note that the harmonics of $z(t)$ can be easily separated from the value corresponding to the DC input. Its fundamental frequency, $1/T$, is inversely proportional to δ (the width of the hysteresis of the Schmitt trigger). Thus, if the maximum DC input signal is given, the minimum value for $1/T$ can be set by δ.

Corollary 1 (Upper and Lower Bounds for Trigger Times) *For all input signals $x = x(t)$, $t \in \mathbb{R}$, with $|x(t)| \le c < 1$, the distance between consecutive trigger times t_k and t_{k+1} is bounded by:*

$$\frac{\delta}{1+c} \le t_{k+1}-t_k \le \frac{\delta}{1-c}, \tag{9}$$

for all k, $k \in \mathbb{Z}$.

Proof: Since $|x(t)| \le c$, it is easy to see that

$$-c(t_{k+1}-t_k) \le \int_{t_k}^{t_{k+1}} x(u)du \le c(t_{k+1}-t_k). \tag{10}$$

By replacing the integral in the inequality above with its value given by equation (6) and solving for $t_{k+1} - t_k$ we obtain the desired result. The lower and upper bounds are achieved for a constant input $x(t) = c$, for all $t,\ t \in \mathbb{R}$, for k even and odd, respectively. A similar relation applies when $x(t) = -c$, for all $t,\ t \in \mathbb{R}$.

Remark 2 If $x(t)$ is a continuous function, by the mean value theorem there exists a $\xi_k \in [t_k, t_{k+1}]$, $k \in \mathbb{Z}$, such that:

$$x(\xi_k)(t_{k+1} - t_k) = (-1)^k[-(t_{k+1} - t_k) + \delta], \tag{11}$$

i.e., the sample $x(\xi_k)$ can be explicitly recovered from information contained in the process $z(t)$, $t_k \leq t \leq t_{k+1}$, $k \in \mathbb{Z}$. Intuitively, therefore, any class of input signals that can be recovered from its samples can also be recovered from $z(t)$.

3 Perfect Recovery

A TDM has the task of recovering the signal $x = x(t)$, $t \in \mathbb{R}$, from $z = z(t)$, $t \in \mathbb{R}$, or a noisy version of the same. Here we will focus on the recovery of the original signal x based on z only. We shall show that a perfect recovery is possible, that is, the input signal x can be recovered from z without any loss of information.

3.1 Recovery Algorithms

Informally, a function of the length of the interval between two consecutive trigger times of $z(t)$ provides an estimate of the integral of $x(t)$ on the same interval. This estimate used in conjunction with the bandlimited assumption on x enables a perfect reconstruction of the signal even though the trigger times are irregular. In order to achieve perfect reconstruction, the distance between two consecutive trigger times has to be smaller then the distance between the uniformly spaced samples in the classical sampling theorem [6], [10].

The mathematical methodology used here for deriving recovery algorithms is based on the theory of frames [3]. We shall construct aa operator on $L^2(\mathbb{R})$, the space of square integrable functions defined on $\mathbb{R}$, and by starting from a good initial guess followed by successive interations, obtain sucessive approximations that converge in the appropriate norm to the original signal x.

Let us assume that $x = x(t), t \in \mathbb{R}$, with $|x(t)| \leq c < 1$, is a finite energy signal on $\mathbb{R}$ bandlimited to $[-\Omega, \Omega]$ and let the operator $\mathcal{A}$ be given by:

$$\mathcal{A}x = \sum_{k \in \mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du \; g(t - s_k), \tag{12}$$

where $g(t) = sin(\Omega t)/\pi t$ and $s_k = (t_{k+1} + t_k)/2$.

The realization of the operator $\mathcal{A}$ above is highly intuitive. Dirac-delta pulses generated at times s_k with weight $\int_{t_k}^{t_{k+1}} x(u)du$ are passed through an ideal low pass filter with unity gain for $\omega \in [-\Omega, \Omega]$ and zero otherwise. The values of $(\int_{t_k}^{t_{k+1}} x(u)du)$, $k \in \mathbb{Z}$, can be obtained from the sequence (t_k), $t \in \mathbb{Z}$, available at the TDM, through equation (6).

Let $x_l = x_l(t)$, $t \in \mathbb{R}$, be a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{A}(x - x_l), \tag{13}$$

for all $l,\; l \in \mathbb{N}$, with the initial condition $x_0 = \mathcal{A}x$.

Note that since the distance between two consecutive trigger times is bounded by $\delta/(1-c)$ (see equation (9)),

$$\| I - \mathcal{A} \| \leq r, \tag{14}$$

where I is the identity operator and $r = \frac{\delta}{1-c}\frac{\Omega}{\pi}$ [4].

Theorem 1 (Operator Formulation) *Let $x = x(t)$, $t \in \mathbb{R}$, be a bounded signal $|x(t)| \leq c < 1$ bandlimited to $[-\Omega, \Omega]$. Let $z = z(t)$, $t \in \mathbb{R}$, be the ouput of a Time Encoding Machine with integrator constant $\kappa = 1$ and Schmitt trigger parameters $(\delta/2, 1)$. If $\delta < (1-c)\frac{\pi}{\Omega}$, the signal x can be perfectly recovered from z as*

$$\lim_{l \to \infty} x_l(t) = x(t), \tag{15}$$

and

$$\| x - x_l \| \leq r^{l+1} \| x \| . \tag{16}$$

Proof: By induction we can show that

$$x_l = \sum_{k=0}^{l} (I - \mathcal{A})^k \mathcal{A}x. \tag{17}$$

Since $\| I - \mathcal{A} \| \leq r < 1$,

$$\lim_{l \to \infty} x_l = \sum_{k \in \mathbb{N}} (I - \mathcal{A})^k \mathcal{A} x = \mathcal{A}^{-1} \mathcal{A} x = x. \tag{18}$$

Also,

$$\begin{aligned} x - x_l &= \sum_{k \geq l+1} (I - \mathcal{A})^k \mathcal{A} x = (I - \mathcal{A})^{k+1} \sum_{k \in \mathbb{N}} (I - \mathcal{A})^k \mathcal{A} x \\ &= (I - \mathcal{A})^{l+1} \mathcal{A}^{-1} \mathcal{A} x = (I - \mathcal{A})^{l+1} x, \end{aligned} \tag{19}$$

and, therefore, $\| x - x_l \| \leq r^{l+1} \| x \|$.

