Long mollifiers of the Riemann zeta-function

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1. Introduction

The best current bounds for the proportion of zeros of $\zeta(s)$ on the critical line are due to Conrey [C], using Levinson's method [Lev]. This method can also be used to detect simple zeros on the critical line. To apply Levinson's method one first needs an asymptotic formula for the mean-square from 0 to T of $\zeta(s)M(s)$ near the $\frac{1}{2}$-line, where

$$M(s) = M(s, h(x)) = \sum_{n \le y} \frac{\mu(n)\, h\!\left(\frac{\log y/n}{\log y}\right)}{n^s},$$

where $\mu(n)$ is the Möbius function, $h(x)$ is a real polynomial with $h(0) = 0$, and $y = T^\theta$ for some $\theta > 0$. It turns out that the parameter θ is critical to the method: having an asymptotic formula valid for large values of θ is necessary in order to obtain good results. For example, if we let κ denote the proportion of nontrivial zeros of $\zeta(s)$ which are simple and on the critical line, then having the formula valid for $0 < \theta < \frac{1}{2}$ yields $\kappa > 0.3562$, having $0 < \theta < \frac{4}{7}$ gives $\kappa > 0.40219$, and it is necessary to have $\theta > 0.165$ in order to obtain a positive lower bound for κ. At present, it is known that the asymptotic formula remains valid for $0 < \theta < \frac{4}{7}$, this is due to Conrey. Without assuming the Riemann Hypothesis, Levinson's method provides the only known way of obtaining a positive lower bound for κ.

On the Riemann Hypothesis there are two other methods presently known for detecting simple zeros. Montgomery's pair correlation method involves considering the function

$$F(\alpha, T) = N(T)^{-1} \sum_{0<\gamma,\gamma'<T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

where $N(T)$ is the number of zeros of $\zeta(s)$ in $0 < \gamma < T$, and $w(u) = \dfrac{4}{4+u^2}$. Montgomery, [M1], showed that on the Riemann Hypothesis

$$F(\alpha, T) = \big(1 + o(1)\big) T^{-2\alpha} \log T + \alpha + o(1)$$

as $T \to \infty$, uniformly for $0 \le \alpha \le 1$; this can be used [M2] to show that $\kappa > 0.6725$. Montgomery's Hypothesis is that $F(\alpha, T) = 1 + o(1)$ as $T \to \infty$, uniformly for $1 \le \alpha \le A$, for any fixed A; this would imply that $\kappa = 1$. Montgomery's Hypothesis essentially says that the sum defining $F(\alpha, T)$ is dominated by the diagonal terms for $\alpha > 1$. The other known method of detecting simple zeros is due to Conrey, Ghosh, and Gonek [CGG1], [CGG2]. Just as in Levinson's method, one starts with $\zeta(s)B(s)$ for some Dirichlet polynomial $B(s)$, but this time one considers discrete mean values involving sums over the zeros of $\zeta(s)$. Assuming the Riemann Hypothesis and the Generalized Lindelöf Hypothesis, this method yields $\kappa > 0.7037$.

The original motivation for this work was to explore the possibility of improving the results obtained by Levinson's method. In particular, the asymptotic formula for the mean value integral is rather complicated, and it would be useful to really understand how the various terms arise. Also, it is desirable to extend the range of θ for which the formula is valid. For some time it has been thought that this formula could not remain valid for $\theta > 1$. This is suggested by a result of Balasubramanian, Conrey, and Heath-Brown [BCH-B]. They obtain an asymptotic formula for the mean square on the $\frac{1}{2}$-line of $\zeta(s)B(s)$ where $B(s)$ is an arbitrary Dirichlet polynomial, and show that their formula is valid for $\theta < \frac{1}{2}$. They conjecture the "$\theta = 1$ conjecture," namely, that their formula remains valid for all $\theta < 1$, and they also exhibit a Dirichlet polynomial $B(s)$ for which their formula *fails* for $\theta > 1$. Similar $\theta = 1$ conjectures are made by Conrey and Ghosh [CG]. The main purpose of this paper is to show that, contrary to the general case, one would expect the formulas involving $\zeta(s)M(s)$ to remain valid for all fixed $\theta > 0$. We will refer to this as the "$\theta = \infty$ conjecture." In the course of doing this we will show that there is a close connection between the mean-value integral required for Levinson's method, Montgomery's function $F(\alpha, T)$, and discrete mean-value sums required for the method of Conrey, Ghosh, and Gonek. As a consequence of the method we use to exhibit this connection we will also obtain asymptotic formulas for several integrals involving $\zeta(s)$ and sums over the zeros of $\zeta(s)$.

In the next section we present the main results of the paper. In Section 3 we present the reasoning behind the $\theta = \infty$ conjectures. Starting with Section 4 we give the proofs of our results. In Section 7 we indicate some consequences of $\theta = \infty$ and give some related conjectures. All of the Propositions and Theorems in the next section are spinoffs of the methods we employed to justify our conjectures.

2. Notation and statement of results

The mollifier $M(s)$, so called because it is designed to dampen the wild behavior of $\zeta(s)$ near the critical line, depends on the function $h(x)$ which is constrained by $h(0) = 0$. For most of this paper we only consider the case $h(x) = x$. Throughout the paper we assume that all of the zeros of $\zeta(s)$ are simple, and we refer to this assumption as SZ. We assume this merely as a convenience, all the manipulations we do can be modified to avoid this assumption. We also assume the Riemann Hypothesis, referred to as RH. We consider the integral

$$I(T, y, a) = \int_0^T |\zeta M(\tfrac{1}{2} + a + it)|^2 dt,$$

which is a simplification of the integral needed for Levinson's method, and the sum

$$S(T, y, a) = \sum_{0<\gamma<T} \zeta M(\rho + a),$$

which is one of the sums used in the method of Conrey, Ghosh, and Gonek. The general versions of these are repeated as (7.1) and (7.2). We have put $y = T^\theta$, which is the length of $M(s)$, and usually a is a small positive number which may depend on T. We consider θ to be fixed and our error terms are not, in general, uniform in θ. In particular, most of our asymptotic formulas do not remain valid if $\theta \to 0$ as $T \to \infty$. As usual, we denote a nontrivial zero of $\zeta(s)$ by $\rho = \frac{1}{2} + i\gamma$, and we assume that our large parameter T satisfies $|T - \gamma| \gg 1/\log T$ for all γ. This does not involve any loss in generality. We write $f \approx g$ for $g \ll f \ll g$. As usual, $s = \sigma + it$.

