Average SNR of a Generalized Diversity Selection Combining Scheme

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***Abstract*— The average signal-to-noise ratio (SNR) of a generalized selection combining scheme, in which the m diversity branches ($m \leq L$, where L is the total number of diversity branches available) with the largest instantaneous SNR's are selected and coherently combined, is derived. A Rayleigh fading channel is assumed, and a simple closed-form expression for the SNR is found which is upper bounded by the average SNR of maximal ratio combining, and lower bounded by average SNR of conventional selection combining.**

I. Introduction

DIVERSITY combining has been considered as an efficient way to combat multipath fading because the combined signal-to-noise ratio (SNR) is increased compared with the SNR of each diversity branch. The optimum combiner is the maximal ratio combiner (MRC) whose SNR is the sum of the SNR's of each individual diversity branch. The conventional selection combiner (CSC) selects the signal from that diversity branch with the largest instantaneous SNR. In this paper, we study a generalized selection scheme. Instead of selecting only the largest (in instantaneous SNR) diversity signal as in CSC, we choose the m largest signals from L total diversity branches and then coherently combine them. The average SNR of this GSC is derived and a simple closed-form expression is found, which is upper bounded by the average SNR of MRC and lower bounded by the average SNR of CSC, as expected.

II. System Model

The received equivalent low-pass signal from the lth diversity branch is

$$r_{lk}(t) = \alpha_l(t)e^{j\phi_l(t)}u_{lk}(t) + z_l(t),$$
$$0 \leq t \leq T;\ l = 1, \cdots, L;\ k = 1, \cdots, K.$$

where $\{u_{lk}(t)\}$ is one of the K transmitted complex low-pass information signals (i.e., K is the alphabet size), T is the symbol duration, and $z(t)$ is zero-mean complex additive white Gaussian noise (AWGN) with a power spectral density of N_0 W/Hz. It is assumed that each of the L branches experiences slow, flat, Rayleigh fading. Thus, $\alpha_l(t)e^{-j\phi_l(t)} = \alpha_l e^{-j\phi_l}$, where α_l is a Rayleigh random variable and ϕ_l is a random variable uniformly distributed in $[0, 2\pi)$. A receiver with MRC will coherently combine the L diversity branches by weighting them by the complex conjugate of their respective fading gains and adding them. Then the envelope of the combined signal of the decision variable is

$$S_c = A_k \sum_{l=1}^{L} \alpha_l^2$$

and the noise component is

$$N_c = \sum_{l=1}^{L} z_l(t)e^{-j\phi_l}\alpha_l.$$

Therefore, the instantaneous SNR for MRC is

$$\gamma_{\text{MRC}} = \frac{|S_c|^2}{2\sum_{l=1}^{L} \alpha_l^2 P_n} = \sum_{l=1}^{L} \frac{|A_k|^2\alpha_l^2}{2P_n} \triangleq \sum_{l=1}^{L} \gamma_l$$

where

$$\gamma_l \triangleq \frac{|A_k|^2\alpha_l^2}{2P_n}$$

which is the sum of the instantaneous SNR's of the L diversity branches. Note that A_k is the kth transmitted low-pass equivalent signal amplitude, and P_n is the noise power in each diversity branch. Also, note that γ_l is exponentially distributed with density function

$$f_{\gamma_l}(y_l) = \frac{1}{\overline{\gamma}_l}e^{-y_l/\overline{\gamma}_l} = a_l e^{-a_l y_l}, \qquad l = 1, 2, \cdots, L \quad (1)$$

where $\overline{\gamma}_l = E\{\alpha_l^2\}\overline{\gamma}_{lo}$, $\overline{\gamma}_{lo} = (|A_k|^2/2P_n)$, $l = 1, 2, \cdots, L$, are the SNR's in the absence of fading, and $a_l = (1/\overline{\gamma}_l)$.

Now instead of combining signals from all L diversity branches (since some of the diversity branches may be too weak to contribute) as in MRC, or only selecting the strongest signal (since too much information may be lost with $L-1$ diversity branches dropped), we select and coherently combine m $(m \leq L)$ diversity branches. The instantaneous SNR of GSC from the above analysis is given by

$$\gamma_{\text{GSC}} = \sum_{l=1}^{L} \gamma_{(l)}$$

where

$$\{\gamma_{(1)},\quad \gamma_{(2)}, \cdots \gamma_{(m)}\}, \gamma_{(1)} \geq \gamma_{(2)}, \geq \cdots \geq \gamma_{(m)}, \quad m \leq L$$

is an ordered SNR set.

