NEW UNIFORM BOUNDS FOR A WALSH MODEL OF THE BILINEAR HILBERT TRANSFORM

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1. Introduction

The notion of bilinear Hilbert transform usually refers to a member of a family of bilinear operators parameterized by a unit vector β perpendicular to $(1,1,1)$. We will write the bilinear operators in this family more symmetrically as dual trilinear forms Λ_β, acting on three test functions on the real line:

$$\Lambda_\beta(f_1,f_2,f_3) := p.v. \int f_1(x-\beta_1 t) f_2(x-\beta_2 t) f_3(x-\beta_3 t)\, \frac{dt}{t} \quad .$$

The interesting case, which we call non-degenerate, is when the three components of β are pairwise different. If two of the components of β are equal the form reduces to the combination of a pointwise product and the dual of the classical linear Hilbert transform. A priori L^p bounds in the non-degenerate case were first shown in [2] and [3]. Namely, for each $1 < p_1, p_2, p_3 \leq \infty$ with $\sum_j 1/p_j = 1$ we have

$$\Lambda_\beta(f_1,f_2,f_3) \leq C_{\beta,p_1,p_2,p_3} \prod_{j=1}^{3} \|f_j\|_{p_j} \quad . \tag{1}$$

The condition $\sum_j 1/p_j = 1$ is necessary by dilation symmetry of the form Λ_β and shall be assumed throughout the rest of this discussion.

If each f_j is bounded by the characteristic function of a set E_j, then inequality (1) implies the restricted type estimate

$$\Lambda_\beta(f_1,f_2,f_3) \leq C_{\beta,\alpha_1,\alpha_2,\alpha_3} \prod_{j=1}^{3} |E_j|^{\alpha_j} \tag{2}$$

where $\alpha_j = 1/p_j$ satisfies $0 \leq \alpha_j < 1$. More generally, the argument in [3] shows that inequality (2) continues to hold in the range $-1/2 < \alpha_j < 1$ under the additional assumption that if $\alpha_j < 0$ then f_j is bounded by the characteristic function of a major subset $E_j{}' \subset E_j$

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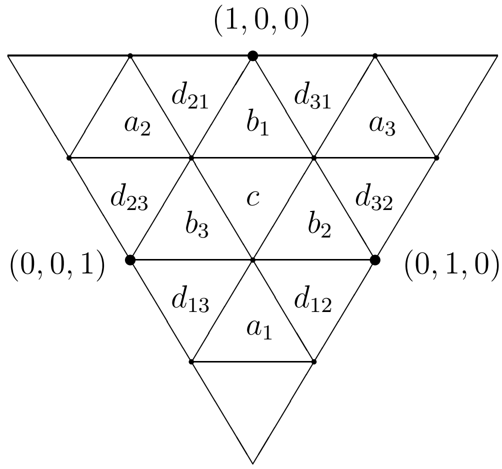

that depends on the sets E_1, E_2, and E_3. Here a major subset is one of measure at least half the measure of the ambient set. The passage to a major subset of E_j is natural and necessary in the setting of negative exponents and was introduced in this context in [6].

The range of triples $(\alpha_1, \alpha_2, \alpha_3)$ for which one has the a priori estimate (2) appears in Figure 1 as the the convex hull of the open triangles a_1, a_2, and a_3. Note that the closed triangle c represents the local L^2 case with $2 \leq p_1, p_2, p_3 \leq \infty$, while the convex hull of the open triangles b_1, b_2, b_3 represents the reflexive Banach triangle where $1 < p_1, p_2, p_3 < \infty$.

In the degenerate case, say $\beta_2 = \beta_3$, a priori estimates estimates for Λ_β follow from Hölder's inequality and bounds for the linear Hilbert transform. One has bounds of the type (2) if $\alpha_1 > 0$ and $\alpha_1, \alpha_2, \alpha_3 < 1$. The intersection of this region with the region of bounds for the non-degenerate case is the convex hull of the open triangles a_2, a_3, b_2, b_3. It is natural to ask whether one has bounds for the non-degenerate case uniformly in the parameter β in a small neighborhood of the degenerate case $\beta_2 = \beta_3$. Several articles have been written on this question: [11] proves inequality (2) uniformly in such β at the two upper corners of the triangle c under the assumption f_j is supported on a major subset $E_j{}' \subset E_j$ when $\alpha_j = 0$. Grafakos and Li [1] show inequality (1) in the triangle c and Li [4] shows (2) uniformly in the open triangles a_2 and a_3. One can interpolate these results to get bounds in the convex hull of the open triangles a_2, a_3, and c, but it remains open to date whether uniform bounds hold in the entire open triangles b_2 and b_3. The current paper presents progress in this direction by proving uniform bounds in b_2 and b_3 for a discrete model of the bilinear Hilbert transform.

The quartile operator was introduced in [10] as a discrete model for the non-degenerate bilinear Hilbert transform. In [12] a family of related operators was introduced that models the set of Hilbert transforms near the degenerate case and allows to address uniformity questions in the model case. Moreover, inequality (2) was shown at the two upper corners of the triangle c under the assumption f_j is supported on a major subset of E_j if $\alpha_j = 0$. In the current paper, we extend these results to the entire convex hull of the open triangles a_2, a_3, b_2, b_3 and thus the full range in which we know bounds both for the degenerate and the non-degenerate case. Our proof simplifies that in [11], using an approach via phase plane projections developed in the continuous case in [7]. It also uses a simple discrete version of the multi-frequency Calderon Zygmund decomposition introduced in [9] as well as a technique of [8] of using BMO bounds for the counting function defined further below. In this sense this article also serves as expository survey of these techniques in the discrete setting. We plan to address the extension of the novel results in this paper to the continuous setting in future work.

