ADVECTION AND DISPERSION IN TIME AND SPACE

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Abstract

Previous work showed how moving particles that rest along their trajectory lead to time-nonlocal advection-dispersion equations. If the waiting times have infinite mean, the model equation contains a fractional time derivative of order between 0 and 1. In this article, we develop a new advection-dispersion equation with an additional fractional time derivative of order between 1 and 2. Solutions to the equation are obtained by subordination. The form of the time derivative is related to the probability distribution of particle waiting times and the subordinator is given as the first passage time density of the waiting time process which is computed explicitly.

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1 Introduction

Continuous time random walks (CTRW) can be used to derive governing equations for anomalous diffusion [7,17,18,20]. The CTRW is a stochastic process model for the movement of an individual particle [21,27]. In the long-time limit, the process converges to a simpler form whose probability densities solve the governing equation, leading to a useful model for anomalous diffusion. For a simple random walk with zero–mean, finite–variance particle jumps, the limit process is a Brownian motion $A(t)$ governed by the classical diffusion equation $\partial p/\partial t = \partial^2 p/\partial x^2$ where $p(x,t)$ is the probability density of the random variable $A(t)$. For symmetric infinite–variance jumps (i.e., those with probability density function tails that fall off like $|x|^{-1-\alpha}$ [16] with some index $0 < \alpha < 2$), the limit process $A(t)$ is an α-stable Lévy motion, and the governing equation becomes $\partial p/\partial t = \partial^\alpha p/\partial|x|^\alpha$ [14]. When waiting times between the jumps are introduced, the limiting process is altered via subordination [28,19,26].

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In this study, we examine the case where the waiting times are independent of jump size, also called an "uncoupled" CTRW. For infinite mean waiting times (whose probability distribution is assumed to decay algebraically with some index $0 < \gamma < 1$) the limit process is $A(E(t))$ where $E(t)$ is the inverse or first passage time process for the γ-stable subordinator. By virtue of its construction, the process $E(t)$ counts the number of particle jumps by time $t \geq 0$, accounting for the waiting time between particle jumps. In the scaling limit, $E(t)$ keeps track of the possibly nonlinear link between real time and the operational time that a particle actually spends in motion. The governing equation becomes $\partial^\gamma p/\partial t^\gamma = \partial^\alpha p/\partial|x|^\alpha$ [17,18]. Some applications [8] seem to indicate a time derivative of order $1 < \gamma \leq 2$. In this paper, we develop one such equation by extending the CTRW approach to processes with finite–mean waiting times, and compute the distribution of the relevant first passage time process.

2 The model

In the usual CTRW formalism, the long-time limit for the waiting time process is a γ-stable subordinator $D(t)$ [17]. Then the inverse Lévy process $E(t) = \inf\{x : D(x) > t\}$ counts the number of particle jumps by time $t \geq 0$, reflecting the fact that the time T_n of the nth particle jump and the number $N_t = \max\{n : T_n \leq t\}$ of jumps by time t are also inverse processes. When the waiting times between particle jumps have heavy tails with $0 < \gamma < 1$ (i.e., infinite mean), subordination of the particle location process $A(t)$ via the inverse Lévy process $E(t)$ is necessary in the long-time limit to account for the amount of time that a particle is not participating in the motion process. The subordination leads to a time derivative of order γ in the governing equation of motion [18]. When waiting times have heavy tails of order $1 < \gamma \leq 2$, meaning that the probability of waiting longer than t falls off like $t^{-\gamma}$, a different model is needed [5]. In this case, convergence of the waiting time process requires centering to the mean waiting time w, which is not necessary when $0 < \gamma < 1$. Accounting for this leads to a waiting time process $W(t) = D(t) + wt$ where $D(t)$ is a completely positively skewed[4] stable Lévy process with index γ, so that $W(t)$ is a Lévy process with drift. The drift ensures that $W(t) \to \infty$ with probability one as $t \to \infty$. Since $\gamma > 1$, the process $W(t)$ is not strictly increasing, so it is proper to use $\mathrm{Max}(t) = \sup\{W(u) : 0 \leq u \leq t\}$ to represent the particle jump times. This substitution is explained as a particle that makes more than one jump without intervening rest periods [5]. Then the inverse or first passage time process $H(t) = \inf\{x : \mathrm{Max}(x) \geq t\}$ counts the number of particle jumps. The first passage time process $H(t)$ serves as the subordinator for (and when $\gamma < 1$ is identical to) $E(t)$. We show this in subsequent sections.

3 First Passage Time Density

According to the previous section we model the waiting time process as a Lévy process with drift; i.e., $W(t)$ has characteristic function

$$E[e^{i\xi W(t)}] = e^{t(iw\xi + a(-i\xi)^\gamma)},$$

where $w > 0$ is the mean waiting time and a is a shape variable akin to the variance. For the remainder of this paper we make the assumption $w = 1$, which entails no loss of generality, since any mean time w can be recovered by

[4] The skew is irrelevant in the normal case $\gamma = 2$.

a simple rescaling in time. In order to compute the density of the process H we use the fact that

$$P\{H(T) > s\} = P\{\mathrm{Max}(s) < T\}$$

for all $s, T \geq 0$. Theorem 1 in [4] shows that the distribution of the maximum process $M(s,T) = P\{\mathrm{Max}(s) < T\}$ satisfies

$$\begin{aligned} u \int_0^\infty \int_0^\infty & e^{-us-\lambda T} d_T(M(s,T)) ds \\ &= \exp\left\{ \frac{1}{2\pi} \int_u^\infty \int_{-\infty}^\infty \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))} \, d\xi \, dx \right\} \end{aligned} \quad (1)$$

for all $\lambda, u > 0$. The integrand has two poles in the upper complex halfplane, at $\xi = i\lambda$ and whenever $x = i\xi + a(-i\xi)^\gamma$. That the second equality holds only once, follows from the following Lemma (for a proof see Appendix) investigating the inverse function of $az^\gamma - z$. This involves the following region. Let $a > 0$, $0 < \alpha \leq \pi/\gamma$ and

$$\Omega(\alpha) := \left\{ re^{i\theta} : -\alpha < \theta < \alpha \text{ and } \frac{\sin(\theta)}{a\sin(\gamma\theta)} < r^{\gamma-1} < \frac{\sin(\alpha)}{a\sin(\gamma\alpha)} \right\}$$

where we take $\sin(\alpha)/a\sin(\gamma\alpha) = \infty$ when $\alpha = \pi/\gamma$.