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III. Combined Average SNR of GSC

The joint density function of the above ordered SNR's was found in [2] and is given by the following expression:

$$f_{\gamma_{(1)}\gamma_{(2)}\cdots\gamma_{(m)}}(y_1, y_2, \cdots, y_m) = \sum_{i=1}^{L} f_{\gamma_i}(y_1) \sum_{\substack{j=1 \\ j\neq i}}^{L} f_{\gamma_j}(y_2) \cdots \sum_{\substack{n=1 \\ n\neq i,j,\cdots}}^{L} f_{\gamma_n}(y_m) \prod_{n'=1}^{L} F_{\gamma'_n}(y_m). \tag{2}$$

Note that the index with a prime refers to the $L - m$ unselected channel outputs (i.e., it excludes all unprimed indexes occurring in any outer summations). If only the largest γ is chosen, then

$$f_{\gamma_{(1)}}(y) = \sum_{i=1}^{L} f_{\gamma_i}(y) \prod_{\substack{j=1 \\ j\neq i}}^{L} F_{\gamma_j}(y). \tag{3}$$

If the largest two are chosen, then

$$f_{\gamma_{(1)}\gamma_{(2)}}(y_1, y_2) = \sum_{i=1}^{L} f_{\gamma_i}(y_1) \sum_{\substack{j=1 \\ j\neq i}}^{L} f_{\gamma_j}(y_2) \prod_{j'=1}^{L} F_{\gamma_j}(y_2). \tag{4}$$

If $\overline{\gamma}_i = \overline{\gamma}_j = \overline{\gamma} = (1/a)$, $\forall i \neq j$, which is the i.i.d. case, the joint density function of (2) reduces to

$$f_{\gamma_{(1)}\gamma_{(2)}\cdots\gamma_{(m)}}(y_1, y_2, \cdots, y_m) = L(L-1)\cdots(L-m+1) f_\gamma(y_1)\cdots f_\gamma(y_m) F_\gamma^{L-m}(y_m). \tag{5}$$

The average SNR of GSC, Γ_{GSC}, is given by

$$\begin{aligned}\Gamma_{\text{GSC}} &= \int_0^\infty \int_{y_m}^\infty \cdots \int_{y_2}^\infty (y_1 + y_2 + \cdots + y_m) \\ &\quad \cdot \sum_{i=1}^{L} f_{\gamma_i}(y_1) \sum_{\substack{j=1 \\ j\neq i}}^{L} f_{\gamma_j}(y_2) \cdots \sum_{\substack{n=1, \\ n\neq i,j,\cdots}} f_{\gamma_n}(y_m) \\ &\quad \cdot \prod_{n'=1}^{L} F_{\gamma_{n'}}(y_m)\, dy_1\, dy_2 \cdots dy_m \\ &= \int_0^\infty \int_{y_m}^\infty \cdots \int_{y_2}^\infty (y_1 + y_2 + \cdots + y_m) \\ &\quad \cdot \sum_{\substack{i,j,\cdots,n=1 \\ i\neq j\cdots\neq n}} a_i e^{-a_i y_1} a_j e^{-a_j y_2} \cdots a_n e^{-a_n y_m} \\ &\quad \cdot \prod_{n'=1}^{L} (1 - e^{-a_{n'} y_m}))\, dy_1\, dy_2 \cdots dy_m. \end{aligned} \tag{6}$$

When the channel fading statistics are iid, then the average combined SNR is shown in the Appendix to be given by

$$\Gamma_c = \frac{1}{a}\left[m + \frac{m}{m+1} + \frac{m}{m+2} + \cdots + \frac{m}{L-1} + \frac{m}{L}\right]. \tag{7}$$

If $m = 1$, $\Gamma_{\text{GSC}} = \Gamma_{\text{CSC}}$ and

$$\Gamma_{\text{CSC}} = \frac{1}{a}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{L-1} + \frac{1}{L}\right) = \overline{\gamma} \sum_{n=1}^{L} \frac{1}{n} \tag{8}$$

which is the same as the result in [1]. If $m = L$, $\Gamma_{\text{GSC}} = \Gamma_{\text{MRC}}$ and from (7)