We proceed to formulate the main theorems of this paper in detail. The Walsh phase plane is the closed first quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ of the plane. A dyadic rectangle is a rectangle in the Walsh phase plane of the form

$$p = I \times \omega = [2^k n, 2^k(n+1)) \times [2^{-k'} l, 2^{-k'}(l+1)) \tag{3}$$

with intergers k, k', n, l and $0 \le n, l$. A tile is a dyadic rectangle of area one, while a bitile is a dyadic rectangle of area two. Each bitile can be split into upper tile P_u and lower tile P_d, or alternatively into left tile P_{left} and right tile P_{right}. Associated to each tile p is a Walsh wave packet w_p, which is a certain function in $L^2(\mathbb{R}_+)$ normalized to have norm one. With the notation as in (3), if $l = 0$, then this wave packet is defined as the appropriate multiple of the characteristic function of I. For other values of l it is defined recursively via the identities

$$w_{P_u} = (w_{P_{\text{left}}} - w_{P_{\text{right}}})/\sqrt{2} \quad ,$$

$$w_{P_d} = (w_{P_{\text{left}}} + w_{P_{rmright}})/\sqrt{2} \quad .$$

By induction on the depth of this recursion one can show ([10]) that w_p is supported on I, it has constant modulus on I, and disjoint tiles correspond to orthogonal wave packets. If S is a subset of the Walsh phase plane that can be written as a disjoint union of a collection **p** of tiles, we define the phase plane projection associated to S to be the orthogonal projection

$$\Pi_S f = \sum_{p\in\mathbf{p}} \langle f, w_p\rangle\, w_p \quad .$$

One can show that this projection is independent of the particular tiling $\mathbf{p}$ of the set S, justifying the notation that ignores the particular choice of tiling. For a subset S in the phase plane and an integer L define $2^L S$ to be the set $\{(x, 2^L\xi), (x,\xi)\in S\}$.

We define the quartile[1] form with parameter $L \geq 2$ as follows:

$$\Lambda_L(f_1, f_2, f_3) := \int \sum_P w_{P_d}(x) \Pi_{P_u} f_1(x) \prod_{j=2}^{3} \Pi_{2^L P_d} f_j(x)\, dx \quad .$$

Here P runs through the set of all bitiles. To avoid technical arguments we shall restrict this set to the set of all bitiles contained in the strip $\mathbb{R}_+ \times [0, 2^N)$ for some very large N. This restriction is equivalent to assuming that f_1 is constant on intervals of length 2^{-N}. The bounds claimed in the following theorems are independent of N. While fixing N destroys the dilation symmetry of the form Λ_L, the family of Λ_L for all such N retains the dilation symmetry and so do the main theorems. We will avoid explicit mentioning of N in most of this paper.

Our main results are the following two theorems:

Theorem 1.1. *For any exponents* $1 < p_1, p_2, p_3 < \infty$ *with*

$$1/p_1 + 1/p_2 + 1/p_3 = 1$$

there is a constant C_{p_1,p_2,p_3} *independent of* L *(and* N*) such that we have the a priori estimate*

$$|\Lambda_L(f_1, f_2, f_3)| \leq C_{p_1,p_2,p_3} \prod_{j=1}^{3} \|f_j\|_{p_j} \quad .$$

Theorem 1.2. *Let* $0 < \alpha_1, \alpha_3 < 1$ *and* $-1/2 < \alpha_2 \leq 0$ *with* $\sum_j \alpha_j = 1$. *For any three measurable subsets* E_j, $j = 1, 2, 3$ *of* $\mathbb{R}_+$ *such that* $|E_2|$ *is maximal among the* $|E_j|$ *there is a major subset* $E_2{}'$ *of* E_2 *such that for any three measurable functions* f_j, *bounded in absolute value by the characteristic function of* E_j *if* $j \neq 2$ *and the characteristic function of* $E_2{}'$ *if* $j = 2$, *we have the following estimate*

$$|\Lambda_L(f_1, f_2, f_3)| \leq C_{\alpha_1,\alpha_2,\alpha_3} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3}$$

uniformly in the parameter L *(and* N*).*

[1] A quartile is a dyadic rectangle of area four. The name quartile form is inherited from the use of quartiles in [10] to define a related model.

Note that if $|E_2|$ is not maximal among the $|E_j|$, the conclusion of Theorem 1.2 follows with $E_2{}' = |E_2|$ from an appliciation of Theorem 1.1 with a different set of exponents. Since the quartile form is symmetric in the indices $j = 2, 3$, one obtains as corollary a symmetric version of Theorem 1.2. The proofs of these theorems are sufficiently robust to allow for a perturbation of the quartile form by an arbitrary bounded sequence $|c_P| \leq 1$:

$$\Lambda_L(f_1, f_2, f_3) := \int \sum_P [c_P w_{P_d}(x)] \, \Pi_{P_u} f_1(x) \prod_{j=2}^{3} \Pi_{2^L P_d} f_j(x) \, dx \quad .$$

This flexibility adds to the usefulness of our arguments as a model situation for bilinear singular integrals.

In Section 2 we prove estimates for trees, which are subcollections of bitiles with lacunary structure, and present a tree selection algorithm. In Section 3 we use these ingredients to assemble the proofs of Theorems 1.1 and 1.2.

2. Trees

A dyadic rectangle $P = I \times \omega$ is less than or equal to another dyadic rectangle $P' = I' \times \omega'$, in writing $P \leq P'$, if $I \subset I'$ and $\omega' \subset \omega$. Two dyadic rectangles of the same area are comparable under this order relation if and only if they have nonempty intersection.

A tree T is a collection of bitiles with a unique maximal element, usually denoted by $P_T = I_T \times \omega_T$.