Lemma 1 *Let $a > 0, 1 < \gamma \leq 2$. There exists a unique holomorphic function $q : \mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}] \to \Omega(\pi/\gamma)$ such that*

$$aq(z)^\gamma - q(z) = z.$$

Furthermore, there exists an analytic function F with $\sup_{t>0} t^{1-1/\gamma} e^{\zeta t} |F(t)| < \infty$ for all $0 < \zeta < a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$ such that $\int_0^\infty e^{-zt} F(t)\, dt = 1/q(z)$ for $z > 0$.

Using the Lemma, we can simplify (1) to (see Appendix B)

$$\int_0^\infty \int_0^\infty e^{-us-\lambda T} d_T(M(s,T))ds = \frac{1-\lambda/q(u)}{u+\lambda-a\lambda^\gamma}.$$

Let $H(s,T) = P\{H(T) \leq s\}$ and recall that $M(s,T) = P\{\text{Max}(s) < T\}$. Then $H(s,T) = P\{H(T) \leq s\} = P\{\text{Max}(s) \geq T\} = 1 - M(s,t)$, and hence we can integrate by parts to get

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-us-\lambda T} d_T(M(s,T))ds &= \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T} M(s,T) dT\, ds \\ &= \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T}(1-H(s,T))ds\, dT \\ &= 1/u - \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T) ds\, dT\end{aligned}$$

In other words, the Laplace-Laplace transform of the distribution function of the first passage time process is given by

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T)\, ds\, dT &= \frac{1}{u\lambda} - \frac{1-\lambda/q(u)}{\lambda(u+\lambda-a\lambda^\gamma)} \\ &= \frac{1-a\lambda^{\gamma-1}+u/q(u)}{u(u+\lambda-a\lambda^\gamma)}.\end{aligned} \tag{2}$$

In the following theorem, we invert this Laplace-Laplace transform to obtain an expression for the cumulative distribution function of the first passage time process (corresponding in our model to the "number of jumps by time t").

Theorem 2 *Let $1 < \gamma \leq 2$, let m be the function with Laplace transform $\tilde{m}(u) = 1/q(u)$ given by Lemma 1 and g_γ be the maximally skewed, standard γ-stable density;i.e., the Fourier transform of g_γ is $\hat{g}_\gamma(k) = e^{(ik)^\gamma}$. Then the cumulative distribution function of the first passage time of a Levy-stable waiting* *time process with drift is given by*

$$H(s,t) = \int_{\frac{t-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u)\,du + \int_0^s \frac{m(s-u)}{(au)^{1/\gamma}} g_\gamma\left(\frac{t-u}{(au)^{1/\gamma}}\right) du. \tag{3}$$

Remark 3 *For $t >> 0$,*

$$H(s,t) \approx \int_{\frac{t-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u)\,du = P(W(s) \geq t) \tag{4}$$

in view of the fact that $W(s)$ is identically distributed with $(as)^{1/\gamma}W_\gamma + s$ where W_γ is the stable random variable with density g_γ. If $W(s)$ were an increasing process, the left-hand and right-hand expressions in (4) would be equal. Hence the second term in (3) accounts for the fact that $W(s)$ is not monotone.

4 The Differential Equations

In this section we show that the density of the uncoupled CTRW limit process solves a fractional partial differential equation with an extra time derivative term of order $1 < \gamma \leq 2$ on the right-hand side whose coefficient is *negative.* This is in contrast to Mainardi and colleagues' [10–13] investigation of the transition via fractional derivatives from the second order hyperbolic equation to the first order parabolic case. Our equations stay parabolic in nature (given an elliptic operator in space).

We begin with a related result concerning the first passage time density. We say that a *mild solution* to a fractional partial differential equation is a function whose Laplace transform solves the equivalent algebraic equation in Laplace-Laplace space. The following theorem employs the Caputo derivative $(d/dt)^\gamma$, which can be defined for $1 < \gamma \leq 2$ by requiring that $(d/dt)^\gamma F(t)$ has Laplace transform $\lambda^\gamma \tilde{F}(\lambda) - \lambda^{\gamma-1}F(0) - \lambda^{\gamma-2}F'(0)$ where $\tilde{F}(\lambda)$ is the Laplace transform of $F(t)$, see for example [6,22].