Let us define $\mathbf{g} = [g(t - s_k)]^T$, $\mathbf{q} = [\int_{t_k}^{t_{k+1}} x(u)\ du]$ and $\mathbf{G} = [\int_{t_l}^{t_{l+1}} g(u - s_k)\ du]$. We have the following

Corollary 2 (Matrix Formulation) *Under the assumptions of Theorem 1 the bandlimited signal x can be perfectly recovered from z as*

$$x(t) = \lim_{l \to \infty} x_l(t) = \mathbf{g} \mathbf{G}^+ \mathbf{q}. \tag{20}$$

where $\mathbf{G}^+$ denotes the pseudo-inverse of $\mathbf{G}$. Furthermore,

$$x_l(t) = \mathbf{g} \mathbf{P}_l \mathbf{q}, \tag{21}$$

where $\mathbf{P}_l$ is given by

$$\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{G})^k. \tag{22}$$

Proof: By induction:

$$\begin{aligned} x_0(t) = \mathcal{A} x &= \sum_{k \in \mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u) du\ g(t - s_k) \\ &= \mathbf{g} \mathbf{q}. \end{aligned} \tag{23}$$

Assume that $x_l(t) = \mathbf{g} \mathbf{P}_l \mathbf{q}$ with $\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{G})^k$. We have

$$\begin{aligned} x_{l+1}(t) &= x_l + \mathcal{A}(x - x_l) \\ &= \mathbf{g}(\mathbf{P}_l + \mathbf{I} - \mathbf{G} \mathbf{P}_l) \mathbf{q} = \mathbf{g} \mathbf{P}_{l+1} \mathbf{q}. \end{aligned} \tag{24}$$

Finally, the interchange of the limit and infinite summation operators in

$$x(t) = \lim_{l\to\infty} x_l(t) = \lim_{l\to\infty} \mathbf{g}\mathbf{P}_l\mathbf{q} = \mathbf{g}\mathbf{G}^+\mathbf{q}. \tag{25}$$

is guaranteed by Theorem 1. The quanta $\mathbf{q}$ can be explicitly derived from $z = z(t)$ since

$$\mathbf{q} = [\int_{t_k}^{t_{k+1}} z(u)du + (-1)^k\delta]. \tag{26}$$

Remark 3 If $\mathbf{c} = [c_k]$ is the vector defined by $\mathbf{c} = \mathbf{G}^+\mathbf{q}$ then the recovery formula (20) becomes

$$x(t) = \sum_{k\in\mathbb{Z}} c_k g(t - s_k). \tag{27}$$

Therefore, the recovery algorithm given by equation (21) has a very simple interpretation. Dirac–delta pulses generated at times s_k with weight c_k are passed through a low pass filter with unity gain for $\omega \in [-\Omega, \Omega]$ and zero otherwise.

Remark 4 While deceptively simple, the signal recovery formula exposed in equation (20) hides the non-linear relationship between the bandlimited signal $x = x(t), t \in \mathbb{R}$, and the trigger times $(t_k), k \in \mathbb{Z}$. Clearly linear operations on the TEM input signal do not translate into linear operations on the (output) trigger times.

3.2 Example

The mathematical formulation of the previous section assumes that the dimensionality of the matrices and vectors used is infinite. In simulations, however, only a finite time window can be used. We briefly investigate two different implementations of the TDM in the finite dimensional case that are, respectively, based on the recursive equation (21) and the closed form formula (20).

In all our simulations, the input signal is given by $x(t) = \sum_{k\in\mathbb{Z}} x(kT)g(t-kT)$ where the samples $x(T)$ through $x(12T)$, are respectively, -0.1961, 0.186965, 0.207271, 0.0987736, -0.275572, 0.0201665, 0.290247, 0.138374, -0.067588, -0.145661, -0.11133, -0.291498, $x(kT) = 0$, for $k \leq 0$ and $k > 12$; $c = 1$, $\Omega = 2\pi \cdot 40$ kHz and $T = \pi/\Omega = 12.5$ μs. The evaluation of the trigger times was carried out in the interval $-2T \leq t \leq 15T$. Fig. 5(a) shows $x(t)$ together

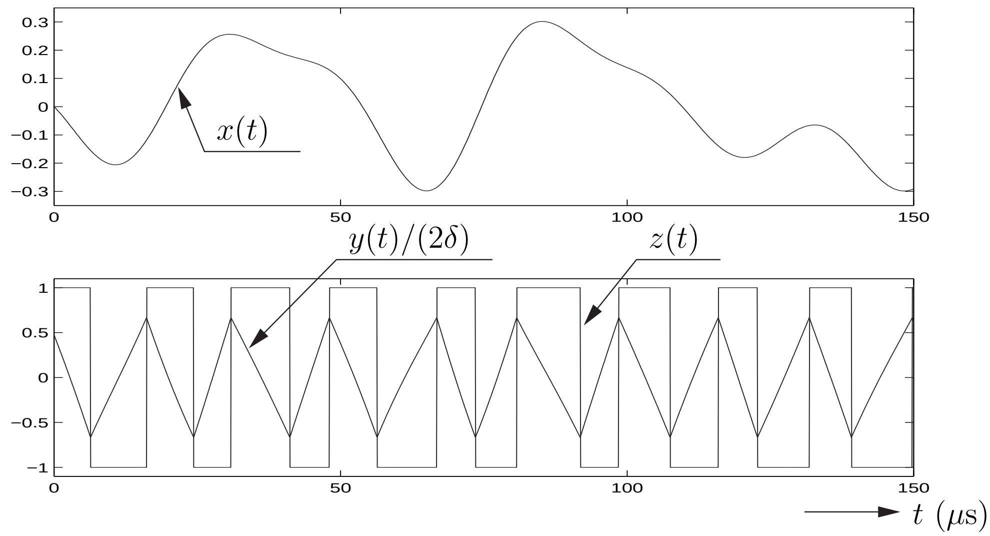

Figure 5: Input signal x(t) (a), integrator and TEM output y(t) and z(t) (b).

with the time window used for simulations. Fig. 5(b) shows the simulation results for $y(t)$ and $z(t)$ with $\delta = 8\mu$s. The 26 trigger times of $z(t)$ (only 18 are shown) were determined with high accuracy using equation (6).

(i) The error signals shown by Fig. 6(a) are defined as $e_l = e_l(t) = x_l(t) - x(t)$, where $x_l(t)$ was calculated based on (21). Instead of applying (22) directly we used the recursion $\mathbf{P}_{l+1} = \mathbf{I} + \mathbf{P}_l(\mathbf{I} - \mathbf{G})$ and calculated $x_l(t)$ iteratively. As shown, $e_l(t)$ decreases in agreement with Theorem 1, since with the parameters introduced $r = 0.914 < 1$.

(ii) Although the matrix $\mathbf{G}$ in (20) is singular, perfect recovery can be achieved using $\mathbf{G}^+$, the pseudo-inverse of $\mathbf{G}$ (if $\mathbf{G}$ is non-singular then $\mathbf{G}^+ = \mathbf{G}^{-1}$). The corresponding error signal defined as $e(t) = \mathbf{g}\mathbf{G}^+\mathbf{q} - x(t)$ is shown Figure 6(b) for $\delta = 8$ μs ($r = 0.914$ for the solid line) and $\delta = 6$ μs ($r = 0.616$ for the dashed line). The improvement is about 10 dB. The remaining small error is due to the finite precision used.