A set $\mathbf{P}$ of bitiles is called convex if for any two elements $P_1, P_2 \in \mathbf{P}$ and any bitile P which satisfies $P_1 < P < P_2$ we have $P \in \mathbf{P}$. It is shown in [10] by induction on the number of bitiles that for any convex set $\mathbf{P}$ of bitiles the union $\bigcup_{P \in \mathbf{P}} P$ may be written as the disjoint union of tiles. If the set $\mathbf{P}$ is a convex tree T, one such a tiling is obtained by decomposing the tree as union of three trees

$$T = \{P_T\} \cup T_u \cup T_d \tag{4}$$

where

$$T_d = \{P \in T : P_T \leq P_d\} \quad ,$$
$$T_u = \{P \in T : P_T \leq P_u\} \quad ,$$

and writing

$$\bigcup_{P \in T} P = (P_T)_u \cup (P_T)_l \cup \bigcup_{P \in T_u} P_d \cup \bigcup_{P \in T_d} P_u \quad . \tag{5}$$

For a convex tree T define the phase plane projection

$$\Pi_T = \Pi_{\bigcup P : P \in T} \quad .$$

For a tree T and a point $\xi \in \omega_T$ define the enlarged tree $T^{(L)}$ to be the set of all bitiles P such that $2^L\xi \in \omega_P$ and P is contained in a rectangle $2^L P'$ with $P' \in T$. The maximal element of $T^{(L)}$ is the unique bitile P in $T^{(L)}$ with $I_P = I_T$. If T is convex then $T^{(L)}$ is also convex. The phase plane projection

$$\Pi_{T^{(L)}} = \Pi_{\bigcup P : P \in T^{(L)}}$$

does not depend on the choice of the frequency ξ, because this choice is only relevant for the bitiles of $T^{(L)}$ which are contained in $2^L P_T$ and these bitiles cover all of $2^L P$ independently of this choice.

Define the trilinear form associated to any subset $\mathbf{P}' \subset \mathbf{P}$ by

$$\Lambda_{\mathbf{P}'}(f_1, f_2, f_3) = \int \sum_{P \in \mathbf{P}'} w_{P_d}(x) \Pi_{P_u} f_1(x) \prod_{j=2}^{3} \Pi_{2^L P_d} f_j(x)\, dx \quad .$$

For a convex collection $\mathbf{P}'$ of tiles define

$$\mathbf{size}(\mathbf{P}', f) = \sup_T |I_T|^{-1/2} \|\Pi_T f\|_2$$

or more gnerally for $L \geq 1$

$$\mathbf{size}^{(L)}(\mathbf{P}', f) = \sup_T |I_T|^{-1/2} \|\Pi_{T^{(L)}} f\|_2 \quad ,$$

where in each case the supremum is taken over all convex trees that are subset of the collection $\mathbf{P}'$.

Lemma 2.1 (Tree Estimate). *For any convex tree T and any three functions f_1, f_2, and f_3 in $L^2(\mathbb{R}_+)$ we have*

$$|\Lambda_T(f_1, f_2, f_3)| \leq C |I_T|\, \mathbf{size}(T, f_1)\, \mathbf{size}(T^{(L)}, f_2)\, \mathbf{size}(T^{(L)}, f_3) \tag{6}$$

with constant C independent of L.

Proof: Following the decomposition (4) it suffices to prove the estimate for the three summands of

$$\Lambda_T = \Lambda_{\{P_T\}} + \Lambda_{T_u} + \Lambda_{T_d}$$

separately. The form Λ_{T_u} is estimated by a double application of Cauchy Schwarz:

$$\Lambda_{T_u}(f_1, f_2, f_3) \leq \left(\sup_{P \in T} \|\Pi_{P_u} f_1\|_\infty \right) \prod_{j=2}^{3} \left(\sum_{P \in T_u} \|\Pi_{2^L P_d} f_j\|_2^2 \right)^{1/2} \quad .$$

To estimate the first factor on the right-hand-side we consider for any individual bitile P the size estimate for the tree $\{P\}$ and obtain

$$\|\Pi_{P_u} f_1\|_\infty \leq |I_P|^{-1/2} \|\Pi_{P_u} f_1\|_2 \leq \mathbf{size}(\{P\}, f_1) \leq \mathbf{size}(T, f_1)$$

where the first inequality follows from the fact that $\Pi_{\mathbf{P}_u}$ is a rank one projection onto the space of multiples of w_{P_u}. To estimate the other two factors we observe

$$\sum_{P\in T_u} \|\Pi_{2^L P_d} f_j\|_2^2 \le \sum_{P\in (T^{(L)})_u} |\langle f_j, w_{P_d}\rangle|^2 \le \mathbf{size}(T^{(L)}, f_j)|I_T|^{1/2} \quad .$$

where the first inequality follows by covering each of the pairwise disjoint rectangles $2^L P_d$ by tiles of the form P_d with $P \in T^{(L)}$. Combining these estimates completes the bound for the tree T_u.

The form $\Lambda_{\{P_T\}}$ is estimated very similarly, so it remains to estimate the form Λ_{T_d}. We first reduce the task to proving the estimate for the special case that T is a complete tree, i.e., it contains all bitiles P with $P \le P_T$. We have the following identity

$$\Lambda_{T_u}(f_1, f_2, f_3) = \Lambda_{T_u}(\Pi_T f_1, \Pi_{T^{(L)}} f_2, \Pi_{T^{(L)}} f_3) \tag{7}$$

by an application of the fact shown in [10] that for any two sets $S_1 \subset S_2$ in the phase plane which can be covered by disjoint unions of tiles we have

$$\Pi_{S_1} = \Pi_{S_1}\Pi_{S_2} \tag{8}$$

Similarly, the right hand side of (6) is equal to

$$C|I_T|\,\mathbf{size}(T, \Pi_T f_1)\,\mathbf{size}(T^{(L)}, \Pi_{T^{(L)}} f_2)\,\mathbf{size}(T^{(L)}, \Pi_{T^{(L)}} f_3) \quad .$$

Define the completion of T to be the tree $\tilde{T}$ of all bitiles which satisfy $P \le P_T$ and let $\tilde{T}_d$ be defined as in (4). If $P \in \tilde{T}_d \setminus T_d$ then

$$\langle w_{P_u}, \Pi_T f_1\rangle = 0 \quad ,$$

hence (7) is also equal to

$$|\Lambda_{\tilde{T}_d}(\Pi_T f_1, \Pi_{T^{(L)}} f_2, \Pi_{T^{(L)}} f_3)| \quad .$$

Next, we have

$$\mathbf{size}(\tilde{T}, \Pi_T f_1) \le \mathbf{size}(T, \Pi_T f_1)$$

because if S is a convex sub-tree of $\tilde{T}$ then

$$\|\Pi_S(\Pi_T f_1)\|_2 = \|\Pi_{S\cap T}(\Pi_T f_1)\|_2 \le |I_S|^{1/2}\mathbf{size}(T, \Pi_T f_1)$$

since the intersection of two convex subtrees of a given tree is either empty or again a convex tree. Similarly we have for $j = 2, 3$

$$\mathbf{size}(\tilde{T}^{(L)}, \Pi_{T^{(L)}} f_1) \le \mathbf{size}(T^{(L)}, \Pi_{T^{(L)}} f_1) \quad .$$

