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Analysis of the Discount Factors in Swap Valuation

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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. The seal features a sun with rays, a crown, and the Latin motto "ALMA MATER VERITAS".

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Abstract

Discount factors are used to discount the cash flows in swap valuation. In my thesis, we study in the two swap valuation methods, the different performances of the discount factors. We lay the foundation for the swap valuation in the first four chapters. We introduce the concepts of the swaps and swaptions, PDEs in finance, how to model the dynamics of the interest rate and some typical interest rate models and then give the mathematical forms of swaps. In the last chapter, we present the general procedure of the swap valuation and introduce the discount factor curves. Then we further study the performances of the discount factor curves in the two different swap valuation methods.

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1 Introduction

Grown dramatically in recent years, the swaps market plays an important part in managing financial risks faced by the firms, since it can offer the firms a flexible way in hedging risks. A swap is a kind of financial derivative between two counterparties which exchange, usually, one stream of cash flows against another cash stream. The dates when to pay the cash flows and the way to calculate them are demonstrated in the swap agreements. Interest rate swaps, foreign currency swaps, equity swaps, commodity swaps and credit default swaps make up the five basic swaps which are determined by an interest rate, foreign exchange rate, equity price, commodity price or some other underlying assets respectively. [1]

1.1 Swaps

(1) Interest rate swaps

The interest rate swaps are the simplest interest rate derivative. In the contract, one party exchanges a loan at a fixed rate of interest, which is called swap rate, for a loan at a floating rate during a given period. In general, the notional principle is not exchanged between the two counterparties. The interest swaps have advantages such as refinancing the debt of the firm and reducing the risk of interest rate fluctuations. For example, one party B, specifically a bank, possesses assets which yield returns at a floating rate referring to LIBOR (the London Interbank Offered Rate), e.g. 30 basis points (.3 percent) above LIBOR, offers the cash flows of the interest periodically to the other party A. Party A possesses assets that yield a fixed rate of return and provides periodic interest cash payments based on a fixed rate, e.g. 10 percent, to the bank. They make the calculations of their own payments based on the principle, a notional amount.

Why we exchange floating rate loans for fixed rate ones between two parties? We just would like to take the comparative advantages of the different firms. For instance, Party A has a comparative advantage in fixed-rate asset markets, while Party B takes a comparative advantage in floating-rate markets. Due to a interest rate swap introduced to the credit markets, transforming a fixed rate capital flow into a floating rate one comes true or vice versa. Therefore, the comparative advantages can be exploited to reduce the interest rate risks effectively.

(2) Currency swaps

Currency swaps, introduced in the 1970s due to foreign exchange controls in Britain, have been an important tool for financing. [2] In a currency swap contract, Party A makes predetermined payments periodically to Party B in one currency like U.S. dollars, meanwhile, Party B pays a certain amount in another currency like Euros. However, in this kind of agreements between two parties, not only the interest payments but also the principal can be exchanged on the two equal loans. [3] It takes advantage of hedging against exchange rate fluctuations. Generally, three kinds of payment flows are involved in a currency swap. First, the two parties get the amount of cash in certain currency they need respectively by the exchange of two different currencies with each other. Second, they pay each other the interests that demonstrate the interest rates level in the home country periodically according to the contract. Finally, the principle is re-exchanged and the swap is completed. [6]

(3) Commodity swaps

It is increasingly common that commodity swaps are used as an important tool in the energy industries and agriculture. Commodity swaps apply the principle of the interest rate swaps to the commodity prices. For a given quantity of some commodity, like crude oil for example, Party A makes payments to Party B at a fixed price per unit during a certain period. Meanwhile, Party B offers cash flows to Party A at a floating price per unit. By means of the commodity swap, the users of the commodity can control the cost at a desirable level; they have to bear some kind of risks such as cost reductions due to the drop of prices. For example, heavy users of oil, such as airlines, will often enter into contracts in which they agree to make a series of fixed payments, for instance, every six months for two years, and receive payments on those same dates as determined by an oil price index. Computations are often based on a specific number of tons of oil in order to lock in the price the airline pays for a specific quantity of oil, purchased at regular intervals over the two-year period. However, the airline will typically buy the actual oil it needs from the spot market.

In the most of the interest rate, currency and equity swaps, the variable payment is based on the price or rate on a specific day. However, in oil swaps it is common to base the variable payment on the average value of an oil index over a period of time, which could be weekly, monthly, quarterly, or the entire period between settlements. This feature removes the effects of an unusually volatile single day and ensures that the payment will more accurately represent the value of the index. Average-price payoff structures are also found in other derivatives, particularly options.

(4) Equity swaps

In an equity swap, one of the two parties exchanges their future returns on the stocks or equity index, for the cash flows of some other financial instruments or returns connected to LIBOR with the other party on set dates. Equity swaps are usually used to reduce the withholding taxes from foreign investments on certain assets.

(5) Credit default swaps

Credit Default Swaps was invented in the early 1990s, and developed in 1997, in order to manage the risk of default to a third party. In a credit default swap, the buyer of the contract pays a premium to the seller periodically, intending to receive a payoff in case a financial instrument like a bond or loan contract undergoes a default, owing to one event demonstrated in the contract happens such as bankruptcy.

Credit default swaps can be used for hedging and arbitrage. Often, the credit risk can be managed by credit default swaps. For instance, if the bond goes into default, the holder will balance their loss from the contract.

1.2 Swaptions

In 1983, William Lawton, the Head Trader for Fixed Income Derivatives at First Interstate Bank in Los Angeles constructed and executed the first known swaption which was for one year. A right was sold to enter into an interest rate swap for five years, in which the holder paid fixed cash flows in order to exchange three-month Libor streams on a principle of \$5 million.

Swaption, that is, 'swap option', typically refers to options on interest rate swaps, is a kind of an option which gives its buyer the right but not the obligation to enter into an underlying swap contract. It is necessary for such kind of contracts to exist in the financial markets for the debt issuers, in order to keep their flexibility. Some debt issuers need the right to withdraw a certain swap in future, while others require the chance to enter into a prespecified swap later. Obviously, we can regard swaptions as calls or puts on coupon bonds.

There are two forms of swaption, the payer swaption and receiver swaption. A payer swaption grants the holder of the swaption the right to get into a swap contract where they pay the fixed leg and receive the floating leg, while the receiver swaption gives the holder a chance which is not obligatory to enter into a swap contract where they pay the floating leg instead of the fixed leg. [5]

Not only can a swaption hedge the holders against risks, it also benefits the buyers, compared to the ordinary swaps.