Theorem 4 *There exists a unique distribution f such that the density $u(t,s) = dH(s,t)/ds$ of the first passage time distribution $H(s,t)$ in* (3) *is the unique mild solution to*

$$-a\left(\frac{d}{dt}\right)^{\gamma} u(t,s) + \frac{d}{dt}u(t,s) = -\frac{d}{ds}u(t,s) + f(s)\delta(t) \tag{5}$$

with conditions: $u(0,s) = \delta(s)$; $u(t,0) = u_t(0,s) = 0\ \forall\ s,t > 0$; and $s \mapsto u(t,s)$ is a probability density for all $t > 0$. For any other distribution f equation (5) *has no solution.*

We now extend this result to incorporate more general spatial derivative operators. When the CTRW particle jumps are in the generalized domain of attraction of an operator stable limit with probability distribution ν, the long-time limiting particle location process $A(t)$ is operator stable with distribution ν^t [15]. For a simple random walk, the density $p(x,t)$ of $A(t)$ defines a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $L^1(\mathbb{R}^d)$ via the formula $T(t)f(x) = \int f(x-y)p(y,t)dy$. If L is the generator of this semigroup, then $C(x,t) = T(t)f(x)$ also solves the abstract Cauchy problem $dC/dt = LC; C(0) = f$. For an uncoupled CTRW with infinite mean waiting times, the limiting particle location is given by the subordinated process $A(E(t))$. The density $h(x,t)$ of this process solves a fractional Cauchy problem $(d/dt)^\gamma h = Lh$ where $0 < \gamma < 1$ [3]. The next theorem extends this result to $1 < \gamma \leq 2$. Theorem 4.1 in [5] shows that the long-time CTRW limit process in this case is $A(H(t))$, and Corollary 4.2 in [5] shows that this limit process has density $\int_0^\infty p(x,s)\, d_s(H(t,s))$. The next theorem shows that this density is the Green's function solution to a fractional partial differential equation (6) with time derivative of order $1 < \gamma \leq 2$. This result extends Theorem 3.1 in [3] showing that these CTRW limits provide stochastic solutions of fractional Cauchy problems with time derivative of order $1 < \gamma \leq 2$.

Theorem 5 *Let L be the generator of an operator stable semigroup $(T(t))_{t\geq 0}$ on $L^1(\mathbb{R}^d)$ such that for $f \in L^1(\mathbb{R}^d)$, $Lf = \mathcal{F}^{-1}(\hat{L}(k)f(k))$. Let $a > 0, 1 < \gamma \leq 2$. Then there exists a unique distribution g such that*

$$-a\left(\frac{d}{dt}\right)^{\gamma} C(t) + \frac{d}{dt}C(t) = LC(t) + \delta(t)g \tag{6}$$

with conditions

$$C(0) = f, C'(0) = 0, C(t) \in L^1(\mathbb{R}^d)$$

for all $t \geq 0$ has a mild solution and this unique solution is given by

$$C(t,x) = \int_0^\infty T(s)f(x)\, d_s(H(t,s)).$$

For any other distribution g equation (6) has no solution.

5 Examples

In this section we give two examples showing how the more differentiated scaling in time influences the resulting model.

Example 5.1 *Consider a random walk composed of zero-mean, unit-variance Gaussian jumps with intervening waiting times that are in the domain of attraction of a Gaussian with unit mean and variance. In this case, the first passage time density of the Gaussian waiting times (with drift) is well known; see, for example, [29]. The governing equation becomes*

$$-a\frac{d^2}{dt^2}C(t,x) + \frac{d}{dt}C(t,x) = \mathcal{D}\frac{d^2}{dx^2}C(t,x) + \delta(t)g$$

with $C(0,x) = f(x)$ and $\partial C/\partial t = 0$ when $t = 0$. The solution is given by

$$C(t,x) = \int_0^\infty \frac{t}{s} \frac{1}{\sqrt{4\pi a s}} \exp\left(-\frac{(t-s)^2}{4as}\right) \frac{1}{\sqrt{4\pi \mathcal{D} s}} \exp\left(-\frac{x^2}{4\mathcal{D}s}\right) ds \star f,$$

with '$\star$' being the convolution operator in x.

Example 5.2 *To generalize the previous example, let the waiting times be regularly varying with unit mean and infinite variance, so the limit of waiting times $D(t)$ converges to a Lévy motion with index $1 < \gamma < 2$. Furthermore, let the jump sizes be symmetric and heavy tailed with index $1 < \alpha \leq 2$. Then the particle position density $C(x,t)$, assuming an initial particle distribution $C(0,x) = f(x)$ and $\partial C/\partial t = 0$ when $t = 0$, follows*

$$-a\left(\frac{\partial}{\partial t}\right)^\gamma C(t,x) + \frac{\partial}{\partial t} C(t,x) = -v\frac{\partial}{\partial x} C(t,x) + \frac{\partial^\alpha}{\partial |x|^\alpha} C(t,x) + \delta(t) g. \quad (7)$$

The mild solution is given by

$$C(t,x) = \int_0^\infty \frac{1}{s^{1/\alpha}} g_\alpha\left(\frac{x - vs}{s^{1/\alpha}}\right) \star f \, d_s(H(t,s)).$$

These two examples show that the particle motion has classical advective (drift) motion with additional dispersion in both space and time. In the first example, the effect of the second time derivative vanishes for large time. Note, however, that when the coefficient on the second time derivative a is larger, the density is more peaked at the origin and has more weight in the tails (Fig. 1). The main effect of the parameter γ, when less than two, is to cause particles to spend heavy-tailed amounts of time in an immobile state that decays very slowly (Figure 2). The density of the first passage time, which provides a map between the number of jumps (a particle's operational time) and the clock time, shows fewer jumps at any time for lower values of γ. For smaller values of γ, the time dispersion is noticeable for prolonged time periods (Figure 2), hence the effect of the γ–order derivative is important for longer periods. The solution to Example 5.2 is obtained by subordinating the shifted Lévy motion against these densities.

6 Conclusions

The study of classical random walks and, more recently, CTRW, has focused on the spatial dispersion of the particles. Temporal dispersion in the limit process is commonly assumed to be restricted to the infinite–mean waiting time CTRW [27,28,7,20]. However, by viewing the time process in a similar mathematical light as the space process, one sees that the time "drift" follows a different scaling procedure than the time dispersion. The time drift measures the linear portion of the map between clock time and the number of jumps, resulting in the familiar first time derivative in the governing equation of motion. If deviations in the waiting times have a power law distribution with finite mean but infinite variance, then the effects of the deviations do not disappear in the limit, and the governing equation has an additional time operator of order $1 < \gamma < 2$. If deviations in the waiting times have finite mean and variance, this procedure leads to an additional second-order time derivative. The resulting model gives a sharper description of space-time diffusive processes.