Now we can state our results. A consequence of Theorem 1 of [CGG1], see (7.1), is

$$I(T, y, a) \sim T + \frac{1}{\log^2 y}\left(T\,\frac{1 - T^{-2a}}{4a^2}\right) - \frac{y^{-2a}}{\log^2 y}\left(T\,\frac{1 - T^{-2a}}{4a^2}\right). \tag{2.1}$$

This formula is valid for $0<\theta<\frac{4}{7}$ and $0<a\ll 1/\log T$. One motivation for this work was to gain a better understanding of how the various terms on the right side of (2.1) arise. Towards this end we have

Proposition 1. *On SZ and RH, if $0<a<\frac{1}{4}$ then*

$$I(T,y,a)=T-\frac{1}{\log^2 y}\big(I_1(T,a)-4\pi\, S_1(T,2a)\big)+\frac{y^{-2a}}{\log^2 y}\big(I_2(T,\theta,a)-4\pi\, S_2(T,\theta,2a)\big)$$

$$+\,O\Big(I_2^{\frac{1}{2}}(T,\theta,a)y^{-\frac{5}{2}-a}+T/\log y+T^{\epsilon}y^{\epsilon}\Big)$$

where

$$I_1(T,a)=\int_0^T\left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right|^2 dt,$$

$$I_2(T,\theta,a)=\int_0^T|\zeta(\tfrac{1}{2}+a+it)|^2\left|\sum_{\rho}\frac{T^{i\theta\gamma}}{\zeta'(\rho)(a+i(t-\gamma))^2}\right|^2 dt,$$

$$S_1(T,a)=\sum_{0<\gamma<T}\frac{\zeta'}{\zeta}(\rho+a),$$

and

$$S_2(T,\theta,a)=\sum_{0<\gamma<T}\sum_{\rho'}T^{i\theta(\gamma'-\gamma)}\,\frac{\zeta(\rho+a)}{\zeta'(\rho')\big(a+i(\gamma-\gamma')\big)^2}.$$

Note that since we are assuming RH and SZ, the above sums and integrals all exist.

For the discrete case we have, by Theorem 3 of [CGG1], see (7.2),

$$S(T,y,a)\sim\frac{1}{2\pi}T\log T-\frac{1}{\log y}\left(\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^2}\right)+\frac{y^{-a}}{\log y}\left(\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^2}\right).\tag{2.2}$$

This formula is valid for $0<\theta<\frac{1}{2}$ and $|a|\ll 1/\log T$. Its counterpart is

Proposition 2. *With the same conditions and notation as in Proposition 1 we have*

$$S(T,y,a)=\frac{1}{2\pi}T\log T-\frac{1}{\log y}S_1(T,a)+\frac{y^{-a}}{\log y}S_2(T,\theta,a)+O\big(y^{-2-a}\big).$$

To reiterate our point, we direct the reader to compare (2.1) and (2.2) to the first line of Proposition 1 and 2, respectively. It is also interesting to compare $S_2(T,\theta,a)$ to Montgomery's function $F(\alpha,T)$. Recall that our goal is to understand $I(T,y,a)$ and $S(T,y,a)$ for large θ. The $\theta=\infty$ conjecture essentially says that $I(T,y,a)$ and $S(T,y,a)$ will not change shape as θ becomes large. Equivalently, we could phrase this as a conjecture about the main terms in Propositions 1 and 2. The integral I_1 has been evaluated by Goldston and Gonek, and this directly leads to an expression for S_1; this is done in Lemmas 3a and 3b. Even without an explicit formula for these terms it is clear that any potential change in shape of the right side of Proposition 1 or 2 is completely governed by the terms $I_2(T,\theta,a)$ and $S_2(T,\theta,a)$. These are evaluated in the next two Theorems.

Theorem 1. *Assume RH and SZ. If $T^{-\frac{1}{2}}\log T \ll a \ll 1/\log T$, and $0<\theta<\frac{1}{2}$ then*

$$\int_0^T |\zeta(\tfrac{1}{2}+a+it)|^2 \left|\sum_\rho \frac{T^{i\theta\gamma}}{\zeta'(\rho)\big(a+i(t-\gamma)\big)^2}\right|^2 dt$$

$$\sim T\left(\frac{1-T^{-2a}}{4a^2} + T^{2a\theta}\log^2 T\int_1^\infty \big(F(\alpha,T)-1\big)T^{-2a\alpha}d\alpha\right).$$

If, in addition, we assume Montgomery's Hypothesis, then for $a \approx 1/\log T$,

$$\int_0^T |\zeta(\tfrac{1}{2}+a+it)|^2 \left|\sum_\rho \frac{T^{i\theta\gamma}}{\zeta'(\rho)\big(a+i(t-\gamma)\big)^2}\right|^2 dt \sim T\,\frac{1-T^{-2a}}{4a^2}.$$

Theorem 2. *Assume RH and SZ. If $T^{-\frac{1}{2}}\log T \ll a \ll 1/\log T$, and $0<\theta<\frac{1}{2}$ then*

$$\sum_{0<\gamma<T}\sum_{\gamma'} T^{i\theta(\gamma'-\gamma)}\frac{\zeta(\rho+a)}{\zeta'(\rho')\big(a+i(\gamma-\gamma')\big)^2}$$

$$\sim \frac{T}{2\pi}\left(\frac{1-T^{-a}}{a^2} + T^{a\theta}\log^2 T\int_1^\infty \big(F(\alpha,T)-1\big)T^{-a\alpha}d\alpha\right).$$

If, in addition, we assume Montgomery's Hypothesis, then for $a \approx 1/\log T$,

(2.3) $$\sum_{0<\gamma<T}\sum_{\gamma'} T^{i\theta(\gamma'-\gamma)}\frac{\zeta(\rho+a)}{\zeta'(\rho')\big(a+i(\gamma-\gamma')\big)^2} \sim \frac{T}{2\pi}\,\frac{1-T^{-a}}{a^2}.$$