The swaption will be exercised as declared in the contract between two counterparties, in the condition that the strike rate of the swaption is more attracting than the present market swap rate. In detail, the swaption gives the holder the benefit of the relatively lower agreed-upon strike rate than the market rates, with the flexibility to enter into the current market swap rates and vice versa.

Typically, such a contract includes the terms:

- (1) the price of the swaption
- (2) the strike rate, identical to the fixed rate of the underlying swap
- (3) length of the option period
- (4) the term of the underlying swap
- (5) notional amount
- (6) frequency of settlement of payments on the underlying swap

Banks, big corporations, and other financial institutions are the leading participants in the swaption market because they intend to manipulate the interest rate risk by swaptions underlying their core business or financing. A bank that buys a receiver swaption attempts to protect itself from low interest rates, since they may hold a mortgage leading to the prepayment. However, a corporation, which buys a payer swaption, wants to prevent itself from rising interest rates risks.

There are three styles of swaptions. Each style reflects a different timeframe in which the option can be exercised.

- (1) American swaption, a swaption which the holder has the opportunity to exercise on any date that falls within the time interval from the lockout end date to the expiry date. The underlying swap works on the condition that the swaption is exercised.

There are two possible types of underlying swap, depending on which you choose.

First, the underlying swap with fixed tenor: its length will not change regardless of the date the swaption is exercised. For instance, an American swaption and the underlying swap both expire in three years. The swaption can be exercised on any date before expiry. If the swaption is exercised in one year, we will end the swap in four years from now, if it is exercised in two years, we will end the swap in five years from today and if you exercise in three years, the swap will end in six years.

Second, the underlying swap with fixed end date: its expiry date is determined in advance, and its actual duration depends on the time that the swap starts, that is, the length of the underlying swap is calculated from the date you exercise the swaption until the fixed end date.

- (2) European swaption, in which the underlying swap into which can be entered only on the exact maturity date.

(3) Bermudan swaption, in which the owner is allowed to enter the swap only on certain dates that fall within a range of the start date and end date. As for a Bermudan payer swaption, the holder makes the fixed coupon payments and receives Libor cash flows; a receiver swaption works in the opposite direction.

Concerning the valuation of swaptions, it is not a simple procedure due to several factors that affect the final results, such as the time to expiration and the length of the underlying swap. Owing to the existing complications in the valuation of swaptions, relative value between different swaptions is calculated, sometimes, in the ways of constructing complex term structure and short rate models.

1.3 Risk-neutral Pricing

The objective probabilities just play a role in determining whether an event is possible or not.

Risk-neutral valuation is a very important topic at present. First let us introduce the concept of risk-neutral world. When we price financial derivatives, the investors are assumed to be risk neutral. That is to say, the investors' preferences cannot determine the value of derivative. To price any derivative, assuming a risk neutral world, we calculate the expected payoff at the expiration time and then discount the expected value by risk-free interest rate. People would like to seek the premium to avoid risks, though, in real life, it is impossible to expect all the people risk neutral.

In the absence of arbitrage, we have another method to calculate the price. We consider taking the expectations under a different probability of future outcomes which incorporate the effects of risk instead of first taking the expectation and adjusting risks. The adjusted probabilities are named risk-neutral probabilities; they constitute the risk-neutral measure. For example, given a market by the equations,

$$dB(t) = rB(t)dt,$$

$$dS(t) = S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))dW(t).$$

Then we define the Q-dynamics of S as

$$dS(t) = r S(t) dt + S(t) \sigma(t, S(t)) dW(t), \text{ where } W \text{ is a Q-Wiener process.}$$

The arbitrage free price of the claim $\Phi(S(T))$ is given by

$$\Pi(t; \sigma) = e^{-r(T-t)} E_{t,s}^Q[\Phi(S(T))]$$

The formula is the one of risk neutral valuation whose economic interpretation is that when we get the present date t and the present price of the stock s , we can calculate the price of the derivative by the expectation of the last payoff $E_{t,s}^Q[\Phi(S(T))]$. And then we discount the expected value to today's value by the factor $e^{-r(T-t)}$. In the process of the calculation of the expectation, we use the measure Q instead of the objective probability measure P . Often, the objective probability measure is called P , and the risk-neutral called Q . The Q -measure is often called martingale measure or risk adjusted measure. [5]

Under the risk-neutral measure, the expected rates of return are the same among all the assets, i.e. the risk-free rate. In the absence of arbitrage, if the markets are complete, the risk-neutral measure is unique.

The big advantage of using the risk-neutral probabilities is that the assets can be priced by its expected value assuming the consumers were risk neutral. Yet what is the situation that we use the physical probabilities? The result will be that each asset requires a different adjustment because of different riskinesses.

2 Partial Differential Equations in Finance

2.1 Diffusion Process

In order to model the assets prices in the financial derivative market, we often use diffusion processes as the building blocks. First, we should talk about what the diffusion is.

Given a stochastic differential equation of the following form,

$$X(t+\Delta t) - X(t) = \mu(t, X(t)) \Delta t + \sigma(t, X(t)) Z(t),$$

to approximate the local dynamics of X , a stochastic process. Then X is called a diffusion.

where $Z(t)$ is a normally distributed disturbance term which is nothing to do with all the events that happened up to time t ; $\mu(t, X(t))$ is a locally deterministic velocity, called the drift term of the process, and $\sigma(t, X(t))$ is used to amplify a Gaussian disturbance term. [5]

2.2 Geometric Brownian Motion

In order to model the asset prices, we often use Geometric Brownian Motion as an important tool. The equation appears as,

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

$$X_0 = x_0$$

The solution to the equation above is $X(t) = x_0 \exp\{(\alpha - 1/2\sigma^2)t + \sigma W_t\}$, and the expected value is $E[X_t] = x_0 e^{\alpha t}$

2.3 PDEs in Finance

To value the financial derivatives, we often need to resort to two main methods (Surely we can approximate the pricing with the help of discrete-time models as well). Given a derivative, we can obtain a valuation formula either by martingales or by partial differential equations (PDEs). Now we just talk about the latter method-- PDE approach. In that way, first we make the descriptions of the state variables with a stochastic differential equation (SDE) and then acquire a PDE, with the coefficients of the given SDE included. On the other hand, following the processes with jumps, the state variables can also be got. Naturally, the additional integral terms will come out. [7]

Constructing the stochastic processes, we can model the behavior of financial market. Partial differential equations act as a valuable part in the process of mathematical modeling. Feynman-Kac formula establishes the relationship between the stochastic analysis and PDEs.