A Proof of Lemma 1

First we show uniqueness. Let $0 < \alpha < \pi/\gamma$,

$$\Gamma_{\pm} = \left\{ \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} e^{\pm i\theta} : 0 \leq \theta \leq \alpha \right\}$$

and

$$\Gamma_r = \left\{ \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)} \right)^{\frac{1}{\gamma-1}} e^{i\theta} : -\alpha \le \theta \le \alpha \right\}.$$

Then $\partial\,\Omega(\alpha) = \Gamma := \Gamma_- + \Gamma_r - \Gamma_+$ is clearly a simply closed path around $\Omega(\alpha)$ (see for example [24], Thm. 10.40). Let

$$p(z) = az^\gamma - z.$$

Then

$$\begin{aligned}
p\left(\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} e^{\pm i\theta} \right) &= a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{\gamma}{\gamma-1}} e^{\pm i\gamma\theta} - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} e^{\pm i\theta} \\
&= a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{\gamma}{\gamma-1}} \cos(\gamma\theta) - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \cos(\theta) \\
&\quad \pm i \left(a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{\gamma}{\gamma-1}} \sin(\gamma\theta) - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \sin(\theta) \right) \\
&= \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\theta)}{\sin(\gamma\theta)} \cos(\gamma\theta) - \cos(\theta) \right) \\
&= -\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\gamma\theta)}.
\end{aligned}$$

A quick calculation shows that for $0 < \theta < \pi/\gamma$, $\theta \mapsto \frac{\sin(\theta)}{\sin(\gamma\theta)}$ is an increasing function which implies that $\theta \mapsto \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\frac{\gamma}{\gamma-1}(\gamma-1)\theta)}$ is also increasing. Since

$$\lim_{\theta\to 0} \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\gamma\theta)} = a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$$

we have that the image of $\Gamma_\pm$ under p is a path on the negative real axis,

$$p(\Gamma_\pm) = \left[-\left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)} \right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\alpha)}{\sin(\gamma\alpha)}, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}} \right].$$

Investigating $p(\Gamma_r)$ we see that for $z \in \Gamma_r$,

$$\begin{aligned} p(z) &= a\left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{\gamma}{\gamma-1}} e^{i\gamma\theta} - \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} e^{i\theta} \\ &= \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)}{\sin(\gamma\alpha)}(\cos(\gamma\theta) + i\sin(\gamma\theta)) - \cos(\theta) - i\sin(\theta)\right) \\ &= \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)\cos(\gamma\theta)}{\sin(\gamma\alpha)} - \cos(\theta) + i\left(\frac{\sin(\alpha)\sin(\gamma\theta)}{\sin(\gamma\alpha)} - \sin(\theta)\right)\right). \end{aligned}$$

Using again that $\theta \mapsto \sin(\theta)/\sin(\gamma\theta)$ is increasing for $\theta > 0$, the imaginary part is positive iff θ is positive. Furthermore,

$$|p(z)| \geq \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)}{\sin(\gamma\alpha)} - 1\right)$$

for all $z \in \Gamma_r$, and this lower bound tends to infinity as $\alpha \to \pi/\gamma$. Hence we obtain for α large enough that $p(\Gamma_r)$ is a closed contour going once counterclockwise around the origin, which implies that $p(\Gamma)$ is a closed contour going once counterclockwise around the origin.

Fix $w \in \mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$, and then choose $\alpha < \pi/\gamma$ (α depends on w) such that $p(\Gamma)$ goes around w. Using the counting formula for zeros and poles, we see that there is exactly one $z \in \Omega(\alpha)$ (see for example [23], Theorem 13.2.2) such that $p(z) = w$. Furthermore (see for example [1], p.153),

$$q(w) := p^{-1}(w) = \frac{1}{2\pi i}\int_\Gamma \zeta \frac{p'(\zeta)}{p(\zeta) - w}\, d\zeta$$

is a holomorphic function on $\mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$.

Next, we show that $1/q$ is the Laplace transform of an analytic function f on $(0, \infty)$ with $t^{1-1/\gamma}e^{-\omega t}f(t)$ bounded for $\omega > -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$.

Since $aq(z)^\gamma - q(z) = z$ we have that for $|z|$ large enough, $(a+1)|q(z)|^\gamma > |aq(z)^\gamma - q(z)| = |z|$. Furthermore, $q(z) \in \Omega(\pi/\gamma)$ and thus $q(z)$ is bounded away from zero. Hence there exists $M > 0$ such that

$$M|q(z)|^{\gamma} \geq |z| \tag{A.1}$$

for all $z \notin (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$. Let

$$r(z) := \frac{d}{dz}\frac{1}{q(z)} = -q(z)^{-2}\frac{1}{a\gamma q(z)^{\gamma-1} - 1}$$

using the fact that $dq/dz = 1/(dp/dz)$. The function $z \mapsto \frac{1}{a\gamma q(z)^{\gamma-1}-1}$ has a single pole at $z = -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$ and is bounded off a neighborhood of that pole. Choose $\omega < 0$ such that $\omega > -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$. Then for $\Sigma_\alpha := \{re^{i\delta} : r > 0, -\alpha < \delta < \alpha\}$ we have that

$$\sup_{z \in \omega + \Sigma_{\delta+\pi/2}} |(z-\omega)r(z)| = \sup_{z \in \omega + \Sigma_{\delta+\pi/2}} \left| \frac{z-\omega}{q(z)^2} \frac{1}{a\gamma q(z)^{\gamma-1} - 1} \right| < \infty$$

for all $0 < \delta < \pi/2$. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$. By the analytic representation theorem for Laplace transforms ([2] Theorem 2.6.1) there exists a holomorphic function $f : \mathbb{C}_+ \to \mathbb{C}$ such that $\sup_{z \in \mathbb{C}_+} |e^{-\omega z} f(z)| < \infty$ and

$$r(z) = \int_0^\infty e^{-zt} f(t)\, dt.$$

Furthermore,

$$\sup_{\Re(z)>0} |z z^{\frac{1}{\gamma}} r(z)| = \left| \frac{z^{2/\gamma}}{q(z)^2} \frac{z^{\frac{\gamma-1}{\gamma}}}{a\gamma q(z)^{\gamma-1} - 1} \right| < \infty.$$