Writing $I_2(T,\theta,a)$ as a complex integral and then moving the path right or left leads to

(2.4) $$I_2(T,\theta,a) = 2\pi\, S_2(T,\theta,2a) + E(T,\theta,a),$$

where $E(T,\theta,a)$ is the resulting integral. By combining Theorems 1 and 2 we see that $E(T,\theta,a)$ is small. A direct proof that $E(T,\theta,a)$ is small would give either of Theorems 1 and 2 as a corollary to the other. Unfortunately it seems that bounding $E(T,\theta,a)$ directly is quite difficult. Finally, to indicate the connection between $I(T,y,a)$ and $S(T,y,a)$ we have

Theorem 3. *Assume RH and SZ. If* $T^{-\frac{1}{2}}\log T \ll a \ll 1/\log T$ *and* $y = T^{\theta}$ *with* $0 < \theta < \frac{1}{2}$ *then*

$$I(T,y,a) \sim T\left(1 + 2\frac{\log T}{\log y}\right) - T\,\frac{1-y^{-2a}}{\log^2 y}\,\frac{1-T^{-2a}}{4a^2} - \frac{4\pi}{\log y} S(T,y,2a).$$

We note that the restriction $a \ll 1/\log T$ in our Theorems is inherited from the same restriction in (7.1) and (7.2). In Section 7, following formula (7.8), we indicate reasons why this restriction may be essential; that is, the main term of (7.1) may change if $a \log T \to \infty$. The restriction $a \gg T^{-\frac{1}{2}}\log T$ originates in Lemma 3a, and removing this restriction would require a careful analysis of the behavior of $\zeta(s)$ near its zeros. This would probably introduce another main term when a is small.

3. The $\theta = \infty$ conjectures

Now we explain the reasoning behind the $\theta = \infty$ conjectures. Recall that we are in the simplest possible case $h(x) = x$. The idea is this: the reasoning which suggests the $\theta = 1$ conjecture also suggests that formula (2.2) and Theorems 1 and 2 should remain valid for $0 < \theta < 1$. On the other hand, "diagonal reasoning," such as leads to Montgomery's Hypothesis, can be used to predict the shape of the various expressions for $\theta > 1$. It then remains to show that the predicted formula for large θ has the same shape as the calculated formula for small θ.

First we consider the discrete case. In Proposition 2 the first two terms, $\frac{1}{2\pi}T\log T$ and $\frac{1}{\log y}S_1(T,a)$, depend on θ in an obvious and explicit way and this dependence will not change shape as θ increases. Thus, any change in shape of $S(T,y,a)$ must come from a corresponding change in shape of $S_2(T,\theta,a)$. For $\theta > 1$ the same reasoning as leads to Montgomery's Hypothesis suggests that S_2 is dominated by those terms where $\rho = \rho'$, and so would not depend on θ. Theorem 2 shows that, if we assume *RH*, *SZ*, and Montgomery's Hypothesis, then S_2 does not depend on θ for $0 < \theta < \frac{1}{2}$. This suggests that it is the diagonal terms which form the entire main term of S_2 in this range, and this is in fact the case, which can be seen as follows. That S_2 is dominated by its diagonal terms means

$$\frac{1}{a^2}\sum_{0<\gamma<T}\frac{\zeta(\rho+a)}{\zeta'(\rho)} \sim \frac{T}{2\pi}\,\frac{1-T^{-a}}{a^2}. \tag{3.1}$$

By Lemma 4 in Section 5 this formula holds only assuming SZ. Thus it is in fact the diagonal terms which give the main term of $S_2(T,\theta,a)$ for $0 < \theta < \frac{1}{2}$. We conclude that the shape of $S_2(T,\theta,a)$ should not change as θ becomes large, and so the same is true for $S(T,y,a)$, whence the discrete $\theta = \infty$ conjecture.

It is perhaps surprising that the sum in (2.3) is dominated by the diagonal terms for small θ, in contrast to Montgomery's function $F(\alpha,T)$. Multiplying (2.3) by a, putting $a = R/\log T$, and expanding in R we find that the coefficient of R^{n+2} on the left side is

$$\sum_{0<\gamma<T}\sum_{\gamma'} T^{i\theta(\gamma'-\gamma)}\, w_R(\gamma-\gamma')\frac{\zeta^{(n)}(\rho)}{\zeta'(\rho')},$$

where $w_R(u) = \dfrac{R^2}{(R+iu\log T)^2}$, and the coefficient of R^{n+2} on the right side is

$$\frac{(-1)^{n+1}}{2\pi}T\log^n T.$$

Although (2.3) is not valid in a sufficiently large range of R to conclude that these two quantities are equal, we note that by Lemma 4 the diagonal terms in the expression from the left side exactly give the expression from the right side. The weight function $w_R(u)$ is much more concentrated around 0 than the weight function in $F(\alpha, T)$, and this may perhaps explain why in our case the diagonal terms dominate immediately.

The mean-square case is more involved, and we have not succeeded in carrying out all of the necessary calculations in this case. We note that Theorem 3 combined with the $\theta = \infty$ conjecture for $S(T, y, a)$ suggests the $\theta = \infty$ conjecture for $I(T, y, a)$, but this assumes that the formula in Theorem 3 remains valid for all θ, and this is essentially what we want to establish. We proceed with the obvious elaboration of the previous argument. Of the five main terms in Proposition 1, we have established that all but $I_2(T, \theta, a)$ should not change shape as θ becomes large. Theorem 2 suggests that $I_2(T, \theta, a)$ should not depend on θ for $0 < \theta < 1$, and we wish to show that I_2 is dominated by its diagonal terms in this range. This is equivalent to

$$\int_0^T |\zeta(\tfrac{1}{2} + a + it)|^2 \sum_\rho \frac{1}{|\zeta'(\rho)|^2 \big(a^2 + (\gamma - t)^2\big)^2} \, dt \sim T \, \frac{1 - T^{-2a}}{4a^2}. \tag{3.2}$$

By writing the above integral as a complex integral, moving the path right, and ignoring the horizontal segments used to move the path, we find that (3.2) is equivalent to