Hence it suffices to prove

$$|\Lambda_{\tilde{T}_d}(\Pi_T f_1, \Pi_{T^{(L)}} f_2, \Pi_{T^{(L)}} f_3)|$$
$$\le C|I_T|\,\mathbf{size}(\tilde{T}, \Pi_T f_1)\,\mathbf{size}(\tilde{T}^{(L)}, \Pi_{T^{(L)}} f_2)\,\mathbf{size}(\tilde{T}^{(L)}, \Pi_{T^{(L)}} f_3) \quad ,$$

which however follows from the special case of the desired inequality for complete trees. This completes the reduction to this special case and we may for the rest of the argument assume that T is a complete tree.

By dilation we may assume $|I_T| = 1$. Choose a frequency $\xi \in \omega_T$. and define for $l \geq 0$ the interval ω_l to be the dyadic interval of length 2^l which contains $2^L\xi$. Define

$$\Pi_l := \Pi_{I_T \times \omega_l}$$

and for $l \geq 1$ define

$$\Pi_l^\Delta = \Pi_l - \Pi_{l-1}$$

Then we may write for $\Lambda_{T_d}(f_1, f_2, f_3)$ by a telescoping argument

$$\int \sum_{l=0}^{\infty} \sum_{P \in T_d : |I_P| = 2^l} w_{P_d}(x) \Pi_{P_u} f_1(x) \prod_{j=2}^{3} \left(\Pi_l f_j(x) + \sum_{m=1}^{L} \Pi_{l+m}^\Delta f_j(x) \right) dx$$

The crucial fact then is that for $|I_P| = 2^l$ and $m \neq m'$ with at least one of m,m' greater than one we have

$$\int w_{P_d}(x) \Pi_{P_u} f_1(x) \Pi_{l+m}^\Delta f_2(x) \Pi_{l+m'}^\Delta f_3(x)\, dx = 0 \quad . \tag{9}$$

Namely, the product $w_{P_d}(x)\Pi_{P_u} f_1$ is a multiple of the Haar function on I_P. On the other hand, the product $\Pi_{l+m}^\Delta f_2 \Pi_{l+m'}^\Delta f_3$ is not constant on either half of I_P, and thus has mean zero on either half. Likewise, (9) holds if $m = m' = 1$ or if $m = m' = 0$ and Π_l^Δ is replaced by Π_l, because then the product of the two factors involving f_2 and f_3 restricted to I_P is the multiple of the square of a Walsh wave packet on this interval and thus constant on the interval.

Hence we can write for $\Lambda_{T_d}(f_1, f_2, f_3)$ as sum of three terms:

$$\int \sum_{l=0}^{\infty} \sum_{P \in T_d : |I_P| = 2^l} w_{P_d}(x) \Pi_{P_u} f_1(x) \Pi_l f_2(x) \Pi_{l+1}^\Delta f_3(x)\, dx$$

$$+ \int \sum_{l=0}^{\infty} \sum_{P \in T_d : |I_P| = 2^l} w_{P_d}(x) \Pi_{P_u} f_1(x) \prod_{j=2}^{3} \Pi_{l+1}^\Delta f_2(x) \Pi_l f_3(x)\, dx$$

$$+ \int \sum_{l=0}^{\infty} \sum_{P \in T_d : |I_P| = 2^l} w_{P_d}(x) \Pi_{P_u} f_1(x) \sum_{m=2}^{L} \prod_{j=2}^{3} \Pi_{l+m}^\Delta f_j(x)\, dx \quad .$$

The first two summands are estimated by a double application of Cauchy Schwarz very similar to the case of T_u. Here the factor involving Π_l is estimated by the supremum over l and Cauchy Schwarz is applied to the other two factors.

To estimate the third summand, we change the order of summation and then estimate by Cauchy Schwarz:

$$|\int \sum_{l=2}^{\infty} \left(\sum_{P \in T_d : 2^{l-L} \le |I_P| \le 2^{l-2}} w_{P_d}(x) \Pi_{P_u} f_1(x) \right) \prod_{j=2}^{3} \Pi_l^{\Delta} f_j(x)\, dx|$$

$$\le \left\| \sup_l |\sum_{P \in T_d : 2^{l-L} \le |I_P| \le 2^{l-2}} w_{P_d} \Pi_{P_u} f_1| \right\|_2 \left\| \sum_{l=2}^{\infty} \prod_{j=2}^{3} |\Pi_l^{\Delta} f_j| \right\|_2 \tag{10}$$

The second factor we estimate by interpolation between L^1 and BMO. We have

$$\left\| \sum_{l=2}^{\infty} \prod_{j=2}^{3} |\Pi_l^{\Delta} f_j| \right\|_1 \le \prod_{j=2}^{3} (\sum_{l=2}^{\infty} \|\Pi_l^{\Delta} f_j\|_2^2)^{1/2} \le \prod_{j=2}^{3} \mathbf{size}(\tilde{T}^{(L)}, f_j) \quad .$$

For the BMO estimate we observe that $\prod_{j=2}^{3} |\Pi_l^{\Delta} f_j|$ is constant on intervals of length 2^{-l} and hence we have

$$\left\| \sum_{l=2}^{\infty} \prod_{j=2}^{3} |\Pi_l^{\Delta} f_j| \right\|_{\mathrm{BMO}} \le \sup_I |I|^{-1} \left\| \sum_{2^{-l} < |I|} \prod_{j=2}^{3} |\Pi_l^{\Delta} f_j| \right\|_{L^1(I)}$$

$$\le \prod_{j=2}^{3} (\sum_{2^{-l} < |I|} \|\Pi_l^{\Delta} f_j\|_{L^2(I)}^2)^{1/2} \le \prod_{j=2}^{3} \mathbf{size}(\tilde{T}^{(L)}, f_j)$$

where we have identified the penultimate term as a product of tree sums for the subtree of $\tilde{T}$ with top interval $I_T = I$. Interpolation shows that the second factor in (10) is bounded by the product of $\mathbf{size}(\tilde{T}^{(L)}, f_j)$ for $j = 2, 3$.