3.3 Relationship to Irregular Sampling

In this section we highlight the relationship between time encoding and irregular sampling, i.e., between two information representations of a bandlimited signal as a discrete time and a discrete amplitude sequence. As in the pre-

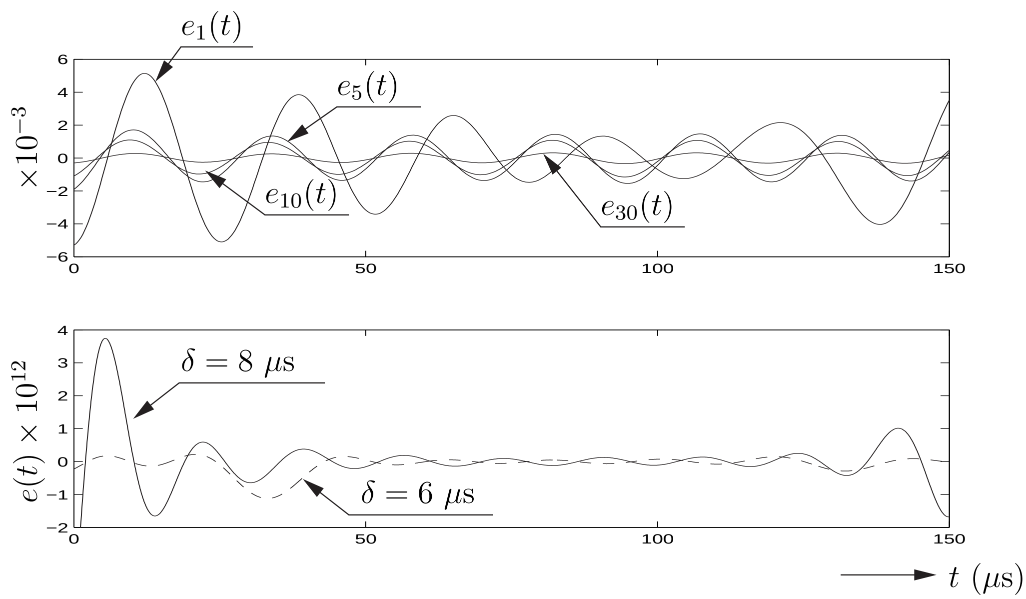

Figure 6: Approximating signals using iteration (a), overall error signals using closed formulas (b).

vious sections $x = x(t)$, $t \in \mathbb{R}$, shall denote a bounded signal $|x(t)| \leq c < 1$ bandlimited to $[-\Omega, \Omega]$. The time sequence will be denoted by $(t_k), k \in \mathbb{Z}$, and the irregular samples $(x(s_k))$, $k \in \mathbb{Z}$, are available at times (s_k), $k \in \mathbb{Z}$.

The operator $\mathcal{A}^*$ defined by

$$\mathcal{A}^* x = \sum_{k \in \mathbb{Z}} x(s_k) f_k(t), \tag{28}$$

where $f_k(t) = (g * 1_{[t_k, t_{k+1}]})(t)$ represents the output of a low pass filter with impulse response g (that is, the $*$ denotes the convolution operation) whose input is the pulse of finite width $1_{[t_k, t_{k+1}]}(t)$. We note the following [4]:

Lemma 2 *$\mathcal{A}$ and $\mathcal{A}^*$ are adjoint operators.*

Proof: $\mathcal{A}$ and $\mathcal{A}^*$ are adjoint if

$$< \mathcal{A}x, y >=< x, \mathcal{A}^* y > \tag{29}$$

for any bandlimited functions x and y, where $< \cdot, \cdot >$ denotes the inner product. Using the linearity properties of the inner product and the fact that $g * x = x$ we have

$$
\begin{aligned}
< \mathcal{A}x, y > &=< \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du \; g(t-s_k), y >= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} x(u)du \; y(s_k) \\
&=< x, \sum_{k\in\mathbb{Z}} 1_{[t_k,t_{k+1}]} \; y(s_k) >=< g * x, \sum_{k\in\mathbb{Z}} 1_{[t_k,t_{k+1}]} \; y(s_k) > \\
&=< x, \sum_{k\in\mathbb{Z}} g * 1_{[t_k,t_{k+1}]} \; y(s_k) >=< x, \mathcal{A}^* y > .
\end{aligned}
\tag{30}
$$

In Theorem 2 below, $x_l = x_l(t)$, $t \in \mathbb{R}$, will denote a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{A}^*(x - x_l), \tag{31}$$

for all $l,\ l \in \mathbb{N}$, with the initial condition $x_0 = \mathcal{A}^* x$. The relevance of $\mathcal{A}^*$ in our context is provided by the following theorem [4]:

Theorem 2 (Reconstruction from Irregular Samples I) *If* $r = \frac{\delta}{1-c} \cdot \frac{\Omega}{\pi} < 1$ *the bandlimited signal* x *can be perfectly recovered from its samples* $(x(s_k))$, $k \in \mathbb{Z}$, *as*

$$\lim_{l\to\infty} x_l(t) = x(t), \tag{32}$$

and

$$\| x - x_l \| \leq r^{l+1} \| x \| . \tag{33}$$

Proof: See [4], Theorem 5.

Let us define $\mathbf{f} = [f_k(t)]^T$, $\mathbf{x} = [x(s_k)]$ and $\mathbf{F} = [f_k(s_l)]$. We have the following

Corollary 3 (Matrix Recovery from Irregular Samples II) *Under the assumptions of Theorem 2 the bandlimited signal* x *can be perfectly recovered from its samples* $(x(s_k))$, $k \in \mathbb{Z}$, *as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{f}\mathbf{F}^+\mathbf{x}. \tag{34}$$

where $\mathbf{F}^+$ *denotes the pseudo-inverse of* $\mathbf{F}$. *Furthermore,*

$$x_l(t) = \mathbf{f}\mathbf{P}_l\mathbf{x}, \tag{35}$$

where $\mathbf{P}_l$ *is given by*

$$\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{F})^k. \tag{36}$$

Proof: The proof closely follows Corollary 2.