(3.3)

$$\frac{2\pi}{4a^2} \sum_{0<\gamma<T} \frac{\zeta(\rho + 2a)}{\zeta'(\rho)} + \frac{1}{i} \int_{2+i}^{2+iT} \zeta(s + a)\zeta(1 - s + a) \sum_\rho \frac{1}{|\zeta'(\rho)|^2 \big(a^2 - (\rho - s)^2\big)^2} \, ds \sim T \, \frac{1 - T^{-2a}}{4a^2}.$$

Write $E(T)$ for the integral in (3.3). By Lemma 4 in Section 5, or (3.1), it suffices to show that $E(T) = o(T \log^2 T)$. Unfortunately, we have not yet succeeded in doing this, but we hope to return to this topic in a later paper. For now we note that (2.4), Theorem 1, and Theorem 3, along with $\theta = \infty$ in the discrete case, all suggest that $I(T, y, a)$ and $I_2(T, \theta, a)$ do not change shape as θ becomes large. This will have to suffice for our justification of the $\theta = \infty$ conjecture in the continuous case.

4. Proof of Propositions 1 and 2

Both proofs are essentially just an application of

Lemma 1. *On SZ we have*

$$M(s) = \frac{1}{\zeta(s)} \left(1 - \frac{1}{\log y} \frac{\zeta'}{\zeta}(s)\right) + \frac{1}{\log y} \sum_\rho \frac{y^{\rho - s}}{\zeta'(\rho)(\rho - s)^2} + O\left(\frac{y^{-2-\sigma}}{1 + t^2}\right),$$

with the error term uniform for s bounded away from the negative even integers.

Proof. Recall that $h(x) = x$. By Perron's formula, for $c > \max\{1, 1 - \sigma\}$,

$$\begin{aligned} M(s) &= \frac{1}{\log y} \sum_{n \le y} \frac{\mu(n) \log y/n}{n^s} \\ &= \frac{1}{\log y} \frac{1}{2\pi i} \int_{(c)} \frac{1}{\zeta(s + w)} \frac{y^w}{w^2} \, dw. \end{aligned}$$

It is clear that we may move the path of integration as far left as we like, so by Cauchy's theorem

$$M(s)=\frac{1}{\zeta(s)}\left(1-\frac{1}{\log y}\frac{\zeta'}{\zeta}(s)\right)+\frac{1}{\log y}\sum_\rho\frac{y^{\rho-s}}{\zeta'(\rho)(\rho-s)^2}+\frac{1}{\log y}\sum_{n=1}^\infty\frac{y^{-2n-s}}{\zeta'(-2n)(s+2n)^2}.$$

Since $\zeta'(-2n)=2\pi^{-2n-\frac12}(2n)!\,\Gamma(2n+\frac12)\zeta(2n+1)$, the sum over n gives the error term claimed. This proves Lemma 1.

Now Proposition 2 follows easily.

Before proving Proposition 1 we collect together various standard facts about the Riemann ζ-function in

Lemma 2. *The Riemann ζ-function satisfies the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$ where*

$$\chi(s)=\frac{\pi^{s-\frac12}\Gamma(\frac12-\frac12 s)}{\Gamma(\frac12 s)}.$$

For $|\sigma|\ll 1$,

$$\frac{\chi'}{\chi}(s)=-\log|t/2\pi|+O(1/(1+|t|)). \tag{4.1}$$

On RH, if $|t-\gamma|\gg 1/\log t$ for all γ, and $-1<\sigma<2$ then

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|\ll\log^2 t. \tag{4.2}$$

On RH, $|\zeta(s)|\ll t^\epsilon$ for $\sigma\ge\frac12$.

All of the above can be found in Titchmarsh [T]. Also, on RH we have $\sum_{n<X}\mu(n)\ll X^{\frac12+\epsilon}$, so for $\sigma\ge\frac12$ we have $M(s)\ll(ty)^\epsilon$.

Now we prove Proposition 1. By Lemma 1 we have

$$\int_0^T\left|\zeta(\tfrac12+a+it)\left(M(\tfrac12+a+it)-\frac{1}{\zeta(\frac12+a+it)}\left(1-\frac{1}{\log y}\frac{\zeta'}{\zeta}(\tfrac12+a+it)\right)\right)\right|^2dt$$

$$=\int_0^T|\zeta(\tfrac12+a+it)|^2\left|\frac{1}{\log y}\sum_\rho\frac{y^{\rho-(\frac12+a+it)}}{\zeta'(\rho)\big(\rho-(\frac12+a+it)\big)^2}+O\left(\frac{y^{-\frac52-a}}{1+t^2}\right)\right|^2dt$$

Putting $\rho=\frac12+i\gamma$ and $y=T^\theta$, squaring out both sides and rearranging we obtain, in the notation of Proposition 1,

$$I(T,y,a) = \frac{y^{-2a}}{\log^2 y} I_2(T,\theta,a) - \frac{1}{\log^2 y} I_1(T,a) + \frac{2}{\log y} Re \int_0^T \frac{\zeta'}{\zeta}\big(\tfrac12 + a + it\big) dt - \int_0^T 1\, dt$$

$$+ 2\, Re \int_0^T \zeta M(\tfrac12 + a + it)\, dt - \frac{2}{\log y} Re \int_0^T \zeta M(\tfrac12 + a + it) \frac{\zeta'}{\zeta}(\tfrac12 + a - it)\, dt$$

$$+ O\left(y^{-5-2a} + y^{-\frac52 - a} \int_0^T \frac{|\zeta(\frac12 + a + it)|^2}{1+t^2} \left| \sum_\rho \frac{T^{i\theta\gamma}}{\zeta'(\rho)(a + i(t-\gamma))^2} \right| dt \right).$$

We now treat the various terms in the above expansion. Recall that we are assuming RH. By Cauchy's theorem and Lemma 2,

$$\frac{2}{\log y} Re \int_0^T \frac{\zeta'}{\zeta}\big(\tfrac12 + a + it\big) dt = \frac{2}{\log y} Re\, \frac{1}{i} \int_{\frac12}^{\frac12 + iT} \frac{\zeta'}{\zeta}(s+a)\, ds$$

$$= \frac{2}{\log y} Re\, \frac{1}{i} \left(\int_{\frac12 + a}^{2} \frac{\zeta'}{\zeta}(s)\, ds + \int_{2+iT}^{\frac12 + a + iT} \frac{\zeta'}{\zeta}(s)\, ds + \int_{2}^{2+iT} \frac{\zeta'}{\zeta}(s)\, ds \right)$$

$$= O(T^\epsilon).$$

The integral over $[\frac12 + a, 2]$ is assumed to have a small indentation so as to miss the pole at $s = 1$; it contributes $O(1)$. The second integral is estimated by (4.2) and our assumption on T; it contributes $\log^2 T \ll T^\epsilon$. The last integral was estimated by expanding the integrand in an absolutely convergent series and integrating term by term, giving a bound of $\ll 1$.