Using the complex representation theorem ([2] Theorem 2.5.1 with $b = 1/\gamma$ and $q(z) = z^{\frac{1}{\gamma}} r(z)$) there exists $g \in C[0,\infty)$ with $\sup_{t>0} t^{-1/\gamma}|g(t)| < \infty$ such that

$$z^{1/\gamma} r(z) = z^{\frac{1}{\gamma}} \int_0^\infty e^{-zt} g(t)\, dt.$$

By the uniqueness of the Laplace transform $f = g$ and hence

$$\sup_{t \geq 0} |t^{-\frac{1}{\gamma}} e^{-\omega t} f(t)| < \infty.$$

Clearly, $F(t) = -f(t)/t \in L^1(0,\infty)$ and hence, using Fubini,

$$\int_0^\infty e^{-zt} F(t)\, dt = -\int_z^\infty \int_0^\infty e^{-st} f(t)\, dt\, ds = -\int_z^\infty r(s)\, ds = 1/q(z)$$

for $z > 0$. Thus the Laplace transform of $F(t)$ is $1/q$ and the growth conditions follow with $\zeta = -\omega$. $\square$

B Simplifying the Laplace-Laplace transform of the Maximum Process

Clearly, $z > 0$ implies $q(z) > 0$. Thus the second pole of the integrand of (1),

$$\frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))},$$

is at $\xi = iq(x)$. For $\Gamma_n = \{ne^{i\theta} : 0 \leq \theta \leq \pi\}$ there exists $M > 0$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} \left| \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))} \right| d\xi \\
&\quad \leq \lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} \frac{M}{n^2}\, d\xi = \lim_{n\to\infty} M/2n = 0.
\end{aligned}$$

Thus, we can apply the residue theorem and obtain

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^\infty \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))}\, d\xi \\
&\quad = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{[-n,n]\cup\Gamma_n} \frac{\lambda}{(\xi - i\lambda)} \frac{-1 + a(-i\xi)^{\gamma-1}}{x(x - (i\xi + a(-i\xi)^\gamma))}\, d\xi \\
&\quad = \lambda \frac{-1 + a\lambda^{\gamma-1}}{x(x - (-\lambda + a\lambda^\gamma))} + \frac{\lambda}{iq(x) - i\lambda} \frac{-1 + a(q(x))^{\gamma-1}}{x(-i + i\gamma a q(x)^{\gamma-1})} \\
&\quad = \left(\frac{-\lambda + a\lambda^\gamma}{x(x + \lambda - a\lambda^\gamma)} + \frac{\lambda}{x(q(x) - \lambda)} \frac{-1 + a(q(x))^{\gamma-1}}{(1 - \gamma a q(x)^{\gamma-1})} \right).
\end{aligned}$$

Since q is an inverse function we can compute its derivative

$$\frac{d}{dx} q(x) = \frac{1}{a\gamma q(x)^{\gamma-1} - 1}$$

as in the proof of Lemma 1. Thus for $u > a\lambda^\gamma - \lambda$,

$$\int_u^\infty \left(\frac{-\lambda + a\lambda^\gamma}{x(x+\lambda - a\lambda^\gamma)}\right) + \frac{\lambda}{x(q(x)-\lambda)}\frac{-1+a(q(x))^{\gamma-1}}{(1-\gamma a q(x)^{\gamma-1})}dx$$

$$\begin{aligned}
&= [\ln(1+(\lambda - a\lambda^\gamma)/x)]_u^\infty - \lambda \int_{q(u)}^\infty \frac{-1+ax^{\gamma-1}}{(ax^\gamma - x)(x-\lambda)}dx \\
&= -\ln(1+(\lambda - a\lambda^\gamma)/u) - [\ln(1-\lambda/x)]_{q(u)}^\infty \\
&= \ln\left(\frac{1-\lambda/q(u)}{1+(\lambda - a\lambda^\gamma)/u}\right) = \ln\left(\frac{u - u\lambda/q(u)}{u+\lambda - a\lambda^\gamma}\right).
\end{aligned}$$

Using L'Hopital's rule, we see that for $a\lambda^\gamma - \lambda > 0$, $\lim_{u\to a\lambda^\gamma-\lambda} \ln\left(\frac{u-u\lambda/q(u)}{u+\lambda-a\lambda^\gamma}\right) < \infty$, since the expression inside the logarithm tends to a finite constant as $u \to a\lambda^\gamma - \lambda$. Then it is not hard to show that the above equality holds for all $u > 0$ (integrate from u to $a\lambda^\gamma - \lambda - \varepsilon$ and $a\lambda^\gamma - \lambda + \varepsilon$ to infinity and then let $\varepsilon \to 0$). Thus, using (1),

$$\int_0^\infty \int_0^\infty e^{-us-\lambda T} d_T(M(s,T))ds = \frac{1-\lambda/q(u)}{u+\lambda - a\lambda^\gamma}. \qquad \Box$$