In what follows we shall use the operator $\mathcal{S}$ defined by

$$\mathcal{S}x = \frac{1}{1+r^2} \cdot \frac{\Omega}{\pi} \sum_{k\in\mathbb{Z}} T_k x(s_k) g(t - s_k), \tag{37}$$

where $T_k = t_{k+1} - t_k$, for all $k, k \in \mathbb{Z}$. In what follows, $x_l = x_l(t)$, $t \in \mathbb{R}$, will denote a sequence of bandlimited functions defined by the recursion:

$$x_{l+1} = x_l + \mathcal{S}(x - x_l), \tag{38}$$

for all $l,\ l \in \mathbb{N}$, with the initial condition $x_0 = \mathcal{S}x$. The relevance of $\mathcal{S}$ in our context is provided by the following theorem [4]:

Theorem 3 (Reconstruction from Irregular Samples II) *If* $r = \frac{\delta}{1-c} \cdot \frac{\Omega}{\pi} < 1$ *the bandlimited signal* x *can be perfectly recovered from its samples* $(x(s_k))$, $k \in \mathbb{Z}$, *as*

$$\lim_{l\to\infty} x_l(t) = x(t), \tag{39}$$

and

$$\| x - x_l \| \leq (\frac{2r}{1+r^2})^{l+1} \| x \| . \tag{40}$$

Proof: See [4], Theorem 6.

Let us define $\mathbf{g} = [g(t - s_k)]^T$, $\mathbf{p} = [T_k x(s_k)]$ and $\mathbf{H} = [T_l g(s_l - s_k)]$. We have the following

Corollary 4 (Matrix Recovery from Irregular Samples II) *Under the assumptions of Theorem 3 the bandlimited signal* x *can be perfectly recovered from its samples* $(x(s_k))$, $k \in \mathbb{Z}$, *as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{g}\mathbf{H}^+\mathbf{p}. \tag{41}$$

where $\mathbf{H}^+$ *denotes the pseudo-inverse of* $\mathbf{H}$. *Furthermore,*

$$x_l(t) = \mathbf{g}\mathbf{P}_l\mathbf{p}, \tag{42}$$

where $\mathbf{P}_l$ *is given by*

$$\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I} - \mathbf{H})^k. \tag{43}$$

Proof: The proof closely follows Corollary 2.

Remark 5 While we have highlighted in this section the similarities between time encoding and irregular sampling from the algorithmic recovery point of view, there are also substantial differences between the two. One key difference mentioned here derives from the functional relationship between the trigger times $(t_k), k \in \mathbb{Z}$, and the associated time sequence $(s_k), k \in \mathbb{Z}$, on the one hand and the bandlimited signal on the other. In the case of time encoding, the t_k's are signal dependent. This is clearly underscored by equation (6). For irregular sampling, however, the s_k's are *signal independent*.

4 Recovery Sensitivity with Respect to δ

In this section we will first demonstrate the high sensitivity of the perfect recovery algorithm with respect to implementation errors of the parameter δ in the TDM. We will then demonstrate how this can be overcome and advance an δ-insensitive recovery algorithm.

We would like to note here that sensitivity issues also arise in the TEM. For example, the Schmitt trigger's implementation might assign the value $1 + \varepsilon$ to the upper threshold and $-1 + \varepsilon$ to the lower threshold. Therefore, the recursive equation describing the TEM becomes

$$\int_{t_k}^{t_{k+1}} [x(u) - \varepsilon] du = (-1)^k [-(t_{k+1} - t_k) + \delta], \tag{44}$$

for all $k,\ k \in \mathbb{Z}$. As a result, $x(t) - \varepsilon$ is recovered instead of $x(t)$. A small DC bias ε is often times acceptable in practice.

4.1 δ with a fixed error ε at the TDM

The model considered in this section is based on the premise that the TEM is employing δ and the TDM implements $\delta + \varepsilon$ and has exact knowledge of the trigger times. The reconstruction algorithm consistently generates an error signal e given by:

$$e(t) = x(t) - \hat{x}(t) = \sum_{k \in \mathbb{N}} (I - \mathcal{A})^k \varepsilon \sum_{l \in \mathbb{Z}} g(t - s_l). \tag{45}$$

In what follows we define a mean-square error measure $\mathcal{E}^2$ as

$$\mathcal{E}^2 = \lim_{n \to \infty} \frac{1}{2nT_{min}} \parallel e1_{[-nT_{min}, nT_{min}]} \parallel^2, \tag{46}$$

where

$$\parallel e1_{[-nT_{min}, nT_{min}]} \parallel^2 = \int_{\mathbb{R}} e^2(u) 1_{[-nT_{min}, nT_{min}]}(u) du, \tag{47}$$

and $T_{min} = \min_{k \in \mathbb{Z}} T_k$.

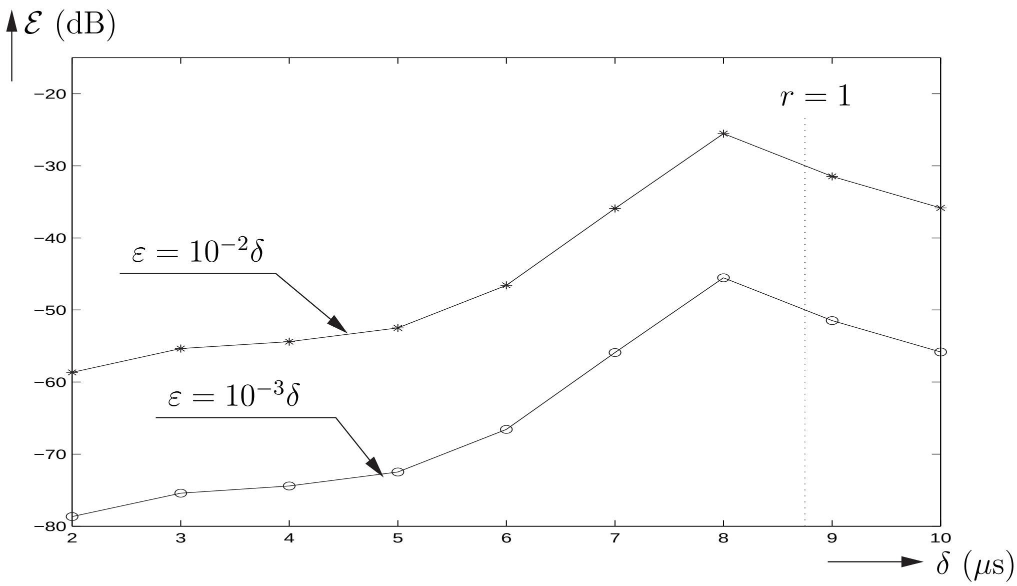

Figure 7: The dependence of $\mathcal{E}$ on δ parameterized by $\varepsilon = 10^{-2}\delta$ (stars) and $\varepsilon = 10^{-3}\delta$ (circles).

Proposition 1 *The mean-square error is proportional with* ε^2.