Next we have, proceeding exactly as above, and using the comment following Lemma 2,

$$2\, Re \int_0^T \zeta M(\tfrac12 + a + it)\, dt = 2\, Re\, \frac{1}{i} \int_{\frac12}^{\frac12 + iT} \zeta M(s+a)\, ds$$

$$= 2\, Re\, \frac{1}{i} \int_{2}^{2+iT} \zeta M(s)\, ds + O(T^\epsilon y^\epsilon)$$

$$= 2T + O(T^\epsilon y^\epsilon),$$

where again we expanded the last integrand and integrated term by term.

Again proceeding as above,

$$-\frac{2}{\log y}\,Re\int_0^T \zeta M(\tfrac12+a+it)\frac{\zeta'}{\zeta}(\tfrac12+a-it)\,dt$$

$$= -\frac{2}{\log y}\,Re\,\frac{1}{i}\int_{\frac12}^{\frac12+iT} \zeta M(s+a)\frac{\zeta'}{\zeta}(1-s+a)\,ds$$

$$= -\frac{4\pi}{\log y}\,Re\sum_{0<\gamma<T}\zeta M(1-\rho+2a) - \frac{2}{\log y}\,Re\,\frac{1}{i}\int_2^{2+iT}\zeta M(s+2a)\frac{\zeta'}{\zeta}(1-s)\,ds + O(T^{\epsilon}y^{\epsilon})$$

$$= -\frac{4\pi}{\log y}\,Re\sum_{0<\gamma<T}\zeta M(\overline{\rho}+2a)$$

$$- \frac{2}{\log y}\,Re\,\frac{1}{i}\int_2^{2+iT}\zeta M(s+2a)\left(\frac{\chi'}{\chi}(s)-\frac{\zeta'}{\zeta}(s)\right)ds + O(T^{\epsilon}y^{\epsilon})$$

$$= -\frac{4\pi}{\log y}\,Re\sum_{0<\gamma<T}\zeta M(\rho+2a) + \frac{2}{\log y}T\log T + O(T/\log y + T^{\epsilon}y^{\epsilon}).$$

The last integral was evaluated and estimated by expanding the integrand and integrating term by term, using formula (4.1). Now we apply Proposition 2 to the first term above and collect the various pieces to obtain the main terms in Proposition 1. The Cauchy-Schwarz inequality gives the remaining error terms. Note: in Proposition 1 there appears S_1 and S_2, not $Re(S_1)$ and $Re(S_2)$. This is because, as can be seen from Lemma 3b and Proposition 2, these quantities are real asymptotically.

5. Asymptotic Formulas

In this section we collect the asymptotic formulas needed in the various proofs. Let

$$D(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)^2}{n^s}$$

where $\Lambda(n) = \log p$ if $n = p^m$, and $\Lambda(n) = 0$ otherwise. It is easy to show that $D(s)$ is regular for $\sigma > \frac12$ except for a double pole at $s = 1$ with Laurent expansion $D(s) = 1/(s-1)^2 + \cdots$.

Lemma 3a. (Goldston and Gonek [GG]). *On RH, if* $T^{-\frac12}\log T \ll a \ll 1$ *with* $Re\,a > 0$ *then*

$$\int_0^T\left|\frac{\zeta'}{\zeta}\left(\frac12+a+it\right)\right|^2 dt \sim T\left(D(1+2a)-\frac{T^{-2a}}{4a^2}\right) + T\log^2 T\int_1^{\infty}\big(F(\alpha,T)-1\big)\,T^{-2a\alpha}d\alpha.$$

If, in addition, $a = o(1)$ then

$$\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+a+it\right)\right|^2 dt \sim T\left(\frac{1-T^{-2a}}{4a^2} + \log^2 T \int_1^\infty \big(F(\alpha,T)-1\big)\, T^{-2a\alpha} d\alpha\right).$$

On Montgomery's Hypothesis the integral involving $F(\alpha,T)$ can be suppressed if $a \approx 1/\log T$.

Note: If $a \log T \to \infty$ then all but the term $T\, D(1+2a)$ can be supressed in the first formula above, and similarly for the first formula in the next lemma.

Lemma 3b. *On RH, if $T^{-\frac{1}{2}} \log T \ll a \ll 1$ with $Re\, a > 0$ then*

$$\sum_{0<\gamma<T} \frac{\zeta'}{\zeta}(\rho+a) \sim \frac{T}{2\pi}\left(D(1+a) - \frac{T^{-a}}{a^2}\right) + \frac{1}{2\pi} T \log^2 T \int_1^\infty \big(F(\alpha,T)-1\big)\, T^{-a\alpha} d\alpha.$$

If, in addition, $a = o(1)$ then

$$\sum_{0<\gamma<T} \frac{\zeta'}{\zeta}(\rho+a) \sim \frac{T}{2\pi}\left(\frac{1-T^{-a}}{a^2} + \log^2 T \int_1^\infty \big(F(\alpha,T)-1\big)\, T^{-a\alpha} d\alpha\right).$$

On Montgomery's Hypothesis the integral involving $F(\alpha,T)$ can be suppressed if $a \approx 1/\log T$.