2.3.1 Feynman-Kac Formula

$$\frac{\partial F}{\partial t}(t, x) + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F}{\partial x^2} = 0,$$
$$F(T, x) = \Phi(x)$$

Provided F is a solution to the above boundary value problem on $[0, T]$.

The formula presents us a probabilistic representation of solutions to PDEs, which has something to do with general SDEs. Consequently, by solving the corresponding PDE, the problem of pricing can be settled. [5]

2.3.2 The Black-Scholes Equation

In a financial market, there are only two kinds of assets, a risky asset whose price is S_t and a risk-free bank account B_t . They have the following equations,

$$dB_t = r B_t dt,$$
$$B_0 = 1$$

$$dS_t = \mu(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t,$$

$$S_0 = s_0$$

where the risk-free interest rate r is constant.

Consider a contingent claim of the form $\Phi(S(T))$, whose price process of the form $\Pi(t) = F(t, S(t))$, for some smooth function F , in the absence of arbitrage in the market, is the unique solution of the following boundary value problem on $[0, T] \times \mathbb{R}^+$.

$$F_t(t, s) + r s F_s(t, s) + \frac{1}{2} s^2 \sigma^2(t, s) F_{ss}(t, s) - r F(t, s) = 0 \quad (1)$$

$$F(T, s) = \Phi(s)$$

The solution of (1) exploits the Feynman-Kac representation below,

$$F(t, s) = e^{-r(T-t)} E_{t,s}^Q [\Phi(S(T))]$$

to obtain the arbitrage free price of a simple claim $\Phi(S(T))$.

In the equation above, S_t is the solution of

$$dS(u) = r S(u) du + \sigma S(u) dW(u)$$

$$S(t) = s,$$

for a Brownian motion $W^\circ(t)$.

Hence, we can obtain $S(T)$ like this,

$$S(T) = s \exp\{(r - 1/2\sigma^2)(T-t) + \sigma(W(T) - W(t))\}$$
 and the pricing formula,

$$F(t,s) = e^{-r(T-t)} \int \Phi(se^z) f(z) dz, \text{ where } z \text{ is a stochastic variable and } f \text{ is the density function.}$$

Now concerning pricing the derivatives, people mainly focus on the Black-Scholes equation.

3 Pricing and Modeling the Interest Rate

3.1 Interest Rate

In the bond market, a lot of interest rates are defined, such as the simple forward rate, the simple spot rate, the continuously compounded forward rate, the continuously compounded spot rate, the instantaneous forward rate and so on. Assume that we are now at time t , and define two future dates $T1$ and $T2$, with the relation $t < T1 < T2$. Now we decide to make an investment and expect to get a deterministic rate of return. Then we can consider taking steps like this,

1. At t , one bond called $T1$ -bond can be sold by us in the market and we get the profit $p(t, T1)$.
2. We use the amount of money from $T1$ -bond, that is, $p(t, T1)$ to buy $T2$ -bond. The share we get is $p(t, T1) / p(t, T2)$.
3. At $T1$, the $T1$ -bond matures and we have to make the payment, 1 SEK for simplicity.
4. At $T2$, the $T2$ -bond matures, and we can get 1 sek per share, i.e. $p(t, T1) / p(t, T2)$ SEK in total. [5]

In the investment shown above, we can draw the definition of forward rate. During the time interval $[T1, T2]$, we finally acquire an interest in a risk-free rate. Then we can name the interest rate as a forward rate. And we have the equation

$$1 + (T2 - T1) L = p(t, T1) / p(t, T2)$$

where L is LIBOR rate which is the solution to the equation.

Next let me introduce some kinds of forward rates of the mathematical form.

(1) The simple forward rate for $[T1, T2]$ has the form as

$$L(t; T1, T2) = - \frac{p(t, T2) - p(t, T1)}{(T2 - T1)p(t, T2)}$$

(2) The simple spot rate for $[T1, T2]$, has the form as

$$L(T1, T2) = - \frac{p(T1, T2) - 1}{(T2 - T1)p(T1, T2)}$$

(3) The continuously compounded forward rate for $[T1, T2]$, has the form as

$$R(t; T1, T2) = - \frac{\ln p(t, T2) - \ln p(t, T1)}{T2 - T1}$$

(4) The continuously compounded spot rate for $[T1, T2]$, $R(T1, T2)$, has the form as

$$R(T1, T2) = - \frac{\ln p(T1, T2)}{T2 - T1}$$

(5) The instantaneous forward rate, with the corresponding bond that matures at T , has the form as

$$f(t, T) = - \frac{\partial \ln p(t, T)}{\partial T}$$

(6) The instantaneous short rate at time t has the denotation

$$r(t) = f(t, t)$$

3.2 Bonds Pricing

The bank account process is described as the following equation,

$$\begin{aligned} dB(t) &= r(t)B(t)dt \\ B(0) &= 1 \end{aligned}$$

Then we have the direct formula of the bond price,

$$p(t, T) = p(t, s) \exp\left\{-\int_s^T f(t, u)du\right\}$$

Some Important Relations

We have short rate dynamics, bond price dynamics and forward rate dynamics equations,

Short rate dynamics

$dr(t) = a(t)dt + b(t)dW(t)$, in which the processes $a(t)$ and $b(t)$ are scalar adapted processes.

Bond price dynamics

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t) \quad (3.1)$$

where $m(t,T)$ and $v(t,T)$ are adapted processes parameterized by the maturity date T .

Forward rate dynamics

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \quad (3.2)$$

$\alpha(t,T)$ and $\sigma(t,T)$ are also adapted processes parameterized by the maturity date T .