To estimate the first factor in (10) we observe that by the triangle inequality it suffices to estimate

$$\left\| \sup_l |\sum_{P \in T_d : |I_P| > 2^l} w_{P_d} \Pi_{P_u} f_1| \right\|_2$$

Since $w_{P_d} \Pi_{P_u} f_1$ is a linear combination of Haar functions at level $|I_P|$, the truncation operator to $|I_P| > 2^l$ can be replaced by an averaging operator to dyadic intervals of length 2^{-l}. By the Hardy Littlewood maximal theorem, we obtain for the last display the upper bound

$$C \left\| \sum_{P \in T_d} w_{P_d} \Pi_{P_u} f_1 \right\|_2 \le \mathbf{size}(\tilde{T}, f_1) \quad .$$

Combining the estimates for the two factors of (10) proves the desired estimate for complete trees and ends the proof of Lemma 2.1.

Lemma 2.2 (Tree Selection). *Assume* $\mathbf{P}$ *is a convex collection of bitiles* $\mathbf{P}$ *contained in the strip* $\mathbb{R}_+ \times [0, 2^N)$ *with* $\mathbf{size}^{(L)}(\mathbf{P}, f) \le 2^{-k}$*. Then we can write* $\mathbf{P}$ *as the union of a convex set of bitiles* $\mathbf{P}'$ *and a collection* $\mathbf{T}$ *of convex trees such that*

$$\| \sum_{T \in \mathbf{T}} 1_{I_T} \|_1 \le C 2^{2k} \|f\|_2^2 \quad , \tag{11}$$

$$\| \sum_{T \in \mathbf{T}} 1_{I_T} \|_{\mathrm{BMO}} \le C 2^{2k} \|f\|_\infty^2 \quad , \tag{12}$$

with constants independent of N *and* L *and* $\mathbf{size}^{(L)}(\mathbf{P}', f) \le 2^{-k-1}$*.*

Note that the case $L = 0$ corresponds to a statement for size($\mathbf{P}$, f).

Proof: By scaling it suffices to prove the lemma for $k = 0$. We shall first see that we may reduce to the case that all bitiles $P \in \mathbf{P}$ satisfy

$$\mathbf{size}^{(L)}(\{P\}, f) \le 2^{-4} \quad . \tag{13}$$

If there is a bitile $P \in \mathbf{P}$ which violates (13), then we pick one such tile P_1 which maximizes I_{P_1} and set $T_1 = \{P' \in \mathbf{P} : P' \le P_1\}$ and $\mathbf{P}_1 = \mathbf{P} \setminus T_1$. Both T_1 and $\mathbf{P}_1$ are convex. Then we iterate this procedure with $\mathbf{P}_1$, provided there is bitile in $\mathbf{P}_1$ which violates (13), and so on. The selected bitiles are all pairwise disjoint. For assume not, then $P_m \le P_k$ for some m, k and by choice of these bitiles we necessarily have $k < m$. But then P_m should have been in the tree T_k and would not have been available for selection at the m-th step, a contradiction. By vertical dilation, the rectangles $2^L P_m$ with P_m a selected bitile are pairwise disjoint. We therefore have

$$\sum_{k=1}^{n} \|\Pi_{2^L P_k} f\|_2^2 \le \|f\|_2^2$$

and hence

$$\sum_{k=1}^{n} |I_{T_k}| \le 2^8 \|f\|_2^2$$

which proves (11) for the set of selected trees. Moreover, for every dyadic interval I we have

$$\sum_{k : I_{P_k} \subset I} \|\Pi_{2^L P_k} f\|_2^2 \le \|f 1_I\|_2^2 \quad .$$

This gives (12) for the set of selected trees. Since all bitiles in the collection $\mathbf{P}$ satisfy $|I_P| \ge 2^{-N}$, estimate (11) shows that the selection process must have stopped after finitely many steps, and the remaining collection has no bitiles violating (13). For the rest of the argument we assume that all bitiles in $\mathbf{P}$ satisfy (13).

By (5) and the triangle inequality it suffices to show that for the collection $\mathbf{P}'$ we are about to construct we have for every tree $T \in \mathbf{P}'$

$$\sum_{P \in T_u} |\Pi_{2^L P_d} f\|_2^2 \leq 2^{-3} |I_T| \|f\|_2^2 \tag{14}$$

$$\sum_{P \in T_d} |\Pi_{2^L P_u} f\|_2^2 \leq 2^{-3} |I_T| \|f\|_2^2 \tag{15}$$

because we already have

$$\|\Pi_{2^L P_T} f\|_2 \leq 2^{-4} \quad .$$

By a symmetry argument it suffices to prove that we can take away a collection $\mathbf{T}$ of trees satifying (11) and (12) such that for the remaining collection $\mathbf{P}'$ of bitiles we have the bound (14) for all convex trees $T \subset \mathbf{P}'$.

To do so, we again iteratively select trees. If there is a tree in the collection $\mathbf{P}$ which violates (14), we choose one such tree S_1 with maximal element P_1 such that the left endpoint of ω_{P_1} is minimal. We may assume f is non-zero and has finite L^2 norm, hence violation of (14) implies an upper bound for I_{S_1} and thus there are only finitley many possible choices for the interval ω_{P_1} contained in $[0, 2^N)$ and thus one of the choices attains the minimum for the left endpoint of ω_{P_1}.

Then we define $T_1 = \{P \in \mathbf{P} : P \leq P_1\}$ and $\mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_1$. Both T_1 and $\mathbf{P}_1$ are convex and T_1 contains S_1. Then we iterate this procedure as long as the remaining collection $\mathbf{P}_n$ contains a tree which violates (14). We prove (11) and (12) for the collection of selected trees.