Proof. Note that we are assuming RH. By Cauchy's theorem and Lemma 2,

$$\int_0^T \left|\frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right|^2 dt = \frac{1}{i}\int_{\frac{1}{2}}^{\frac{1}{2}+iT} \frac{\zeta'}{\zeta}(s+a)\frac{\zeta'}{\zeta}(1-s+a)\, ds$$

$$= 2\pi \sum_{0<\gamma<T} \frac{\zeta'}{\zeta}(\rho+2a) + \frac{1}{i}\int_2^{2+iT} \frac{\zeta'}{\zeta}(s+2a)\frac{\zeta'}{\zeta}(1-s)\, ds + O(T^\epsilon)$$

$$= 2\pi \sum_{0<\gamma<T} \frac{\zeta'}{\zeta}(\rho+2a) + \frac{1}{i}\int_2^{2+iT} \frac{\zeta'}{\zeta}(s+2a)\left(\frac{\chi'}{\chi}(s) - \frac{\zeta'}{\zeta}(s)\right) ds + O(T^\epsilon)$$

$$= 2\pi \sum_{0<\gamma<T} \frac{\zeta'}{\zeta}(\rho+2a) + O(T^\epsilon).$$

The last integral was estimated by expanding the integrand and integrating term by term, using (4.1). Now Lemma 3b follows from Lemma 3a.

Lemma 4. *On SZ, if $T^{-\frac{1}{2}} \ll |a| \ll 1/\log T$ then*

$$\sum_{0<\gamma<T} \frac{\zeta(\rho+a)}{\zeta'(\rho)} \sim \frac{1}{2\pi} T\left(1-T^{-a}\right)$$

as $T \to \infty$.

Proof. Let

$$I(T) = \frac{1}{2\pi i} \int_{-1+i}^{-1+iT} \frac{\zeta(1-s+a)}{\zeta(1-s)}\, ds.$$

Setting

$$b_n = \sum_{\alpha\beta=n} \mu(\alpha)\beta^{-a}$$

we have $b_1 = 1$ and

$$\frac{\zeta(s+a)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \tag{5.2}$$

with the series absolutely convergent for $\sigma > 1+|a|$. By a change of variables we find

$$I(T) = -\frac{1}{2\pi i} \int_{2-i}^{2-iT} \frac{\zeta(s+a)}{\zeta(s)}\, ds$$

$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} b_n \int_{2-i}^{2-iT} n^{-s}\, ds$$

$$= \frac{1}{2\pi} T + O(1).$$

On the other hand, by moving the path of integration right and using the functional equation,

$$I(T) = \sum_{0<\gamma<T} \frac{\zeta(\rho+a)}{\zeta'(\rho)} + \frac{1}{2\pi i} \int_{2+i}^{2+iT} \frac{\chi(1-s+a)\zeta(s-a)}{\chi(1-s)\zeta(s)}\, ds + O(T^{\epsilon}).$$

Using (5.2) again, along with Stirlings formula, we find

$$\frac{1}{2\pi i} \int_{2+i}^{2+iT} \frac{\chi(1-s+a)\zeta(s-a)}{\chi(1-s)\zeta(s)}\, ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} b_n \int_{2+i}^{2+iT} t^{-a} n^{-s} \left(1 + O\big((1+|t|)^{-1}\big)\right) ds$$

$$= \frac{1}{2\pi} \frac{T^{1-a}}{1-a} + O(T^{\epsilon}).$$

Thus,

$$\sum_{0<\gamma<T} \frac{\zeta(\rho+a)}{\zeta'(\rho)} = \frac{1}{2\pi} T \left(1 - \frac{T^{-a}}{1-a}\right) + O(T^{\epsilon})$$

$$= \frac{1}{2\pi} T \left(1 - T^{-a}\right) + O\big(aT^{1-a} + T^{\epsilon}\big),$$

which proves Lemma 4.

6. Proof of the Theorems

We evaluate $I_2(T,\theta,a)$ and $S_2(T,\theta,a)$.

In the course of proving Proposition 1 it was shown that, on RH and SZ,

$$\begin{aligned} I(T,y,a) &= \frac{y^{-2a}}{\log^2 y} I_2(T,\theta,a) - \frac{1}{\log^2 y} I_1(T,a) - \frac{4\pi}{\log y}\, Re\, S(T,y,a) \\ &\quad + T\left(1 + 2\frac{\log T}{\log y}\right) + O(T/\log y). \end{aligned} \tag{6.1}$$

Combining (6.1) with (2.1), Lemma 3a, and (2.2) yields the asymptotic formula for $I_2(T,\theta,a)$ given in Theorem 1. Then combine Theorem 1, (6.1), and Lemma 3a to obtain Theorem 3.

Combine Proposition 2 with Lemma 3b and formula (2.2) to obtain Theorem 2. Note that this gives the same result as combining (2.4) and Theorem 1, on the assumption that $E(T,\theta,a)$ is small.

7. Consequences of $\theta = \infty$, and some related conjectures

To make things explicit, we reproduce here the formulas for which we have made a $\theta = \infty$ conjecture. The first is Theorem 1 of [CGG1], where we put $h_a(x) = T^{\theta a(x-1)} h(x)$,

$$\begin{aligned} &\int_1^T \zeta(\tfrac{1}{2}+u+it)\,\zeta(\tfrac{1}{2}+v-it)\, M(\tfrac{1}{2}+a+it, h(x))\, M(\tfrac{1}{2}+b-it, g(x))\, dt \\ &\sim T\left(h(1)g(1) + \frac{1}{\theta}\int_0^1 T^{-\xi(u+v)} d\xi\, \frac{d}{d\alpha}\frac{d}{d\beta} T^{-\theta(\alpha u+\beta v)} \int_0^1 h_a(x+\alpha) g_b(x+\beta)\, dx \right)\Bigg|_{\alpha=\beta=0} \end{aligned} \tag{7.1}$$

The second is Theorem 3 of [CGG1],

$$\sum_{0<\gamma<T} \zeta(\rho+u)\, M(\rho+a, h(x)) \sim \frac{T}{2\pi}\left(h(1)\log T + \big(1 - T^{-u}\big)\Big(\frac{1}{u} + \theta \log T \int_0^1 h_a(x)\, dx\Big)\right). \tag{7.2}$$

The above formulas are valid for $0 < \theta < \frac{4}{7}$ and $0 < \theta < \frac{1}{2}$, respectively, for $h(x)$ and $g(x)$ real polynomials with $h(0) = g(0) = 0$, uniformly for $|u| + |v| + |a| + |b| \ll 1/\log T$. This uniformity is important because it implies that we may differentiate the formulas via Cauchy's theorem. It is easy to see that differentiating the main term will increase the main term by a factor of $\log T$, so when using Cauchy's theorem we must integrate around circles of radius $\gg 1/\log T$. This restriction will become significant momentarily. Subject to the requirements on $h(x)$ and $g(x)$ mentioned above, and the restrictions on $u,\ v,\ a,\ b$ described in the next paragraph, our "$\theta = \infty$" conjectures are that (7.1) and (7.2) hold for all $\theta > 0$.