(1) If $p(t,T)$ satisfies (3.1), then the following equations work,

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where α and σ satisfy,

$$\alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T),$$

$$\sigma(t,T) = -v_T(t,T).$$

(2) If $f(t,T)$ satisfies (3.2), then the following equations work,

$$dr(t) = a(t)dt + b(t)dW(t),$$

where

$$a(t) = f_T(t,t) + \alpha(t,t)$$

$$b(t) = \sigma(t,t)$$

(3) If $f(t,T)$ satisfies (3.2), then

$$dp(t,T) = p(t,T) \{r(t) + A(t,T) + 1/2 \|S(t,T)\|^2\} dt + p(t,T)S(t,T)dW(t),$$

where

$$A(t,T) = -\int_t^T \alpha(t,s)ds$$

$$S(t,T) = -\int_t^T \sigma(t,s) ds. \quad [5]$$

3.3 Short Rate Models

The short rate can be modelled as the solution to an stochastic differential equation (SDE) of the following form,

$$dr(t) = \mu(t,r(t)) dt + \sigma(t,r(t)) dW(t)$$

3.3.1 The Term Structure Equation

In an arbitrage free market, there exist T-bonds with all maturities, for each T, we can assume the price of a T-bond is,

$$p(t,T) = F(t,r(t);T) \quad (3.3)$$

And when the bond matures, the T-bond is valued 1 SEK, i.e. $F(T,r;T) = 1$, for any r.

Let me fix two maturity dates T_1 and T_2 , and apply Ito formula to (3.3). We can obtain the equations that express T_1 -bond and T_2 -bond like this,

$$dF^T = F^T \alpha_T dt + F^T \sigma_T dW,$$

in which r and t satisfies

$$\begin{aligned} \alpha_T &= \frac{F_t^T + \mu F_r^T + \frac{1}{2} \sigma^2 F_{rr}^T}{F^T}, \\ \sigma_T &= \frac{\sigma F_r^T}{F^T}. \end{aligned} \quad (3.4)$$

We make a relative portfolio by (x_{T_1}, x_{T_2}) , then the dynamics of the portfolio is that,

$$dV = V \left\{ x_{T_1} \frac{dF^{T_1}}{F^{T_1}} + x_{T_2} \frac{dF^{T_2}}{F^{T_2}} \right\}.$$

We combine the equation with (3.4) and finally we get

$$dV = V \{ x_{T_1} \alpha_{T_1} + x_{T_2} \alpha_{T_2} \} dt + V \{ x_{T_1} \sigma_{T_1} + x_{T_2} \sigma_{T_2} \} dW \quad (3.5)$$

Note: the equations

$$x_{T1} + x_{T2} = 1 \quad (3.6) \text{ and}$$

$$x_{T1}\sigma_{T1} + x_{T2}\sigma_{T2} = 0 \quad (3.7) \text{ work.}$$

At last we obtain the value dynamics from (3.5),

$$dV = V \{ x_{T1}\alpha_{T1} + x_{T2}\alpha_{T2} \} dt \quad (3.8)$$

From (3.7) and (3.6), we can solve

$$x_{T1} = - \frac{\sigma_{T2}}{\sigma_{T1} - \sigma_{T2}}$$

$$x_{T2} = - \frac{\sigma_{T1}}{\sigma_{T1} - \sigma_{T2}}$$

and further get the equation of the value dynamics

$$dV = V \left\{ \frac{\alpha_{T2}\sigma_{T1} - \alpha_{T1}\sigma_{T2}}{\sigma_{T1} - \sigma_{T2}} \right\} dt.$$

A process λ in a non-arbitrage market can be defined as,

$$\frac{\alpha_{T(t)} - r(t)}{\sigma_{T(t)}} = \lambda(t) \quad \text{for any choice of } T.$$

In a matter of fact, the process λ is the market price of risk; please refer to [5].

There exists term structure equation in the non-arbitrage market,

$$F_t^T + (\mu - \lambda\sigma)F_r^T + 1/2 \sigma^2 F_{rr}^T - rF^T = 0,$$

$$F^T(T,r) = 1. \quad [5]$$

From the series of the equations listed above, we can obviously conclude that the term structure equation is a standard PDE. When the process λ is determined, the bond prices of all can be determined with the help of the term structure equation.

3.3.2 Some Standard Martingale Models for the Short Rate

(1) The Vasicek Model

For this model, we have $dr = (b-ar)dt + \sigma dW$, and the term structure equations

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

$$B(t,T) = \frac{1}{a} \{1 - e^{-a(T-t)}\}$$

$$A(t,T) = \frac{\{B(t,T) - T + t\}(ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{a^2 B^2(t,T)}{4a}$$

Then the bond prices are determined by the equations above.

(2) The Ho-Lee Model

For this model, we have the term structure equations

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\{(T-t) f^*(0,t) - 1/2 \sigma^2 t(T-t)^2 - (T-t)r(t)\}$$

p^* is the observed bond prices and f^* is the observed forward rates in the market.

From the equation above, we can obtain the bond prices.

(3) The CIR Model

For this model, we have the term structure equations

$$F^T(t,r) = A_0(T-t)e^{-B(T-t)r},$$

where,

$$B(x) = \frac{2(e^{\gamma x} - 1)}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}$$

$$A_0(x) = \left[\frac{2\gamma e^{(a+\gamma)x/2}}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma} \right]^{2ab/\sigma^2}$$

(4) The Hull-White Model

The dynamics of the short rate under Q-measure is

$$dr = \{\Theta(t) - ar\} dt + \sigma dW(t), \text{ where } a \text{ and } \sigma \text{ are constant.}$$

Its term structure is

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp \left\{ B(t,T) f^*(0,t) - \frac{\sigma^2}{4a} B^2(t,T) (1 - e^{-2at}) - B(t,T) r(t) \right\}$$

$$B(t,T) = \frac{1}{a} \{ 1 - e^{-a(T-t)} \}$$

And then we can get the bond prices by the help of the equation.

4 Swap

Since we have introduced the concepts of several basic swaps in the first chapter, now we just need to give the mathematical form of the swaps.

4.1 Interest Rate Swap

In a time interval between T_0 and T_n , we divide the time into equally spaced dates $T_0, T_1, T_2, \dots, T_n$, and each date is 'pay time' except T_0 .

At T_i , the cash flow of the fixed rate is $K\epsilon R$, where K is the nominal principal, R is the swap rate and ϵ is time interval we divide; and the cash flow of the floating rate is $K\epsilon L(T_{i-1}, T_i)$.

Thus, we can easily get the net cash flow at T_i ,

$$K\epsilon [L(T_{i-1}, T_i) - R]. \quad (5.1)$$

Concerning the valuing problem, we can convert it into solving floating rate bond problem.