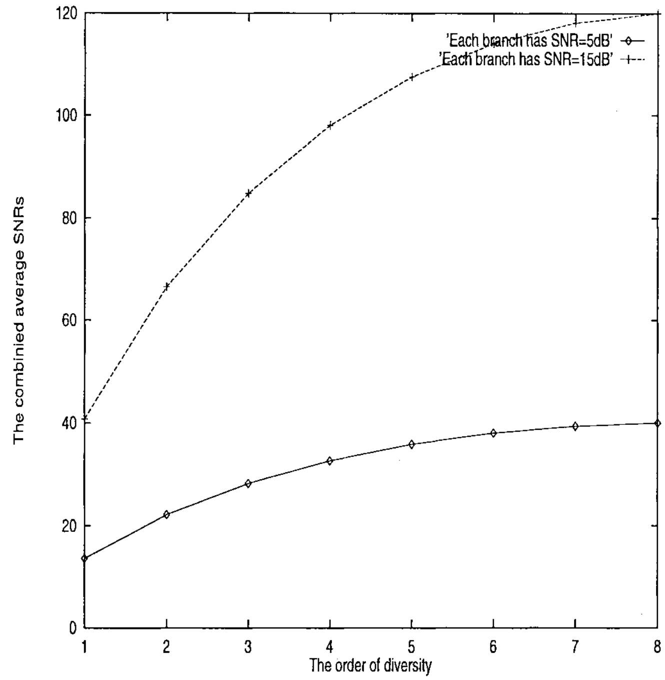

Fig. 1. Combined average SNR as a function of the number of branches chosen.

$$\Gamma_{\text{MRC}} = L\frac{1}{a} = L\overline{\gamma}. \tag{9}$$

Note that for MRC, not only the combined instantaneous SNR is the sum of the individual instantaneous SNR's, but also the combined average SNR is the sum of the individual average SNR's.

It can be easily shown that (7) is a monotonic function of m, therefore

$$\Gamma_{\text{CSC}} \leq \Gamma_{\text{GSC}} \leq \Gamma_{\text{MRC}}. \tag{10}$$

Fig. 1 is the plot of the combined average SNR's as a function of the number of selected branches with $L = 8$.

IV. Conclusions

In this paper, we derived the SNR of a generalized selection combining scheme. We presented a simple closed-form expression for the average SNR of this generalized seletion combining for independent identically distributed (i.i.d.) diversity channels, which is upper bounded by that of maximal ratio combining and lower bounded by that of conventional selection combining.

Appendix

The derivation of (7) is as follows. Let

$$\Gamma_{\text{GSC}} = \frac{L!}{(L-m)!}\Gamma'_{\text{GSC}}$$

where

$$\begin{aligned}\Gamma'_{\text{GSC}} &= \int_0^\infty \int_{y_m}^\infty \cdots \int_{y_2}^\infty (y_1 + y_2 + \cdots + y_m) \\ &\quad \cdot a e^{-a y_1} a e^{-a y_2} \cdots a e^{-a y_m} \\ &\quad \cdot (1 - e^{-a y_m})^{L-m}\, dy_1\, dy_2 \cdots dy_m. \end{aligned} \tag{11}$$

$$J(m, L-m) = \frac{1}{m}\,\frac{(L-m)!(m-1)!}{L!} + \frac{L-m}{m}\,\frac{1}{m+1}\,\frac{(L-m-1)!m!}{L!} + \cdots + \frac{L-m}{m}\,\frac{L-m-1}{m+1}\cdots\frac{1}{L-1}\,\frac{1!(L-2)!}{L!} + \frac{L-m}{m}\,\frac{L-m-1}{m+1}\cdots\frac{2}{L-2}\,\frac{1}{L-1}\,\frac{1}{L^2}.$$

Consider the following transformation:

$$\begin{aligned}
&x_1 = y_1 - y_2, \quad x_2 = y_2 - y_3, \cdots, x_{m-1} = y_{m-1} - y_n \\
&x_m = y_m \\
\Gamma'_{\mathrm{GSC}} &= \int_0^\infty \cdots \int_0^\infty (x_1 + 2x_2 + \cdots + mx_m) \\
&\quad \cdot e^{-ax_1 - 2ax_2 - \cdots - max_m}(1 - e^{-ax_m})^{L-m} \\
&\quad dax_1\, dax_2 \cdots dax_m \\
&\triangleq \sum_{i=1}^{m} t_i
\end{aligned}$$