Proof: We note that

$$
\begin{aligned}
\mathcal{E}^2 &= \lim_{n\to\infty} \frac{1}{2nT_{min}} \parallel \sum_{k\in\mathbb{N}} (I-\mathcal{A})^k \varepsilon \sum_{l\in\mathbb{Z}} g(t-s_l)\cdot 1_{[-nT_{min},nT_{min}]}(t) \parallel^2 \\
&\le \varepsilon^2 \parallel \sum_{k\in\mathbb{N}} (I-\mathcal{A})^k \parallel^2 \lim_{n\to\infty} \frac{1}{2nT_{min}} \parallel \sum_{l\in\mathbb{Z}} g(t-s_l)\cdot 1_{[-nT_{min},nT_{min}]}(t) \parallel^2 \\
&\le \frac{\varepsilon^2}{(1-r)^2} \frac{1}{T_{min}} \lim_{n\to\infty} [\sum_{l\in\mathbb{Z}} (\frac{1}{2n} \int_{-nT_{min}}^{nT_{min}} g^2(u-s_l)du)^{1/2}]^2
\end{aligned}
\tag{48}
$$

since by the CBS inequality:

$$
\frac{1}{2n} \int_{-nT_{\min}}^{nT_{\min}} \sum_{l\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} g(u-s_l)g(u-s_m)du \le [\sum_{l\in\mathbb{Z}} (\frac{1}{2n} \int_{-nT_{min}}^{nT_{min}} g^2(u-s_l)du)^{1/2}]^2. \tag{49}
$$

The dependance of the mean square recovery error on δ parameterized by ε is shown in Figure 7. The TEM input signal used and the details of the simulations are the same with the ones described in section 3.2

4.2 δ-Insensitive Recovery Algorithm

As shown in Figure 7, the implementation of the TDM recovery algorithm given in Theorem 3.1, is highly sensitive to the exact knowledge of the parameter δ. Remedy is provided by the following

Lemma 3 (The Compensation Principle)

$$
\int_{t_l}^{t_{l+2}} x(u)du = \int_{t_l}^{t_{l+2}} z(u)du, \tag{50}
$$

for all $l \in \mathbb{Z}$.

Proof: The desired result is obtained by adding equations (6) for $k = l$ and $k = l + 1$.

Remark 6 Note that the Compensation Principle provides for an estimate of the amplitude of the input signal $x(t)$ on a very small time scale that does not explicitly depend on δ. Note also that the Compensation Principle can be easily extended to subsets of or to the entire real line. Thus the DC component of the input can be recovered from $z(t)$ even for non-bandlimited input signals $x(t), t \in \mathbb{R}$.

The Compensation Principle suggests the construction of the following operator:

$$\begin{aligned}\mathcal{A}x &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_{t_k}^{t_{k+2}} x(u)du\ g(t-t_{k+1}) \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_{t_k}^{t_{k+1}} x(u)du\ [g(t-t_k)+g(t-t_{k+1})].\end{aligned} \tag{51}$$

Theorem 4 (δ-insensitive perfect recovery) *If $r = \frac{\delta}{1-c}\cdot\frac{\Omega}{\pi} < 1/\sqrt{2}$ the bandlimited signal x can be perfectly recovered from the trigger times (t_k), $k \in \mathbb{Z}$, as*

$$\lim_{l\to\infty} x_l(t) = x(t), \tag{52}$$

and

$$\| x - x_l \| \leq (\sqrt{2}r)^{l+1} \| x \| . \tag{53}$$

Proof: Following the method of proof in Lemma 2 we can readily find that the adjoint operator to $\mathcal{A}$ is

$$\mathcal{A}^* = \frac{1}{2}\sum_{\in\mathbb{Z}}[x(t_k)+x(t_{k+1})]\mathcal{P}1_{[t_k,t_{k+1}]}, \tag{54}$$

where $\mathcal{P}$ is the projection operator defined as $\mathcal{P}1_{[t_k,t_{k+1}]} = (g * 1_{[t_k,t_{k+1}]})(t)$.

Note that:

$$\begin{aligned}\| x - \mathcal{A}^*x \|^2 &= \| \mathcal{P}x - \frac{1}{2}\sum_{k\in\mathbb{Z}}[x(t_k)+x(t_{k+1})]\mathcal{P}1_{[t_k,t_{k+1}]} \|^2 \\ &\leq \| \sum_{k\in\mathbb{Z}}\{x - \frac{1}{2}[x(t_k)+x(t_{k+1})]\}1_{[t_k,t_{k+1}(t)]} \|^2 \\ &= \sum_{k\in\mathbb{Z}}\int_{t_k}^{t_{k+1}} |x(u) - \frac{1}{2}[x(t_k)+x(t_{k+1})]|^2\ du\end{aligned} \tag{55}$$

By applying Wirtinger's inequality, we obtain

$$\begin{aligned}\int_{t_k}^{t_{k+1}} &|x(u)-x(t_k)+x(u)-x(t_{k+1})|^2\ du \\ &\le \int_{t_k}^{t_{k+1}} |x(u)-x(t_k)|^2\ du + \int_{t_k}^{t_{k+1}} |x(u)-x(t_{k+1})|^2\ du \\ &\le \frac{4}{\pi^2}T_k^2 \int_{t_k}^{t_{k+1}} |x^{'}(u)|^2\ du + \frac{4}{\pi^2}T_k^2 \int_{t_k}^{t_{k+1}} |x^{'}(u)|^2\ du\end{aligned} \quad (56)$$

and therefore

$$\| x - \mathcal{A}^* x \|^2 \le \frac{2}{\pi^2}(\frac{\delta}{1-c})^2 \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} |x^{'}(u)|^2\ du \le \frac{2}{\pi^2}(\frac{\delta}{1-c})^2 \| x^{'} \|^2 . \quad (57)$$

Finally, by applying Bernstein's inequality we have

$$\| x - \mathcal{A}^* x \| \le \sqrt{2} r \| x \| . \quad (58)$$

Therefore,

$$\| x - \mathcal{A} x \| \le \sqrt{2} r \| x \| . \quad (59)$$

and the theorem follows.