Since the valuation formula for the floating rate bond is,

$$p(t,T) = p(t,T_n) + \sum_{i=1}^n [p(t, T_{i-1}) - p(t, T_i)] = p(t, T_0) \quad (5.2)$$

and the coupon c_i has the relation,

$$c_i = (T_i - T_{i-1}) L(T_{i-1}, T_i) K = 1/p(T_{i-1}, T_i) - 1 \quad (5.3)$$

Combine (5.1)-(5.3), we can obtain the net cash flow representation as

$$Kp(t, T_{i-1}) - K(1+\varepsilon R)p(t, T_i)$$

And we can further get the total value of the swap as,

$$\Pi(t) = K \sum_{i=1}^n [p(t, T_0) - (1 + \varepsilon R)p(t, T_i)].$$

Since when the contract is made, the net cash flow is 0, the swap rate R can be got,

$$R = \frac{p(0, T_0) - p(0, T_n)}{\varepsilon \sum_{i=1}^n p(0, T_i)}. \quad [5]$$

4.2 The Forward Swap Rate

At time t, for the holder of the receiver swap, the total value he/she pays is,

$$\sum_{i=n}^{N-1} [p(t, T_i) - p(t, T_{i+1})] = p_n(t) - p_N(t)$$

Meanwhile, the holder of the payer swap, makes the total payments worth $R \sum_{i=n+1}^N \alpha_i p_i(t)$.

The par, i.e. the forward swap rate $R_n^N(t)$ of a swap with tenor $T_N - T_n$, equals to the value of R when the net cash flow is zero in the swap. Then we can get

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^N \alpha_i p_i(t)}$$

5 Swap Valuation

In this section, let me first introduce the basic concepts of the swap valuation and take one numerical example and analyze the currency swap valuation to better understand the procedure of the swap valuation.

5.1 Common Procedure of the Swap Valuation

Concerning the valuation, swaps can be regarded as the pricing of bonds. The factors that are related are, fluctuations in the interest rate levels, the swap spreads changing, the shape of the yield curve, foreign exchange rates valuation, notional principals, the maturity date, payment frequency, and floating rate reference index, etc..

How to measure the value of the interest rate swap in the market? Let us take the following four steps:

- (1) Construct a yield curve of zero coupon bond
- (2) Predict the future interest rates and calculate the future floating-rate cash flow
- (3) Derive the discount factors to value each swap fixed and floating rate cash flow
- (4) Discount the future cash flows and get the present values

Construct a Yield Curve

To build a yield curve, we need to combine current deposit rates, futures prices, treasury yields, and interest rate swap spreads together. Those factors make up today's coupon yield curve.

Predict the Future Short Rates

Due to the uncertainty of the forward rates dynamics, the floating leg of a swap becomes hard to price. Although what we know is just the present floating rate, predicting the future floating rates, i.e. a forward yield curve of short rate is necessary. Thus we can calculate the prospective floating leg. If the zero coupon curve has been

developed, it can be easily transformed into the forward curve, by which we can use to obtain floating rate cash flows.

Derive Discount Factors

Discount factors are the tool that converts the future cash flows into the present value. Similar to the forward interest rates, discount factors are a transformation of zero coupon rates. In the following part, we will emphasize the discussions on discount factors and take the cases in cross currency swaps as examples.

Discount and Value the Swap

The last step is to use the discount factors to find the present values of the fixed and floating swap cash flows. Then the net values determine the current price of the swap. It is possible for the values to be positive, zero, or even negative, depending on different interest rates prevailed in the market when the swap contract is signed.

5.2 Analysis of the Discount Factors in Two Different Pricing Methods of Cross Currency Swap

5.2.1 What is the Cross Currency Swap?

As an effective tool for managing foreign currency exposures, Cross Currency Swap (CCS) is a swap contract in which two interest rates being swapped on two legs are in two different currencies.

To better understand the mechanics of a cross currency swap, we start with the forward contract in the foreign exchange market. In a foreign exchange forward, the exchange of two different currencies on a future date will be realized at a forward price fixed today. Such a transaction can be thought of being made up of two processes, one is an exchange of the principal and the other is an exchange of the interest payments. The forward price is defined as the ratio of the total currency amounts exchanged finally.

Similar to a forward contract, both the exchanges of principals and interest payments are involved in a CCS. Unlike a forward, multiple exchanges of interest payments

exist even as well as the principals in a CCS.

In the Cross Currency Swap contract, the borrowers are allowed to raise funds in the financial markets where they have the opportunities to take loans at relatively lower interest rates, yet meanwhile, their liabilities are converted into the currency of their choice as well. That is, the currency denomination of assets and liabilities can be changed by CCS. In that way, not only can the counterparties balance the expected interest earnings and loss, but they can also manage the foreign currency risks involved in those assets and liabilities.

For example, Company A owns a SEK-denominated bond. To reduce the cost of loans, the company may consider turning to the lower interest rate JPY market for help. A CCS is used to create a JPY-denominated debt. The initial exchange converts the SEK-bond proceeds to yen, and the subsequent cash flows convert the interest payments and principal from SEK to yen. As shown in the example above, Cross Currency swaps provide companies with opportunities to reduce borrowing costs in both domestic and foreign markets. However, since currency swaps involve exchange risk on principal, the credit risk associated with these transactions is substantially greater than with interest rate swaps.

A note must be mentioned that exchanging Principals is usually included when the swap expires. A cross currency swap thus can be regarded as an exchange of interest rate payments between the bonds. Take the most common (vanilla) cross currency swap as an example, it is an exchange of floating or fixed-rate payments on principals P_1 and P_2 in domestic and foreign currencies respectively. First, we define an time interval $[T_0, T_N]$, and then divide it into N parts equally. The domestic and foreign payments P_d and P_f based on LIBOR are made with the frequency $(T_N - T_0)/N$. The swap value $V(t)$ is the accumulation of the payments $V_i(t)$, $i=1,2,\dots,N$, which is a PDE.