where

$$\begin{aligned}
t_i &= \int_0^\infty \cdots \int_0^\infty ix_i e^{-ax_1 - 2ax_2 - \cdots - max_m} \\
&\quad \cdot (1 - e^{-ax_m})^{L-m}\, dax_1\, dax_2 \cdots dax_m \\
&= 1 \cdot \frac{1}{2} \cdots \frac{1}{ia} \cdots \frac{1}{m-1} \\
&\quad \cdot \int_0^\infty e^{-max_m}(1 - e^{-ax_m})^{L-m}\, dax_m \\
&= \frac{1}{(m-1)!a} \int_0^\infty e^{-mx_m}(1 - e^{-x_m})^{L-m}\, dx_m \\
&\triangleq \frac{1}{(m-1)!a} I(m, L-m), \qquad i < m \\
t_m &= \int_0^\infty e^{-ax_1}\, dax_1 \cdots m \int_0^\infty x_m e^{-max_m} \\
&\quad \cdot (1 - e^{-ax_m})^{L-m}\, dax_m \\
&= 1 \cdot \frac{1}{2} \cdots \frac{1}{i} \cdots \frac{1}{m-1}\, \frac{1}{a} \int_0^\infty x_m e^{-mx_m} \\
&\quad \cdot (1 - e^{-x_m})^{L-m}\, dx_m \\
&\triangleq \frac{m}{(m-1)!a} J(m, L-m).
\end{aligned}$$

Note that

$$\begin{aligned}
I(m, L-m) &= \int_0^\infty e^{-mx}(1 - e^{-x})^{L-m}\, dx \\
&= -\frac{1}{m} \int_0^\infty (1 - e^{-x})^{L-m}\, de^{-mx} \\
&= \frac{L-m}{m} \int_0^\infty e^{-(m+1)x}(1 - e^{-x})^{L-m-1}\, dx \\
&= \frac{L-m}{m} I(m+1, L-m-1).
\end{aligned}$$

Since

$$I(L, 0) = \int_0^\infty e^{-Lx}\, dx = \frac{1}{L}$$

we have

$$\begin{aligned}
I(m, L-m) &= \frac{L-m}{m} I(m+1, L-m-1) \\
&= \frac{L-m}{m}\, \frac{L-m-1}{m+1} I(m+2, L-m-2) \\
&= \cdots \frac{L-m}{m}\, \frac{L-m-1}{m+1} \cdots \frac{1}{L-1} I(L, 0) \\
&= \frac{(L-m)!(m-1)!}{L!}.
\end{aligned}$$

Also

$$\begin{aligned}
J(m, L-m) &= \int_0^\infty xe^{-mx}(1 - e^{-x})^{L-m}\, dx \\
&= -\frac{1}{m} \int_0^\infty x(1 - e^{-x})^{L-m}\, de^{-mx} \\
&= \frac{1}{m} \int_0^\infty e^{-mx} d[x(1 - e^{-x})^{L-m}] \\
&= \frac{1}{m} I(m, L-m) + \frac{L-m}{m} \\
&\quad \cdot J(m+1, L-m-1) \\
J(L, 0) &= \int_0^\infty xe^{-Lx}\, dx = \frac{1}{L^2}.
\end{aligned}$$

Hence, see the equation shown at the top of the page. Therefore

$$\begin{aligned}
\Gamma'_{\mathrm{GSC}} &= \frac{m-1}{(m-1)!a}\, \frac{(L-m)!(m-1)!}{L!} + \frac{m}{(m-1)!a} \\
&\quad \cdot \Bigg[\frac{1}{m}\, \frac{(L-m)!(m-1)!}{L!} + \frac{L-m}{m} \\
&\quad \cdot \frac{1}{m+1}\, \frac{(L-m-1)!m!}{L!} + \cdots \\
&\quad + \frac{L-m}{m}\, \frac{L-m-1}{m+1} \cdots \frac{1}{L-1}\, \frac{1!(L-2)!}{L!} \\
&\quad + \frac{L-m}{m}\, \frac{L-m-1}{m+1} \cdots \frac{2}{L-2}\, \frac{1}{L-1}\, \frac{1}{L^2}\Bigg] \\
&= \frac{1}{a}\, \frac{(L-m)!}{L!} \Bigg[(m-1) + 1 + \frac{m}{m+1} + \frac{m}{m+2} \\
&\quad + \cdots + \frac{m}{L-1} + \frac{m}{L}\Bigg],
\end{aligned}$$

or

$$\Gamma_{\mathrm{GSC}} = \frac{1}{a}\left[m + \frac{m}{m+1} + \frac{m}{m+2} + \cdots + \frac{m}{L-1} + \frac{m}{L}\right].$$

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