C Proof of Theorem 2

Clearly, the inverse in u of (2) is given by [5]

$$\begin{aligned}
\tilde{H}(s,\lambda) =&(1-a\lambda^{\gamma-1})\int_0^s \exp(-r(\lambda - a\lambda^\gamma))\,dr \\
&+ \int_0^s m(s-r)\exp(-r(\lambda - a\lambda^\gamma))\,dr \\
=&\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda} + \int_0^s m(s-r)\exp(-r(\lambda - a\lambda^\gamma))\,dr.
\end{aligned}$$

Inverting with respect to λ is a bit more delicate. Using the complex inversion formula (see for example [31], Thm 7.6) we obtain

[5] As the Laplace transform in λ has to stay bounded as $\lambda \to \infty$, we see that convolution with the function m has to have the following effect for all $s > 0$:

$$m \star \exp(-s(\lambda - a\lambda^\gamma)) \approx \exp(-s(\lambda - a\lambda^\gamma))/\lambda.$$

$$
\begin{aligned}
\int_0^t H(s,r)dr &= \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\Bigg(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda} \\
&\qquad + \int_0^s m(s-r)\exp(-r(\lambda - a\lambda^\gamma))\,dr\Bigg)\,d\lambda \\
&= \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\left(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda}\right)d\lambda \\
&\quad + \int_0^s m(s-r)\frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\exp(-r(\lambda - a\lambda^\gamma))\,d\lambda\,dr
\end{aligned}
$$

the second equality holding due to Fubini since

$$
\begin{aligned}
\int_0^s |m(s-r)| \int_{c+i\mathbb{R}} & \frac{|e^{\lambda t}|}{|\lambda|}|\exp(-r(\lambda - a\lambda^\gamma))|\,d\lambda\,dr \\
&= e^{ct}\int_0^{s/2} |m(s-r)| \int_{c+i\mathbb{R}} \frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda\,dr \\
&\quad + e^{ct}\int_{s/2}^s |m(s-r)| \int_{c+i\mathbb{R}} \frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda\,dr \\
&\le e^{ct}\sup_{s/2<r<s}|m(r)|\int_{c+i\mathbb{R}}\int_0^{s/2}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,dr\,d\lambda \\
&\quad + e^{ct}\|m\|_1 \sup_{s/2<r<s}\int_{c+i\mathbb{R}}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda \\
&< \infty.
\end{aligned}
$$

Making a change of variables we see that on the right hand side we have an expression akin to the formula for the inversion of the Fourier transform.

$$
\begin{aligned}
\frac{1}{2\pi i} & \int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\left(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda}\right)d\lambda \\
&\quad + \int_0^s m(s-r)\frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\exp(-r(\lambda - a\lambda^\gamma))\,d\lambda\,dr \\
&= \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\left(\frac{1-\exp(-s(iu+c-a(iu+c)^\gamma))}{(iu+c)^2}\right)du \\
&\quad + \int_0^s m(s-r)\frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\frac{\exp(-r(iu+c-a(iu+c)^\gamma))}{iu+c}\,du\,dr \quad \text{(C.1)}
\end{aligned}
$$

Now the Fourier transform of a shifted γ-stable distribution is

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(ar)^{1/\gamma}} g_\gamma \left(\frac{x-r}{(ar)^{1/\gamma}} \right) dx = e^{-ikr+ra(ik)^\gamma}.$$

Since $e^{-cx} g_\gamma(x)$ is bounded for all $x \in \mathbb{R}$ for some $c \geq 0$ (e.g., [30], Theorem 4.7.1) we obtain that

$$\int_{-\infty}^{\infty} e^{-iux} \frac{e^{-cx}}{(ar)^{1/\gamma}} g_\gamma \left(\frac{x-r}{(ar)^{1/\gamma}} \right) dx = e^{-r(iu+c+a(iu+c)^\gamma)}.$$

Hence, the expressions in (C.1) are indeed inverse Fourier transforms and

$$\begin{aligned}
\frac{e^{ct}}{2\pi} \quad & \int_{-\infty}^{\infty} e^{iut} \left(\frac{1-\exp(-s(iu+c-a(iu+c)^\gamma))}{(iu+c)^2} \right) du \\
& + \int_0^s m(s-r) \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{iut} \frac{\exp(-r(iu+c-a(iu+c)^\gamma))}{iu+c} \, du \, dr \\
= & \, t - \int_{-\infty}^{t} \int_{-\infty}^{w} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx \, dw \\
& + \int_0^s m(s-r) \int_{-\infty}^{t} \frac{1}{(ar)^{1/\gamma}} g_\gamma \left(\frac{x-r}{(ar)^{1/\gamma}} \right) dx \, dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
H(s,t) &= 1 - \int_{-\infty}^{t} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx + \int_0^s \frac{m(s-r)}{(ar)^{1/\gamma}} g_\gamma \left(\frac{t-r}{(ar)^{1/\gamma}} \right) dr \\
&= \int_{t}^{\infty} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx + \int_0^s \frac{m(s-r)}{(ar)^{1/\gamma}} g_\gamma \left(\frac{t-r}{(ar)^{1/\gamma}} \right) dr \\
&= \int_{\frac{t-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u) \, du + \int_0^s \frac{m(s-u)}{(au)^{1/\gamma}} g_\gamma \left(\frac{t-u}{(au)^{1/\gamma}} \right) du. \qquad \text{(C.2)}
\end{aligned}$$

□

D Proof of Theorem 4

Assume there exists a solution to (5). Using the Caputo derivative and the fact that $u(0,s) = \delta(s)$, the Laplace-Laplace transform of $\left(\frac{d}{dt}\right)^\gamma u(t,s)$ is $\lambda^\gamma \tilde{\tilde{u}}(\lambda, r) -$

$\lambda^{\gamma-1}$. Then it follows easily that

$$\tilde{\tilde{u}}(\lambda, r) = \frac{1 - a\lambda^{\gamma-1} + \tilde{f}(r)}{r + \lambda - a\lambda^{\gamma}}.$$

Since $s \mapsto u(t,s)$ is a probability density, $|\tilde{u}(t,r)| \leq 1$. Hence, the Laplace-Laplace transform for each r is analytic in the right halfplane in λ. Thus $1 - a\lambda^{\gamma-1} + \tilde{f}(r) = 0$ if $r + \lambda - a\lambda^{\gamma} = 0$ or equivalently, $\lambda = q(r)$, q given by Lemma 1. Hence

$$\tilde{f}(r) = \frac{-\lambda + a\lambda^{\gamma}}{\lambda} = r/q(r).$$

Since the range of $\lambda \mapsto a\lambda^{\gamma} - \lambda$ contains the right halfplane, $\tilde{f}$ is uniquely determined, and so is f.