5.2.2 Two Valuation Methods for Cross Currency Swap

First we define a fixed set of increasing maturities $T_0, T_1, T_2, \dots, T_N$, $\alpha_i = T_i - T_{i-1}$, p_i is zero coupon bond price $p(t, T_i)$ and $L_i(t)$ denote the LIBOR forward rate contracted at t during the interval $[T_{i-1}, T_i]$. Then we have

$$L_i = \frac{1}{\alpha_i} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)} \quad (6)$$

In order to get the dynamics of the L_i , we must get the dynamics of p_i . That can be done by choosing an appropriate short rate model arbitrarily introduced in Section 4, for instance, the Ho-Lee Model. Finally we can obtain a PDE which describes the dynamics of L_i . By choosing some numerical method like Monte Carlo, we can

simulate L_i in discrete time because LIBOR rate is a discrete market rate instead of the continuously time market rate like instaneously interest rate.

However, in my thesis, we mainly focus on the performance of the discount factors in two different cross currency swap valuation methods. Therefore, for simplicity, we just directly cite the data from [8].

Of all types of cross currency swaps, the floating to floating cash flows type is called a basis swap. That is, a basis swap can be regarded as a contract in which two floating rate bonds are exchanged. Since the two principals cannot be equal all the time because of the change in the exchange rate. Then, how can the market manage the problem of making basis swaps to be fair? The answer is the marker introduces so-called cross currency basis spread. The cross currency basis spreads usually refer to a liquidity benchmark, e.g. USD LIBOR.

The discount factors are used to discount the cash flows of the respective currency in the procedure of the swap valuation. We have to involve the cross currency basis spread in the valuation methodology, in order to avoid the probability of arbitrage. Let us denote the basis spread s_i . If s_i is the fair spread, it means that a floating rate bond that matures on T_i in a certain currency, pays Libor plus spread s_m values to par. [8]

(1) The First Valuation Method

Let us first introduce a very popular methodology. In this approach, two discount factor curves are applied, one is used for projecting forward rates and the other is used for discounting all the cash flows.

The standard curve of discount factors $DF(t)$ can be derived from the recursive bootstrapping relationship,

$$DF(T_n) = \frac{1 - C_n \sum_{i=1}^{n-1} \delta_i DF(T_i)}{1 + \delta_n C_n}, n= 1,2,\dots \quad (6.2)$$

where δ_n is the interval between T_n and T_{n-1} , and C_n is the fair swap rate in the market.

$DF(t)$ is used to project forward rates according to the following formula,

$$L_i^0 = \frac{\frac{DF(T_{i-1})}{DF(T_i)} - 1}{\delta_i} \quad (6.3)$$

The formula above is the definition of the forward rate L_i^0 , which is projected from the discount curve, for the interval $[T_{i-1}, T_i]$.

Introducing $DF^*(t)$ as the other curve of discount factor, we intend to ensure that Libor plus spread s_n is at par in one floater. $DF^*(t)$ is used to discount any cash flows, fixed or floating ones. We define $DF^*(t)$ as the form of

$$1 = \sum_{i=1}^n \delta_i(L_i^0 + s_n) DF^*(T_i) + DF^*(T_n), n=1, 2, \dots$$

Further we can obtain the representation of DF^* as

$$DF^*(T_n) = \frac{1 - \sum_{i=1}^{n-1} \delta_i(L_i^0 + s_n) DF^*(T_i)}{1 + \delta_n(L_n^0 + s_n)}, n=1, 2, \dots \quad (6.4)$$

(6.4) is actually the recursive bootstrapping relation.

In this method, we obtain different values of the discount factor $DF^*(T_n)$ by adjusting the parameters C_n and s_n . Let us list them as follows. (The data come from [8]). For simplicity, we assume that the payment frequency is one year and set a 30/360 day count convention.

| T_n | C_n | s_n | DF^* from (6.4) |
|-------|-------|--------|-------------------|
| 1 | 5.00% | -0.10% | 0.953289 |
| 2 | 5.10% | -0.12% | 0.907339 |
| 3 | 5.20% | -0.14% | 0.862218 |
| 4 | 5.30% | -0.16% | 0.817985 |
| 5 | 5.40% | -0.18% | 0.774694 |
| 6 | 5.50% | -0.20% | 0.732392 |
| 7 | 5.60% | -0.22% | 0.691121 |
| 8 | 5.70% | -0.24% | 0.650917 |
| 9 | 5.80% | -0.26% | 0.611810 |
| 10 | 5.90% | -0.28% | 0.573823 |

(2) The Second Valuation Method

In the second method, we still use two discount factors curves. $DF(t)$ is used for discounting the fixed cash flows, while $DF^*(t)$ is used to discount the floating cash streams.

Remark: This approach is also applied to single currency swaps, while the first one is not. There are two conditions that should be satisfied. One is that the value of a floating rate bond equals to the value of a bond whose coupon is the swap rate C_n . The other is that a floating rate bond, which make payments in terms of Libor plus cross currency basis spread s_n , is valued to par.

The two conditions mentioned above imply that a fixed coupon bond is valued par if it pays the holder the coupon C_n plus the cross currency basis spread s_n . That is, assuming the payment frequencies of the fixed and floating legs are the same, we have:

$$1 = \sum_{i=1}^n \delta_i C_n DF(T_i) + \sum_{i=1}^n \delta_i s_n DF(T_i) + DF(T_n)$$

We can draw the curve $DF(t)$ from that equation. And we have the following simple bootstrapping equation

$$DF(T_n) = \frac{1 - \sum_{i=1}^{n-1} (\delta_i C_n + \delta_i s_n) DF(T_i)}{1 + \delta_n C_n + \delta_n s_n}$$

Definition 6.1:

The value today of the floating interest rate (LIBOR rate) cash flow for period $[T_{i-1}, T_i]$ is denoted as,

$$DF^*(T_{i-1}) - DF^*(T_i)$$

Now we restate that a floating rate bond, which make payments in terms of Libor plus cross currency basis spread s_n , is valued par. In this condition, it now implies that,

$$1 = DF^*(T_0) - DF^*(T_n) + \sum_{i=1}^n \delta_i s_n DF(T_i) + DF(T_n)$$

If we set $DF^*(T_0) = 1$, $T_0=0$, then we have,

$$DF^*(T_n) = DF(T_n) + s_n \sum_{i=1}^n \delta_i DF(T_i) \quad (6.5)$$

And then we can determine $DF^*(r)$, the second curve of discount factors.