In this paper we have so far just considered the case $u = v = b = a$ with $a > 0$, and $h(x) = g(x) = x$. We see no reasonable objection to extending the conjecture all $u,\ v,\ a,\ b$, each $\ll 1/\log T$ and with positive real part. Having done so we observe that

$$\frac{d}{ds}M(s,h(x)) = \theta \log T\, M(s,(x-1)h(x)),$$

so by differentiating with respect to a and b and taking linear combinations we can obtain any polynomial which vanishes at 0 just by starting with $h(x) = g(x) = x$. Thus we are led to believe that the formulas continue to hold for large θ with the only added restriction that the small parameters u, v, a, b have positive real part. This restriction on a and b is natural because, on RH, as $\theta \to \infty$, $M(s)$ converges to $1/\zeta(s)$ for $\sigma > \frac{1}{2}$, and it does not converge for $\sigma < \frac{1}{2}$. Requiring the parameters u, v, a, b to have positive real part also carries with it another hidden restriction. As noted above, to differentiate our formulas we integrate over a circle of radius $\gg 1/\log T$. If this circle is confined to the right half-plane then the new error term dominates unless the appropriate parameter u, v, a, b is also of size $\gg 1/\log T$. So, in the formulas for which we have made a $\theta = \infty$ conjecture and in all other conjectured formulas we assume that each of u, v, a, b has positive real part and is of size $\approx 1/\log T$.

We will now give an application of our conjecture. Recall that we denote by κ the proportion of nontrivial zeros of $\zeta(s)$ which are simple and on the critical line. We will use Levinson's method to show that $\theta = \infty$ implies $\kappa = 1$. In addition to $h(x)$, Levinson's method [Lev], as modified by Conrey [C], has two free parameters: R and λ. The idea is that given θ one can use the calculus of variations to choose $h(x)$ optimally, then it is an easy manner to choose R and λ optimally. The details of passing from (7.1) to a bound for κ can be found in [C]. The result is that with $Q(x) = 1 + \lambda x$, $w(y) = e^{Ry}Q(y)$, and

$$A = \int_0^1 e^{2Ry}Q^2(y)\,dy$$

and

$$C = \int_0^1 e^{2Ry}\big(RQ(y) + Q'(y)\big)^2 dy$$

then

$$\kappa \geq 1 - \frac{1}{R}\log\left(\frac{1}{2}\big(1 + w^2(1)\big) + \sqrt{AC}\coth\theta\sqrt{\frac{C}{A}}\right). \tag{7.3}$$

For example, by choosing $R = 1.132$, $\lambda = -1.028$, and $\theta = \frac{4}{7} - \epsilon$, $\epsilon \to 0^+$, we obtain Conrey's bound $\kappa > 0.40219$. Our concern is with the behavior for large θ. We find that if we choose $R = -\lambda = \theta^{-\frac{1}{5}}$ then $A = 1 + O(\theta^{-\frac{2}{5}})$, $C = \frac{4}{3}\theta^{-\frac{4}{5}} + O(\theta^{-1})$, $w^2(1) = 1 + O(\theta^{-\frac{2}{5}})$, and the right side of (7.3) equals

$$1 - \theta^{\frac{1}{5}}\log\Big(1 + O(\theta^{-\frac{2}{5}})\Big) = 1 + O(\theta^{-\frac{1}{5}}).$$

In particular, as $\theta \to \infty$ we find $\kappa \geq 1 + o(1)$. That is, on the $\theta = \infty$ conjecture almost all of the zeros of $\zeta(s)$ are simple and on the critical line.

Next we exploit the observation that $M(s)$ converges to $1/\zeta(s)$ as $\theta \to \infty$ to conjecture some mean value formulas. Having assumed the $\theta = \infty$ conjecture we should expect that we may allow $\theta \to \infty$ as a function of T, assuming that θ grows sufficiently slowly. Then from (7.1) we obtain the conjecture

$$\int_1^T \frac{\zeta(\frac{1}{2}+u+it)\zeta(\frac{1}{2}+v-it)}{\zeta(\frac{1}{2}+a+it)\zeta(\frac{1}{2}+b-it)}\,dt \sim T\left(1+\left(1-T^{-(u+v)}\right)\frac{(u-a)(v-b)}{(u+v)(a+b)}\right), \tag{7.4}$$

and from (7.2) we conjecture

$$\sum_{0<\gamma<T} \frac{\zeta(\rho+a)}{\zeta(\rho+b)} \sim \frac{T}{2\pi}\left(\log T + \left(1-T^{-a}\right)\left(\frac{1}{b}-\frac{1}{a}\right)\right). \tag{7.5}$$

These formulas may be differentiated with respect to u, v, a, and b, allowing one to deduce many other formulas. As an example, (7.5) actually follows from (7.4), and we can also obtain the following conjectures: for a real, $a \approx 1/\log T$,

$$\int_1^T \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right|^2 dt \sim T\left(\frac{1-T^{-2a}}{4a^2}\right), \tag{7.6}$$

$$\int_1^T \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right|^4 dt \sim T\left(\frac{1-T^{-2a}}{8a^4} - \frac{T^{-2a}\log^2 T}{4a^2}\right), \tag{7.7}$$

and

$$\int_1^T \left|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right|^6 dt \sim T\left(\frac{3\left(1-T^{-2a}\right)}{32a^6} - \frac{3T^{-2a}\log^2 T}{8a^4} + \frac{T^{-2a}\log^3 T}{8a^3} - \frac{T^{-2a}\log^4 T}{16a^2}\right). \tag{7.8}$$

It is interesting to note that (7.6) combined with Theorem 1 of [GG] (see Lemma 3a) implies that $F(\alpha,T) \sim 1$ for $\alpha > 1$, that is, $\theta = \infty$ implies Montgomery's pair correlation hypothesis. Much more may actually be true. It is possible that the methods in [GG] could be used to evaluate expressions like

$$\int_1^T \left(\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a+it)\right)^n \left(\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a-it)\right)^m dt$$

for values other than $m = n = 1$. If so, then the result should involve the (m+n)-correlation function. For small values of m and n the above integral can be evaluated by (7.4) and the result obtained could be checked against that given by assuming the GUE hypothesis, for example. We note that combining (7.6) and Lemma 3a also suggests that the formulas in this section, including (7.1) and (7.2), are not valid for u, v, a, b outside the range $\ll 1/\log T$.