For each selected tree S_m, consider the pruned convex tree $\tilde{S}_m$ which is the convex S_m without the minimal bitiles in S_m. We claim that

$$\sum_{P \in (\tilde{S}_m)_u} \|\Pi_{2^L P_d} f\|_2^2 \geq 2^{-4} |I_T| \|f\|_2^2 \quad . \tag{16}$$

Since the minimal elements P_j, $j = 1, \ldots, J$ in a tree are pairwise disjoint but all less than the maximal element of the tree, the intervals I_j are pairwise disjoint and we have

$$\sum_{j=1}^{J} |I_j| \leq |I_T| \quad .$$

Using the size estimate for every individual bitile we obtain

$$\sum_{j=1}^{J} \|\Pi_{2^L(P_j)_d} f\|_2^2 \leq 2^{-4}|I_T|\|f\|_2^2 \quad .$$

This together with the fact that S_m violates (14) shows that the pruned tree $\tilde{S}_m$ satisfies (16).

Let S_k and S_m be two different selected trees. We claim that if P_k and P_m are bitiles in the respective pruned trees, we have that $(P_k)_d$ and $(P_m)_d$ are disjoint. For assume not, then without loss of generality

$$(P_k)_d \leq (P_m)_d \ .$$

Then also $(P_k)_u \leq (P_m)_d$ because P_k and P_m have to be different. This implies that $m < k$ by the choice of trees. Since P_k is in the pruned tree, there is a different element $P \in S_k$ such that $P < P_k$. Then we have

$$P \leq P_m$$

which implies that P should have been selected for the tree T_m and should not have been available for S_k. This is the desired contradiction and establishes that $(P_k)_d$ and $(P_m)_d$ are disjoint. Hence we have for the collection of selected trees

$$\sum_m \left(\sum_{P \in \tilde{S}_m} \|\Pi_{2^L P_d} f\|_2^2 \right) \leq \|f\|_2^2 \quad .$$

Combining this with (16) shows the desired estimate (11). Estimate (12) then follows again by localization.

By virtue of (11) and the lower bound $2^{-N} \leq |I_T|$ we are guaranteed that the tree selection process stops and the remaining collection does not contain a tree that violates (14). This completes the proof of Lemma 2.2 .

3. Proof of the Main Theorems

We first prove Theorem 1.1 in the open triangle c of Figure 1. Then we prove certain restricted weak type bounds in the open diamond $b_3 \cup d_{23}$ and use interpolation techniques to obtain Theorem 1.1 in the open triangle b_3 and Theorem 1.2 in the open triangle d_{23}. Symmetric arguments can be applied to the diamonds $b_1 \cup d_{21}$, $b_1 \cup d_{31}$ and $b_2 \cup d_{32}$. The argument is however not entirely symmetric, it does not apply to the forbidden diamonds $b_2 \cup d_{1,2}$ or $b_3 \cup d_{13}$. The full extend of Theorems 1.1 and 1.2 then follows by interpolation.

Proposition 3.1 (Triangle c). *For each $i = 1, 2, 3$ let E_i be a subset of $\mathbb{R}_+$ and f_i a measurable function bounded by the characteristic function of E_i. Let $1/2 \le \alpha_i \le \infty$ and assume $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Let j be an index such that $|E_j|$ is maximal. Then there is a major subset $E_j{}'$ of E_j depending on E_1, E_2, E_3 such that if f_j is also supported in $E_j{}'$ we have*

$$\Lambda_L(f_1, f_2, f_3) \le C_{\alpha_1,\alpha_2,\alpha_3} |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3}$$

with a constant $C_{\alpha_1,\alpha_2,\alpha_3}$ independent of L. If the three sets E_i, have measure within a factor of four of each other, we may choose $E_j{}' = E_j$.

Proof: Dilating by a powers of 2 we may assume $1 \le |E_j| < 2$. Define the exceptional set

$$F = \bigcup_{i \ne j} \{x : M_2(1_{E_i}/|E_i|^{1/2})(x) > 2^{10}\} \quad .$$

By the Hardy Littlewood maximal theorem the measure of F is less than one half and we may define the major subset $E_j{}' = E_j \setminus F$. The set F is empty if the measure of all E_i is at least one fourth.

Given functions f_i as in the proposition define the normalized functions $g_i = f_i |E_i|^{-1/2}$ for $i = 1, 2, 3$. It suffices to show

$$\Lambda_L(g_1, g_2, g_3) \le C \quad .$$

Let $\mathbf{P}$ be the convex set of all bitiles P in the strip $\mathbb{R}_+ \times [0, 2^N)$ such that I_P is not contained in F. Since g_j vanishes on F we have

$$\Lambda_{\mathbf{P}}(g_1, g_2, g_3) = \Lambda_L(g_1, g_2, g_3) \quad .$$

Outside the set F, the M_2 maximal function of each of the three functions g_i is bounded by a universal constant, hence we have

$$\mathbf{size}(\mathbf{P}, g_i) \le C, \quad \mathbf{size}^{(L)}(\mathbf{P}, g_i) \le C \quad .$$

Applying Lemma 2.2 repeatedly, we define a decreasing nested sequence of convex subsets $\mathbf{P}_k$ of the set $\mathbf{P}_0 := \mathbf{P}$ such that

$$\mathbf{size}(\mathbf{P}_k, g_i) \le C2^{-k}, \quad \mathbf{size}^{(L)}(\mathbf{P}_k, g_i) \le C2^{-k}$$

for each i and $\mathbf{P}_{k-1}$ is the disjoint union of $\mathbf{P}_k$ and a collection $\mathbf{T}_k$ of convex trees with

$$\sum_{T \in \mathbf{T}_k} |I_T| \le C2^{2k} \quad .$$

We then have

$$\Lambda_{\mathbf{P}}(g_1, g_2, g_3) = \sum_{k \ge 1} \sum_{T \in \mathbf{T}_k} \Lambda_T(g_1, g_2, g_3)$$

$$\le \sum_{k \ge 1} \sum_{T \in \mathbf{T}_k} C|I_T| \mathbf{size}(T, g_1) \mathbf{size}(T^{(L)}, g_2) \mathbf{size}(T^{(L)}, g_3)$$

$$\leq \sum_{k\geq 1} \sum_{T\in \mathbf{T}_k} C2^{-3k}|I_T| \leq \sum_{k\geq 1} C2^{-k} \leq C$$

This completes the proof of the proposition.