Now think of the example taken in the previous method and use the new method to illustrate it again by the same two parameters C_n and s_n . And calculate the discount factor curve DF^* . We obtain,

| T_n | C_n | s_n | DF* from (6.4) |
|-------|-------|--------|----------------|
| 1 | 5.00% | -0.10% | 0.952336 |
| 2 | 5.10% | -0.12% | 0.905108 |
| 3 | 5.20% | -0.14% | 0.858412 |
| 4 | 5.30% | -0.16% | 0.812335 |
| 5 | 5.40% | -0.18% | 0.766959 |
| 6 | 5.50% | -0.20% | 0.722358 |
| 7 | 5.60% | -0.22% | 0.678601 |
| 8 | 5.70% | -0.24% | 0.635750 |
| 9 | 5.80% | -0.26% | 0.593860 |
| 10 | 5.90% | -0.28% | 0.552980 |

5.3 Simulation and Conclusion

By MATLAB, we can graph the relationship between the swap rate C_n and the discount factor DF^* in Figure 1, and the relationship between the cross currency basis swap spread rate s_n and the discount factor DF^* in Figure 2. In addition, we can display and compare the performances of $DF^*(t)$ on the condition that both C_n and s_n are the same in the two different methods in Figure 3. The source codes can be found in Appendix. A fact to be noted is that though we graph the relations in lines or curves, the values of $DF^*(t)$ are discrete. Yet, for us, it is easy to get an insight into the relations in the way that we perform them in continuous form.

Now let us compare the values of $DF^*(t)$ in different methods first. As is shown in Figure 1, it is obvious that both the values of $DF^*(t)$ in different methods decrease with the increase of the swap rate C_n . The disparities of $DF^*(t)$ values in the two methods seem to become bigger and bigger, with the rise of C_n . And the values of $DF^*(t)$ in the first method is always a little higher than the ones in the second method. That means, when we discount the floating rate cash flows in swap valuation, we always obtain a higher today's value of the floating rate cash flows, by means of the first method, supposed that the other conditions are the same.

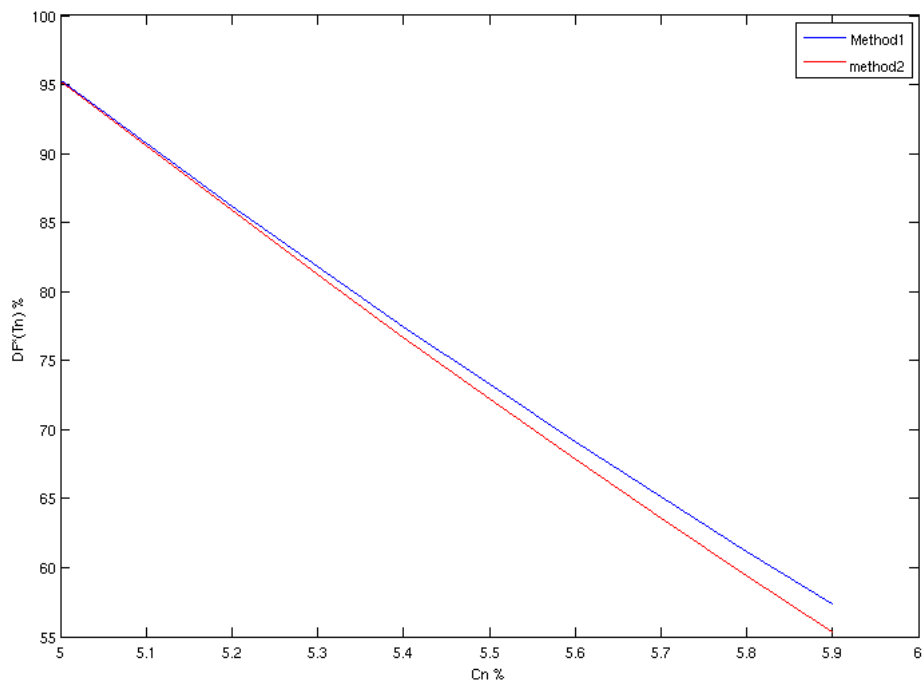


Figure 1

In Figure Two, we can see that both the values of $DF^*(t)$, in different methods, increase with the rise of the cross currency basis swap spread rate s_n . The disparities of $DF^*(t)$ values in the two methods seem to become smaller and smaller, with the increase of s_n . And the values of $DF^*(t)$ in the first method is always a little higher than the ones in the second method. That also means, assuming that the other conditions are identical, we always obtain a higher today's value of the floating rate cash flows, in the first method, when we discount the floating rate cash flows in swap valuation.

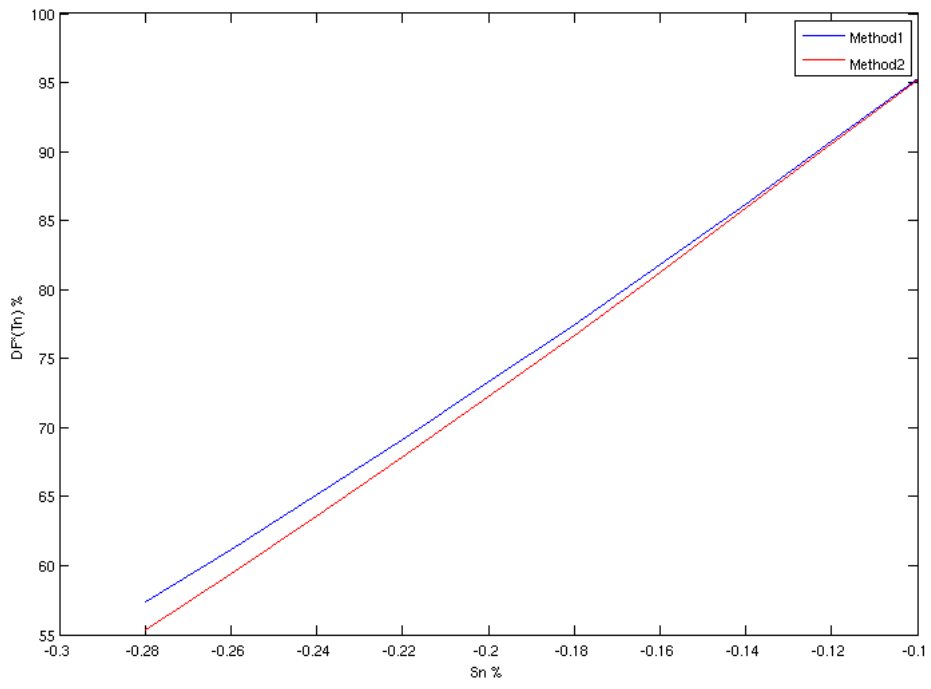


Figure 2

In Figure Three, we compare the values of $DF^*(t)$ in the same conditions, i.e. the same C_n and s_n by the two different methods. We can easily observe that despite some small disparities, the values of $DF^*(t)$ in different methods are very close. Yet, if we do some deeper research, it is not difficult to find that the values of $DF^*(t)$ in the first method is a little higher than the ones in the second method all the time. That is to say, a higher today's value of the floating rate cash flows seems to occur if we use the first method instead of the second when the floating rate cash flows discounted in swap valuation.