The first passage time distribution has Laplace-Laplace transform

$$\int_0^\infty \int_0^\infty e^{-rs-\lambda T} H(s,T)\, ds\, dT = \frac{1 - a\lambda^{\gamma-1} + r/q(r)}{r(r + \lambda - a\lambda^{\gamma})}$$

by (2). Since $H(0,T) = 0$ for all $T > 0$ we have that

$$\int_0^\infty \int_0^\infty e^{-rs-\lambda T}\, d_s(H(s,T))\, dT = \frac{1 - a\lambda^{\gamma-1} + r/q(r)}{r + \lambda - a\lambda^{\gamma}} \qquad \text{(D.1)}$$

and hence the density of the first passage time distribution has the same Laplace-Laplace transform as the solution to the differential equation and it is therefore a mild solution. □

E Proof of Theorem 5

Taking the Fourier transform of (6) we obtain

$$-a\left(\frac{d}{dt}\right)^{\gamma} \hat{C}(t,k) + \frac{d}{dt}\hat{C}(t,k) = \hat{L}(k)\hat{C}(t,k) + \delta(t)\hat{g}(k).$$

Taking the Laplace transform in t yields

$$-a\lambda^\gamma \tilde{\hat{C}}(\lambda,k) + a\lambda^{\gamma-1}\hat{f}(k) + \lambda\tilde{\hat{C}}(\lambda,k) - \hat{f}(k) = \hat{L}(k)\tilde{\hat{C}}(\lambda,k) + \hat{g}(k).$$

Thus

$$\tilde{\hat{C}}(\lambda,k) = \frac{\hat{f}(k) - a\lambda^{\gamma-1}\hat{f}(k) + \hat{g}(k)}{-a\lambda^\gamma + \lambda - \hat{L}(k)}.$$

Since $\tilde{C}(\lambda) \in L^1(\mathbb{R}^d)$ we know that its Fourier transform has to be bounded for all $\Re(\lambda) > 0$. This implies that the numerator of the above equation has to be zero whenever the denominator is equal to zero, or whenever $q(-\hat{L}(k)) = \lambda$. Hence

$$\hat{g}(k) = (-1 + aq(-\hat{L}(k))^{\gamma-1})\hat{f}(k) = -\hat{L}(k)\hat{f}(k)/q(-\hat{L}(k))$$

is uniquely determined. Thus,

$$\tilde{\hat{C}}(\lambda,k) = \frac{1 - a\lambda^{\gamma-1} - \hat{L}(k)/q(-\hat{L}(k))}{-a\lambda^\gamma + \lambda - \hat{L}(k)}\hat{f}(k).$$

To see that $C(t) = \int_0^\infty T(s)f\, d_s(H(t,s))$ has the same Fourier-Laplace transform, take the Fourier transform and observe that

$$\hat{C}(t,k) = \int_0^\infty e^{s\hat{L}(k)}\, d_s(H(t,s))\hat{f}(k).$$

But this is the Laplace transform in s of the first passage time density evaluated at $-\hat{L}(k)$ times $\hat{f}(k)$, i.e. taking the Laplace transform in t, using (D.1) we see that

$$\begin{aligned}\tilde{\hat{C}}(\lambda,k) &= \int_0^\infty \int_0^\infty e^{-\lambda t - s(-\hat{L}(k))}\, d_s(H(t,s))\, dt\, \hat{f}(k) \\ &= \frac{1 - a\lambda^{\gamma-1} - \hat{L}(k)/q(-\hat{L}(k))}{-a\lambda^\gamma + \lambda - \hat{L}(k)}\hat{f}(k),\end{aligned}$$

and therefore $C(t,x)$ is indeed the mild solution of (6).

Finally we prove that g is a distribution. The proof depends on the fact that for some positive real constant C we have

$$|\hat{L}(k)| \leq C \max\{\|k\|, \|k\|^2\} \quad \text{for all } k \in \mathbb{R}^d. \tag{E.1}$$

To see this, use the Lévy Representation [16] to write

$$\hat{L}(k) = ik \cdot a - \frac{1}{2} k \cdot Ak + \int_{x \neq 0} \left(e^{ik \cdot x} - 1 - \frac{ik \cdot x}{1 + \|x\|^2} \right) \phi(dx)$$

where $a \in \mathbb{R}^d$, A is a nonnegative definite matrix, and ϕ is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{x \neq 0} \min\{1, \|x\|^2\} \phi(dx) < \infty. \tag{E.2}$$