By interpolation as in [6] this proposition proves Theorem 1.1 in the region $2 < p_j < \infty$. First on proves that the proposition holds in this region for f_j not necessarily supported on $E_j{}'$. Namely one splits f_j into $f_j 1_{E_j{}'} + f_j 1_{E_j \setminus E_j{}'}$. On the first summand the conclusion of the proposition gives the desired bound, while on the second summand one iterates the proposition with E_j replaced by $E_j \setminus E_j{}'$. One continues the iteration until all three sets are of comparable size at which time the proposition holds already with the major subset being the full set. The various estimates throughout the iteration process are summable provided $\alpha_j > 0$ for all j. With this variant of the proposition established, one applies standard multilinear Marcinkiewicz interpolation to obtain the strong type estimate in the region $2 < p_j < \infty$.

Proposition 3.2 (**Diamond** $b_3 \cup d_{23}$). *Let $0 < \epsilon, \alpha < 1/2$. For each $i = 1, 2, 3$ let E_i be a measurable subset of $\mathbb{R}_+$ and let f_i be a measurable function bounded by the characteristic function of E_i. Assume $|E_3| < |E_2|$ and $1 \leq |E_2| \leq 2$. Then there is a major subset $E_2{}'$ of E_2 depending on E_1, E_2, E_3 such that if f_2 is supported in $E_2{}'$ we have*

$$\Lambda_L(f_1, f_2, f_3) \leq C_{\alpha,\epsilon}|E_1|^\alpha |E_3|^{1-\epsilon}$$

with a constant $C_{\alpha,\epsilon}$ independent of L.

Proof: Define the exceptional set F to be

$$\{x : M_2(1_{E_1}/|E_1|^\alpha)(x) > 2^{10}\} \cup \{M_{1/(1-\epsilon)}(1_{E_3}/|E_3|^{\epsilon-1})(x) > 2^{10}\} \quad .$$

We may assume that $|E_3|$ is sufficiently small so that E_3 is containd in F, or else the desired estimate is trivial from the already established case of Theorem 1.1 in the vicinity of $\alpha_2 = 1/2$ and $\alpha_3 = 1/2 - \alpha$.

By the Hardy Littlewood maximal theorem the measure of F is less than $1/2$ and we may define the major subset $E_2{}' = E_2 \setminus F$. Given a triple of functions as in the proposition define the normalized functions

$$g_1 = f_1|E_1|^{-\alpha}, \quad g_2 = f_2, \quad g_3 = f_3|E_3|^{\epsilon-1} \quad .$$

Let $\mathbf{P}$ be the convex collection of all bitiles in the strip $\mathbb{R}_+ \times [0, 2^N)$ such that I_P is not contained in F. We have

$$\mathbf{size}(\mathbf{P}, g_1) \leq C \quad .$$

Applying Lemma 2.2 repeatedly we obtain a nested sequence $\mathbf{P}_k$ of convex subsets of $\mathbf{P} = \mathbf{P}_0$ such that we have

$$\mathbf{size}(\mathbf{P}_k, g_1) \leq C2^{-k}$$

and, interpolating between (12) and (11), the set $\mathbf{P}_{k-1}$ is the disjoint union of $\mathbf{P}_k$ and a collection $\mathbf{T}_k$ of convex trees with

$$\| \sum_{T\in\mathbf{T}_k} 1_{I_T}\|_p \leq C2^{2k} \tag{17}$$

for $1/p = 2\alpha$. We shall fix a $k \geq 1$ and prove

$$| \sum_{T\in\mathbf{T}_K} \Lambda_T(g_1, g_2, g_3)| \leq Ck2^{-\epsilon k} \quad , \tag{18}$$

which will clearly finish the proof of the proposition.

For $P \in \mathbf{P}_k$ define $\mathbf{p}_P$ to be the set of all minimal tiles (those with spatial interval of length $2^{-L}|I_P|$) contained in $2^L P_d$ for which I_p is not contained in F. Then we may write

$$\int w_{P_d}(x)\Pi_{P_u} g_1(x) \prod_{j=2}^{3} \Pi_{2^L P_d} g_j(x)\, dx$$

$$= \int w_{P_d}(x)\Pi_{P_u} g_1(x) \sum_{p\in\mathbf{p}_P} \prod_{j=2}^{3} \langle g_j, w_p\rangle\, w_p(x)\, dx \quad .$$

because the wave packets w_p are disjointly supported as p runs through $\mathbf{p}_P$ and $\langle g_2, w_p\rangle = 0$ if $I_p \subset F$. Note that in this argument the symmetry between the three indices $1, 2, 3$ is broken. The role played by the index 2 in this argument, namely the use of vanishing of g_2 on F, could symmetrically be taken by the index 3, but not by the index 1. This is the only place in the proof of this proposition, where the symmetry is broken in an essential way.

Let $\mathbf{I}$ be the collection of maximal dyadic intervals contained in the set F. For an interval $I \in \mathbf{I}$ let $\mathbf{p}_I$ be the collection of tiles p with time interval I which intersect a tile p' in $\mathbf{p}_P$ for some $P \in \bigcup_{T\in\mathbf{T}_k} T$. We have $p' \leq p$ for such tiles, and the relation is strict in the sense $p' \neq p$. Then we have also $2^L P_T \leq p$ and hence there is at most one element in $\mathbf{p}_I$ which intersects with a given tree in $\mathbf{T}_k$. Hence $\mathbf{p}_I$ has at most N_I elements where N_I is the constant value of the function

$$\sum_{T\in\mathbf{T}_k} 1_{I_T}$$

on the interval I.