Finally we indicate a method of conjecturing special cases of the above formulas, providing a check on the mutual consistency of our conjectures. Setting $h(x) = x^2$ and mimicing the proof of Proposition 2 we find

$$\begin{aligned}\sum_{0<\gamma<T} \zeta(\rho+a)M(\rho+a,x^2) \sim & \frac{1}{2\pi}T\log T - \frac{2}{\log y}\sum_{0<\gamma<T}\frac{\zeta'}{\zeta}(\rho+a) \\ & + \frac{2}{\log^2 y}\left(\sum_{0<\gamma<T}\left(\frac{\zeta'}{\zeta}\right)^2(\rho+a) - \frac{1}{2}\sum_{0<\gamma<T}\frac{\zeta''}{\zeta}(\rho+a)\right) \\ & - \frac{2y^{-a}}{\log^2 y}\sum_{0<\gamma<T}\sum_{\rho'} T^{i\theta(\gamma'-\gamma)}\frac{\zeta(\rho+a)}{\zeta'(\rho')(a+i(\gamma-\gamma'))^3}.\end{aligned} \tag{7.9}$$

And by (7.2), for $0 < \theta < \frac{1}{2}$ and $|a| \ll 1/\log T$,

$$\begin{aligned}\sum_{0<\gamma<T} \zeta(\rho+a)M(\rho+a,x^2) \sim & \frac{1}{2\pi}T\log T - \frac{2}{\log y}\left(\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^2}\right) \\ & + \frac{2}{\log^2 y}\left(\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^3}\right) - \frac{2y^{-a}}{\log^2 y}\left(\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^3}\right).\end{aligned} \tag{7.10}$$

By "matching up" the terms in the above formulas we obtain conjectures about the third and fourth terms on the right side of (7.9). On the other hand, by differentiating the formula in Lemma 3b we obtain, on RH, SZ, and Montgomery's Hypothesis,

$$\sum_{0<\gamma<T}\frac{\zeta''}{\zeta}(\rho+a) - \sum_{0<\gamma<T}\left(\frac{\zeta'}{\zeta}\right)^2(\rho+a) \sim \frac{T}{2\pi}\left(\frac{T^{-a}\log T}{a^2} - 2\,\frac{1-T^{-a}}{a^3}\right).$$

Combining this with the formula obtained by matching up the $2/\log^2 y$ terms in (7.9) and (7.10) we have

$$\sum_{0<\gamma<T}\left(\frac{\zeta'}{\zeta}\right)^2(\rho+a) \sim \frac{T\log T}{2\pi}\,\frac{T^{-a}}{a^2}$$

and

$$\sum_{0<\gamma<T}\frac{\zeta''}{\zeta}(\rho+a) \sim \frac{T}{\pi}\left(\frac{T^{-a}\log T}{a^2} - \frac{1-T^{-a}}{a^3}\right).$$

One can easily check that these formulas are consistent with (7.5). By repeating this analysis for an arbitrary polynomial $h(x)$ it is possible to show that all formulas obtained in this way are consequences of conjecture (7.5).

We end by considering the "pair correlation" terms which have arisen: for example, the last expression in (7.9). By applying the method of Proposition 2 to the polynomial $h(x) = x^N$ and matching terms as above we obtain the following conjecture:

$$\sum_{0<\gamma<T}\sum_{\rho'} T^{i\theta(\gamma'-\gamma)}\frac{\zeta(\rho+a)}{\zeta'(\rho')(a+i(\gamma-\gamma'))^{N+1}} \sim (-1)^{N+1}\frac{T}{2\pi}\,\frac{1-T^{-a}}{a^{N+1}}, \tag{7.12}$$

for $N \geq 1$, $\theta > 0$, and $a \approx 1/\log T$ with $Re\, a > 0$. This says that the left side of (7.12) is dominated by its diagonal terms for all fixed $\theta > 0$.

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8. References

[BCH-B] *R. Balasubramanian, J. B. Conrey*, and *D. R. Heath-Brown*, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, J. reine angew. Math. **357** (1985), 161-181.

[C] *J. B. Conrey*, More than two fifths of the zeros of the Riemann zeta function are on the critical line, J. reine angew. Math. **399** (1989), 1-26.

[CG] *J. B. Conrey* and *Ghosh*, Mean Values of the Riemann zeta-function, III, preprint.

[CGG1] *J. B. Conrey, A. Ghosh*, and *S. M. Gonek*, Mean values of the Riemann zeta-function with application to distribution of zeros, Number Theory, Trace Formulas and Discrete Groups, (1989), 185-199.

[CGG2] *J. B. Conrey, A. Ghosh*, and *S. M. Gonek*, Simple zeros of the Riemann zeta-function, preprint

[GG] *D. Goldston* and *S. M. Gonek*, Mean values of $\frac{\zeta'}{\zeta}(s)$ and primes in short intervals, preprint

[G] *S. M. Gonek*, On negative moments of the Riemann zeta-function, Mathematika **36** (1989), 71-88.

[Lev] *N. Levinson*, More than one-third of the zeros of Riemann's zeta-function are on $\sigma = \frac{1}{2}$, Adv. in Math., **13** (1974), 383-436.

[M1] *H. L. Montgomery*, The pair correlation of zeros of the zeta function, Proc. Symp. Pure Math. **24** (1973), 181-193.

[M2] *H. L. Montgomery*, Distribution of the zeros of the Riemann zeta function, Proc. of the Int. Cong. for Math., Vancouver, BC **1** (1974), 379-381.

[T] *E. C. Titchmarsh*, The Theory of the Riemann Zeta-Function, Second Ed., Oxford 1986.

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