The integral term I in the Lévy Representation satisfies $|I| \leq |I_1| + |I_2|$ with

$$\begin{aligned} I_1 &= \int_{0 < \|x\| < 1} \left(e^{ik \cdot x} - 1 - \frac{ik \cdot x}{1 + \|x\|^2} \right) \phi(dx) \\ &= I_{11} + I_{12} \end{aligned}$$

where

$$\begin{aligned} |I_{11}| &= \left| \int_{0 < \|x\| < 1} \left(e^{ik \cdot x} - 1 - ik \cdot x \right) \phi(dx) \right| \\ &\leq \int_{0 < \|x\| < 1} \frac{1}{2} \|k\|^2 \|x\|^2 \phi(dx) \\ &\leq C_1 \|k\|^2 \end{aligned}$$

and

$$\begin{aligned} |I_{12}| &= \left| ik \cdot \int_{0 < \|x\| < 1} \left(x - \frac{x}{1 + \|x\|^2} \right) \phi(dx) \right| \\ &\leq \|k\| \int_{0 < \|x\| < 1} \left(\frac{\|x\|^3}{1 + \|x\|^2} \right) \phi(dx) \\ &\leq C_2 \|k\| \end{aligned}$$

while

$$|I_2| = \int_{\|x\|\geq 1} \left(e^{ik\cdot x} - 1 - \frac{ik \cdot x}{1+\|x\|^2} \right) \phi(dx)$$
$$\leq D + D + D\|k\|$$

where $D = \phi\{x : \|x\| \geq 1\} < \infty$ using the fact that $|e^{ik\cdot x}| = 1$ and $\|x\|/(1 + \|x\|^2) \leq 1$ for $\|x\| \geq 1$. Then (E.1) holds.

Since $\hat{g}(k) = -\hat{L}(k)\hat{f}(k)/q(-\hat{L}(k))$ and $|q(z)| \geq M_0|z|^{1/\gamma}$ for almost all $z \in \mathbb{R}^d$ by (A.1) we have

$$\left| \frac{-\hat{L}(k)}{q(-\hat{L}(k))} \right| \leq \frac{1}{M_0} |\hat{L}(k)|^{1-1/\gamma}$$

and note that $1 - 1/\gamma > 0$. Using (E.1) we obtain

$$|\hat{g}(k)| \leq M_1 |\hat{f}(k)| \max\{\|k\|^{1-1/\gamma}, \|k\|^{2-2/\gamma}\},$$

and then it follows easily that for some $D > 0$ we have

$$\int (1 + \|k\|^2)^{-D} \hat{g}(k) dk < \infty.$$

Now Example 7.12 (b) on p. 191 of [25] shows that g is a tempered distribution. $\square$

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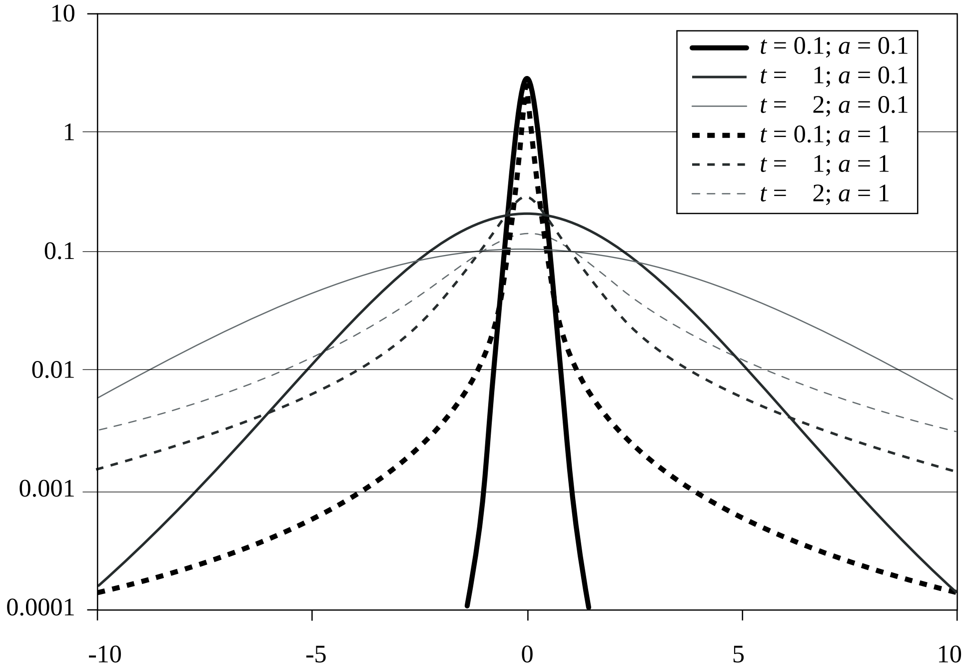

Fig. 1. Solutions to Example 5.1 with pulse initial condition. Solid lines are with $a = 0.1$ and dashed lines are for $a = 1$, shown for various t and with diffusion parameter $\mathcal{D}$ set to 1.

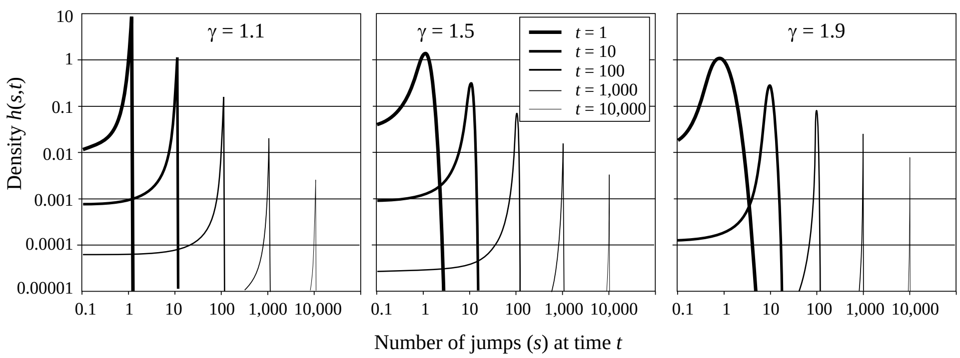

Fig. 2. First passage time densities $d_s(H(s,t))$ corresponding to γ–stable waiting time processes with a=0.1 and unit drift. In the model, the first passage time tracks the random number of jumps at time t.