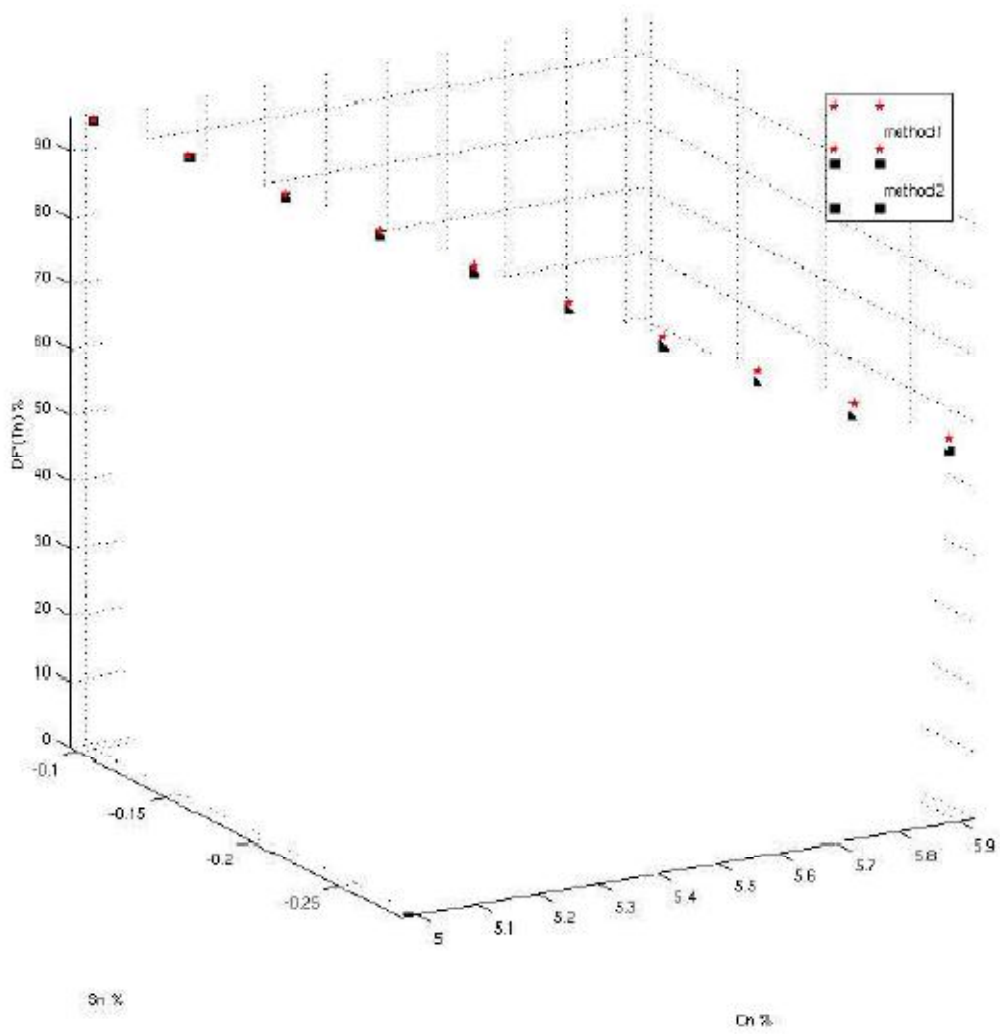


Figure 3

In general, we cannot judge which method is more effective in swap valuation just by the simulation of the discount factors $DF^*(t)$. Yet, since there are always different values of DF^* yielded, when we use the different methods to value the swap. At least, we can speculate that there are arbitrage opportunities due to that. Further research has to be carried out to show which methodology the market participants prefer to adopt.

Appendix

Matlab Source Codes

```
%%  
clear all  
clc  
  
x=5 : 0.1 :5.9;  
y=-0.1 : -0.02 : -0.28;  
%%plot 3D  
[X,Y]=meshgrid(x,y);  
  
Z=[  
    95.3289,0,0,0,0,0,0,0,0;  
    0,90.7339,0,0,0,0,0,0,0;  
    0,0,86.2218,0,0,0,0,0,0;  
    0,0,0,81.7985,0,0,0,0,0;  
    0,0,0,0,77.4594,0,0,0,0;  
    0,0,0,0,0,73.2392,0,0,0;  
    0,0,0,0,0,0,69.1121,0,0;  
    0,0,0,0,0,0,0,65.0917,0;  
    0,0,0,0,0,0,0,0,61.1810;  
    0,0,0,0,0,0,0,0,0,57.3823;  
    ];  
mesh(X,Y,Z);hold on
```

```
Z1=[  
95.2336,0,0,0,0,0,0,0,0;  
0,90.5108,0,0,0,0,0,0,0;  
0,0,85.8412,0,0,0,0,0,0;  
0,0,0,81.2335,0,0,0,0,0;  
0,0,0,0,76.6959,0,0,0,0;  
0,0,0,0,0,72.2358,0,0,0;  
0,0,0,0,0,0,67.8601,0,0;  
0,0,0,0,0,0,0,63.5750,0;  
0,0,0,0,0,0,0,0,59.3860;  
0,0,0,0,0,0,0,0,0,55.2980;  
    ];
```

```
mesh(X,Y,Z1);
%%plot 2D
%
z1=[95.3289,90.7339,86.2218,81.7985,77.4594,73.2392,69.1121,65.0917,61.1810,57.
3823];
%
%
z2=[95.2336,90.5108,85.8412,81.2335,76.6959,72.2358,67.8601,63.5750,59.3860,55.
2980];
%
% plot(x,z1);hold on
% plot(x,z2,'r');
% figure
% plot(y,z1);hold on
% plot(y,z2,'r');
```

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