Let a_I be the orthogonal projection of g_3 onto the span of wave packets associated to the tiles in $\mathbf{p}_I$, and let $a = \sum a_I$. Since g_3 is supported on the union of the intervals $I \in \mathbf{I}$, we have for every tree $T \in \mathbf{T}_k$

$$\Lambda_T(g_1, g_2, g_3) = \Lambda_T(g_1, g_2, a) \quad .$$

We have (Hölder and Hausdorff Young on I),

$$\|a_I\|_2^2 = \sum_{p\in\mathbf{p}_I} |\langle g_3, w_p\rangle|^2$$

$$\leq N_I^{1-2\epsilon}(\sum_{p\in\mathbf{p}_I} |\langle g_3, w_p\rangle|^{1/\epsilon})^{2\epsilon}$$

$$\leq N_I^{1-2\epsilon}\|g_3\|_{1/(1-\epsilon)}^2 |I|^{2\epsilon-1} \leq C N_I^{1-2\epsilon}|I|$$

where in the last inequality we used the bound for the $M_{1/(1-\epsilon)}$ - function of g_3 at some point of the dyadic parent of I. Hence we have

$$\|a\|_2^2 \leq C\sum_I N_I^{1-2\epsilon}|I|$$

$$\leq C(\sum_I N_I^p|I|)^{(1-2\epsilon)/p}(\sum_I |I|)^{1-(1-2\epsilon)/p} \leq C2^{2k(1-2\epsilon)} \quad .$$

Define $\mathbf{I}_0$ to be the set of all intervals in $\mathbf{I}$ such that N_I is at most 2^{2k}. For $m > 0$ define $\mathbf{I}_m$ to be the set of all intervals in $\mathbf{I}$ such that N_I is between $2^{2(k+m-1)}$ and $2^{2(k+m)}$. Split

$$a = \sum_{m\geq 0} a_m$$

accordingly, i.e., a_m is supported on the union of intervals in $\mathbf{I}_m$. We have

$$\|a_0\|_2^2 \leq C2^{2k(1-2\epsilon)}$$

and for $m > 0$

$$\|a_m\|_2^2 \leq C\sum_{I\in\mathbf{I}_m} 2^{2(k+m)(1-2\epsilon)}|I|$$

$$\leq C2^{2(k+m)(1-2\epsilon)}2^{-2p(k+m)}\|\sum_I N_I 1_I\|_p^p$$

$$\leq C2^{2(k+m)(1-2\epsilon)}2^{-2pm} \quad .$$

Moreover, we have for every tree T contained in $\bigcup_{T\in\mathbf{T}_k} T$

$$\|a_m\|_{L^2(I_T)}^2 \leq C\sum_{I\in\mathbf{I}_m: I\subset I_T} 2^{2(k+m)(1-2\epsilon)}|I| \leq C2^{2(k+m)(1-2\epsilon)}|I_T|$$

and hence

$$\mathbf{size}^{(L)}(\bigcup_{T\in\mathbf{T}_k} T, a_m) \leq C2^{(k+m)(1-2\epsilon)} \quad .$$

We shall fix m and prove

$$|\sum_{T\in\mathbf{T}_k} \Lambda_k(g_1, g_2, a_m)| \leq Ck2^{-2\epsilon k}2^{-2\epsilon m} \ , \tag{19}$$

which will imply (18) and finish the proof of the proposition. Normalize

$$\tilde{a}_m = a_m 2^{-(k+m)(1-2\epsilon)} 2^{pm}$$

so that $\|\tilde{a}_m\|_2 \le C$, then it clearly suffices to prove

$$|\sum_{T\in\mathbf{T}_k} \Lambda_k(g_1, g_2, \tilde{a}_m)| \le Ck2^{-k} \ ,$$

With Lemma 2.2 we decompose $\bigcup_{T\in\mathbf{T}_k} T$ into collections $\bigcup_{T\in\mathbf{T}_{k,l}} T$ with $-pm \le l < pk$ and a remainder set $\mathbf{P}_{k,pk}$ such that each tree in $\mathbf{T}_{k,l}$ satisfies

$$\mathbf{size}^{(L)}(T, g_2) \le C2^{-l}, \ \ \mathbf{size}^{(L)}(T, a_m) \le C2^{-l}$$

and we have

$$\sum_{T\in\mathbf{T}_{k,l}} |I_T| \le 2^{2l}$$

and we have for every tree in the remainder set $\mathbf{P}_{k,pk}$

$$\mathbf{size}^{(L)}(T, g_2) \le C2^{-pk}, \ \ \mathbf{size}^{(L)}(T, a_m) \le C2^{-pk} \ \ .$$

Note that we also have

$$\mathbf{size}^{(L)}(T, g_2) \le C$$

for every selected tree since g_2 is bounded by a universal constant.

By Lemma 2.1 we obtain

$$|\sum_{T\in\mathbf{T}_k} \Lambda_{T_k}(g_1, g_2, \tilde{a}_m)|$$
$$\le C \sum_{-pm\le l<pk} 2^{-k}\min(1, 2^{-l})2^{-l}2^{2l}$$
$$+|\Lambda_{\mathbf{P}_{k,pk}}(g_1, g_2, \tilde{a}_m)|$$

The first term is a geometric series for $l < 0$ and a constant series for $l > 0$ and hence

$$C \sum_{-pm\le l<pk} 2^{-k}\min(1, 2^{-l})2^{-l}2^{2l} \le Ck2^{-k}$$

with constant C depending on p. To estimate the term involving $\mathbf{P}_{k,pk}$ we use that $\mathbf{P}_{k,pk}$ is a subset of the union of trees in $\mathbf{T}_k$. For every tree T' in $\mathbf{T}_k$ we can write $\mathbf{P}_{k,pk} \cap T'$ as a union of trees T with $\sum |I_T| \le |I_{T'}|$. By (17) we can write $\mathbf{P}_{k,pk}$ as the union of trees with

$$\sum_T |I_T| \le 2^{2pk} \ \ .$$

Hence we have

$$|\Lambda_{\mathbf{P}_{k,pk}}(g_1, g_2, \tilde{a}_m)| \le C2^{-k}2^{-pk}2^{-pk}2^{2pk} \le C2^{-k} \ \ .$$

This proves (19) and completes the proof of the proposition.

By scaling we can remove the restriction $1 \leq |E_2| \leq 2$. As detailed in the proof, we may also omit the restriction $|E_3| < |E_2$ by an application of Theorem 1.1 in the region where it is already proven. This proves Theorem 1.2 in the open triangle d_{23}. In the open triangle b_3, we may iterate Proposition 3.2 as in the discussion of the previous proposition, and as a result obtain that the proposition holds for $E_2{}' = E_2$. Then Theorem 1.1 holds in the open triangle b_3 by multilinear Marcinkiewicz interpolation.

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