Let us define $\mathbf{h} = \frac{1}{2}[g(t-t_k)+g(t-t_{k+1})]^T$, $\mathbf{q} = [\int_{t_k}^{t_{k+1}} x(u)du]$ and $\mathbf{H} = \frac{1}{2}[\int_{t_l}^{t_{l+1}}[g(t-t_k)+g(t-t_{k+1})]du]$. We have the following

Corollary 5 (δ-insensitive Matrix Recovery) *Under the assumptions of Theorem 5 the bandlimited signal x can be perfectly recovered from its associated trigger times (t_k), $k \in \mathbb{Z}$, as*

$$x(t) = \lim_{l\to\infty} x_l(t) = \mathbf{h}\mathbf{H}^+\mathbf{q}. \quad (60)$$

where $\mathbf{H}^+$ denotes the pseudo-inverse of $\mathbf{H}$. Furthermore,

$$x_l(t) = \mathbf{h}\mathbf{P}_l\mathbf{q}, \quad (61)$$

where $\mathbf{P}_l$ is given by

$$\mathbf{P}_l = \sum_{k=0}^{l} (\mathbf{I}-\mathbf{H})^k. \quad (62)$$

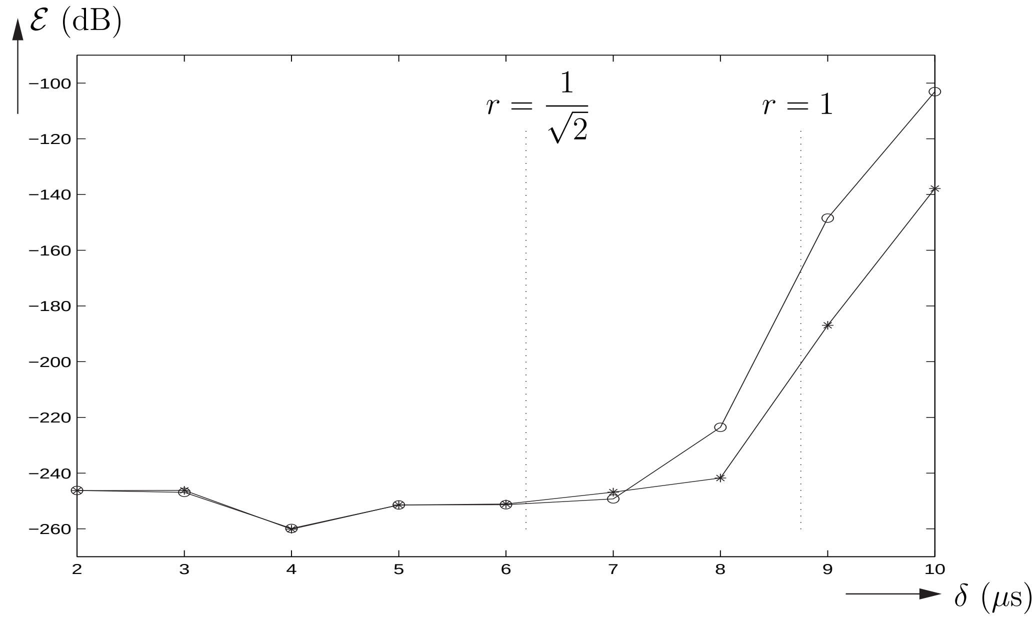

Figure 8: Mean square error for the δ-sensitive (stars) and δ-insensitive algorithms (circles).

The δ-insensitive recovery algorithm achieves perfect recovery provided that $r < 1/\sqrt{2}$. Simulation results for the δ-sensitive and δ-insensitive recovery algorithms are shown in Figure 8 and are denoted by stars and circles, respectively. Dotted vertical lines correspond to the values of δ for which $r = 1/\sqrt{2}$ and $r = 1$, respectively.

Remark 7 The Compensation Principle suggests other operator constructions as well. An example is given by

$$\begin{aligned} \mathcal{A}x &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_{t_k}^{t_{k+2}} x(u)du\ g(t - \frac{t_{k+2}+t_k}{2}) \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\int_{t_k}^{t_{k+1}} x(u)du\ [g(t - s_k - \frac{T_{k+1}}{2}) + g(t - s_k + \frac{T_{k-1}}{2})]. \end{aligned} \tag{63}$$

The details of the correspoding δ-insensitive recovery algorithm can be found in a manner similar to the one above.

5 Recovery Sensitivity with Respect to Time Quantization

In this section we shall assume that the sequence of trigger times (t_k), $k \in \mathbb{Z}$, is measured with finite precision and the actual values available for recovery are $\hat{t}_k$, $k \in \mathbb{Z}$. We shall denote by $T_k = t_{k+1} - t_k$ and $\hat{T}_k = \hat{t}_{k+1} - \hat{t}_k$ for all $k \in \mathbb{Z}$.

5.1 An Upper Bound on a Measure of Error Recovery

The key point of our analysis is the observation that, if the condition $\max_k(\hat{T}_k < T)$ is satisfied, then

$$x = \sum_{k \in \mathbb{N}} (I - \hat{\mathcal{A}})^k \hat{\mathcal{A}} x,$$

where $\hat{\mathcal{A}}$ is defined by

$$\hat{\mathcal{A}} x = \sum_{k \in \mathbb{Z}} \int_{\hat{t}_k}^{\hat{t}_{k+1}} x(u) du \; g(t - \hat{s}_k) \tag{64}$$

and $\hat{s}_k = (\hat{t}_k + \hat{t}_{k+1})/2$. Since the reconstructed signal is given by

$$\hat{x} = \sum_{k \in \mathbb{N}} (I - \hat{\mathcal{A}})^k \sum_{l \in \mathbb{Z}} [(-1)^l (-\hat{T}_l + \delta)] g(t - \hat{s}_l)$$

the error signal amounts to

$$e(t) = \sum_{k \in \mathbb{N}} (I - \hat{\mathcal{A}})^k \sum_{l \in \mathbb{Z}} \epsilon_l g(t - \hat{s}_l), \tag{65}$$

where

$$\epsilon_k = (-1)^k (-\hat{T}_k + \delta) - \int_{\hat{t}_k}^{\hat{t}_{k+1}} x(u) du. \tag{66}$$

Proposition 2 *Assuming that the quantization error $d_k = \hat{T}_k - T_k$, $k \in \mathbb{Z}$, is a sequence of i.i.d. random variables on $[-\Delta/2, \Delta/2]$, the expected MSE is bounded by:*

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{1+c}{\delta T} \cdot (\frac{1+c}{1-r})^2 \cdot \frac{\Delta^2}{12} \; . \tag{67}$$

Proof: Using a method similar to the one developed in (48):

$$\mathcal{E}^2 \leq \frac{1}{(1-r)^2} \lim_{n\to\infty} \frac{1}{2nT_{min}} \parallel \sum_{l\in\mathbb{N}} \epsilon_l g(t-\hat{s}_l) \cdot 1_{[-nT_{min},nT_{min}]}(t) \parallel^2 .$$

Since the norm above is increasing in n

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{1}{(1-r)^2} \lim_{n\to\infty} \frac{1}{2nT_{min}} \mathbb{E} \parallel \sum_{l\in\mathbb{N}} \epsilon_l g(t-\hat{s}_l) \cdot 1_{[-nT_{min},nT_{min}]}(t) \parallel^2 \quad (68)$$

The expectation on the right hand side can be bounded as follows:

$$\begin{aligned} \mathbb{E} \parallel \sum_{l\in\mathbb{N}} \epsilon_l g(t-\hat{s}_l) \cdot 1_{[-nT_{min},nT_{min}]}(t) \parallel^2 = \\ = \int_{-nT_{\min}}^{nT_{\min}} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} g(t-\hat{s}_k) g(t-\hat{s}_m) dt \cdot \mathbb{E}\{\epsilon_k \epsilon_m\} \\ = \int_{-nT_{\min}}^{nT_{\min}} \sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k) dt \cdot (\frac{\delta}{T_k})^2 \frac{\Delta^2}{12} \leq \frac{2n}{T} \cdot (1+c)^2 \cdot \frac{\Delta^2}{12} \end{aligned} \quad (69)$$

since

$$\mathbb{E}\{\epsilon_k \epsilon_m\} = (\frac{\delta}{T_k})^2 \frac{\Delta^2}{12} \delta_{k,m} \quad (70)$$

and

$$\frac{1}{2n} \int_{-nT_{\min}}^{nT_{\min}} \sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k) dt \leq \frac{1}{T} \quad (71)$$

as shown in the Result 3 and 2, respectively, in the Appendix. Finally, substituting the upper bound derived in (69) into (68) gives the desired result.

5.2 Example

The solid line in Figure 9 and Figure 10 shows the mean square recovery error $\mathcal{E}$ as a function of δ and the number of quantization bits, N, respectively. The same figures also depict the upper bound arising in inequality (67). More details about these figures are given in the next section.

6 A Comparison of Time and Amplitude Quantization

In this section we shall compare the effects of quantization in the time and amplitude domains. Since time encoding and irregular sampling are different discrete representations of information contained in a bandlimited function, the signal recovery of quantized version of the trigger times and irregular samples, respectively, is of great interest in practice.

6.1 Upper Bound for the Amplitude Quantization Error

Assume that the instances s_k are exactly known and the amplitudes $x(s_k)$ are corrupted to $x(s_k) + \epsilon_k$.

Proposition 3 *If the random variables* (ϵ_k), $k \in \mathbb{Z}$, *are independent uniformly distributed within* $[-\varepsilon/2, \varepsilon/2]$ *then*

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{r}{(1-r)^2}\frac{1+c}{1-c}\frac{\varepsilon^2}{12} \tag{72}$$

Proof: The the error signal due to amplitude quantization is:

$$e(t) = \hat{x}(t) - x(t) = \sum_{k=0}^{\infty}(I-\mathcal{S})^k \sum_{l\in\mathbb{Z}} T_l \epsilon_l g(t-s_l)$$

Following the same derivation as in Proposition 2 we obtain

$$\mathbb{E}\{\mathcal{E}^2\} \leq \frac{1}{(1-r)^2}\frac{T_{\max}^2}{T_{\min}T}\frac{\varepsilon^2}{12} = \frac{r}{(1-r)^2}\frac{1+c}{1-c}\frac{\varepsilon^2}{12} \tag{73}$$

since $\| \mathcal{S}^{-1} \| \leq (1-r)^{-1}$ and

$$\mathbb{E}\{\epsilon_k \epsilon_m\} = \delta_{k,m}\mathbb{E}\{\epsilon_k^2\} = \delta_{k,m}\int_{-\varepsilon/2}^{\varepsilon/2}\frac{1}{\varepsilon}x^2 dx = \delta_{k,m}\frac{\varepsilon^2}{12} \tag{74}$$

and finally

$$\begin{aligned}\int_{-nT_{\min}}^{nT_{\min}} \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} T_m T_k g(t-s_k) g(t-s_m) dt \cdot \mathbb{E}\{\epsilon_k \epsilon_m\} \\ \leq T_{\max}^2 \sum_{k\in\mathbb{Z}} g^2(t-s_k)\frac{\varepsilon^2}{12}.\end{aligned} \tag{75}$$

6.2 Example

A reasonable comparison between the effects of amplitude and time quantization can be established if we assume that the quantized amplitudes and quantized trigger times are transmitted at the same bitrate. Since $x(s_k)$ and T_k are associated with the trigger times t_k and t_{k+1}, the same transmission bitrate is achieved if $x(s_k)$ and T_k are represented by the same number of bits N. Since $-c \leq x \leq c$, the amplitude quantization step amounts to $\varepsilon = 2c/2^N$.

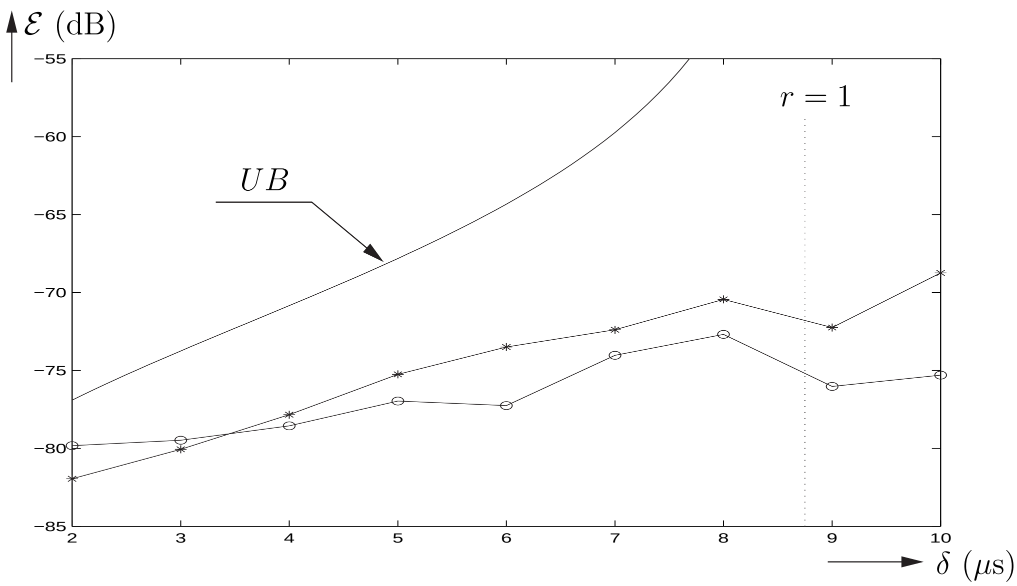

Figure 9: The dependence of $\mathcal{E}$ on δ for time encoding (stars) and irregular sampling (circles).

For time encoding $T_{\min} \leq T_k \leq T_{\max}$, or equivalently $0 \leq T_k - T_{\min} \leq T_{\max} - T_{\min}$. Thus, theoretically, if $T_{\min}$ is exactly known, then only measuring T_k as $T_k = T_{\min} + (T_k - T_{\min})$ is needed for obtaining a quantized version of $T_k - T_{\min}$ in the range $(0, T_{\max} - T_{\min})$:

$$\Delta = \frac{T_{\max} - T_{\min}}{2^N} = \frac{1}{2^N}\left(\frac{\delta}{1-c} - \frac{\delta}{1+c}\right) = \frac{\delta\varepsilon}{1-c^2}.$$

Substituting the values of ε and Δ above into (67) and (72) results exactly in the same upper bound for both the expected mean square error for time encoding and irregular sampling, respectively.

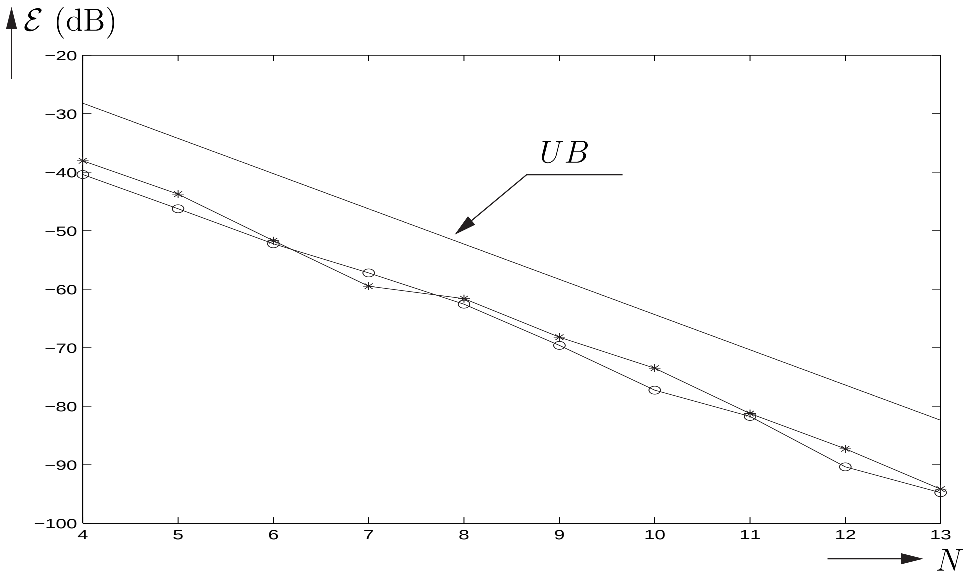

Figure 10: The dependence of $\mathcal{E}$ on the number of quantization bits for time encoding (stars) and irregular sampling (circles).

Result 1 *For the same number of bits N, the upper bound for time quantization is equal to the upper bound for amplitude quantization and amounts to:*

$$UB = \frac{1}{3 \cdot 2^{2N}} \cdot \frac{r}{(1-r)^2} \cdot \frac{1+c}{1-c} \cdot c^2.$$

Figure 9 and Figure 10 show the mean square error $\mathcal{E}$ as a function of δ and the number of quantization bits, N, respectively. The details of the simulation are as before.

7 Conclusions

In this paper we have established time encoding as an alternative information representation modality for bandlimited signals. We have shown that a simple Time Encoding Machine can be used to generate a sequence of trigger times and demonstrated an algorithm for perfect signal recovery. We have shown how to construct a TDM that does not depend on the key parameter of the TEM.

We derived an upper bound on the expect mean square error of signal recovery when a quantized version of the trigger times ia available. We have also shown that quantization in the time and amplitude domain lead to largely equivalent information representation capabilities.

8 Appendix

Result 2

$$\frac{1}{2n}\int_{-nT_{min}}^{nT_{min}}\sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k)dt \leq \frac{1}{T} \tag{76}$$

and

$$\frac{1}{2n}\int_{-nT_{max}}^{nT_{max}}\sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k)dt \geq \frac{1}{T} \tag{77}$$

and in the limit equality is achieved.

Proof: Increasing the density of packing in the interval $[-nT_{min}, nT_{min}]$ implies:

$$\int_{-nT_{\min}}^{nT_{\min}}\sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k)dt \leq \int_{-nT_{\min}}^{nT_{\min}}\sum_{k\in\mathbb{Z}} g^2(t-T_{\min}/2+kT_{\min})dt$$

since the function $g^2(t-\hat{s}_k)$ is positive for all $t, t\in\mathbb{R}$. The infinite sum $\sum_{k\in\mathbb{Z}} g^2(t-T_{\min}/2-kT_{\min})$ represents a periodic function and

$$\frac{1}{2nT_{min}}\int_{-nT_{\min}}^{nT_{\min}}\sum_{k\in\mathbb{Z}} g^2(t-T_{\min}/2+kT_{\min})dt$$

represents its zero'th order Fourier coefficient. This coefficient amounts to:

$$\frac{1}{T_{min}}\int_{\mathbb{R}} g^2(t-T_{min}/2)dt = \frac{1}{T_{min}}\frac{1}{2\pi}\int_{\mathbb{R}} 1_{[-\Omega,\Omega]}d\omega = \frac{1}{T_{min}}\frac{\Omega}{\pi} = \frac{1}{T_{min}T}.$$

The second inequality is similarly derived by noting that decreasing the packing in the interval $[-nT_{max}, nT_{max}]$ implies:

$$\int_{-nT_{\max}}^{nT_{\max}}\sum_{k\in\mathbb{Z}} g^2(t-\hat{s}_k)dt \geq \int_{-nT_{\max}}^{nT_{\max}}\sum_{k\in\mathbb{Z}} g^2(t-T_{\max}/2+kT_{\max})dt$$

The rest of the proof is as above.

Result 3

$$\mathbb{E}\{\epsilon_k \epsilon_m\} = \left(\frac{\delta}{T_k}\right)^2 \frac{\Delta^2}{12} \delta_{k,m} \tag{78}$$

Proof: With $d_k = \hat{T}_k - T_k$

$$\epsilon_k = \int_{t_k}^{t_{k+1}} x(u)du - \int_{\hat{t}_k}^{\hat{t}_{k+1}} x(u)du - (-1)^k d_k.$$

Using the mean-value theorem we obtain

$$\epsilon_k = x(\xi_k)T_k - x(\hat{\xi}_k)\hat{T}_k - (-1)^k d_k,$$

where $\xi_k \in (t_k, t_{k+1})$ and $\hat{\xi}_k \in (\hat{t}_k, \hat{t}_{k+1})$. For Δ small enough, $\xi_k \simeq \hat{\xi}_k$ and

$$\epsilon_k \simeq (-x(\xi_k) - (-1)^k)d_k.$$

Since

$$x(\xi_k) = \frac{1}{T_k}\int_{t_k}^{t_{k+1}} x(t)dt = -(-1)^k + (-1)^k \frac{\delta}{T_k}$$

we have

$$\epsilon_k \simeq= -(-1)^k \frac{\delta}{T_k} d_k$$

and thus

$$\mathbb{E}\{\epsilon_k \epsilon_m\} = (-1)^{k+m} \frac{\delta}{T_k}\frac{\delta}{T_m}\mathbb{E}\{d_k d_m\} = \left(\frac{\delta}{T_k}\right)^2 \frac{\Delta^2}{12}\delta_{k,m}.$$

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