First-Order Methods in Nonlinear Model Predictive Control

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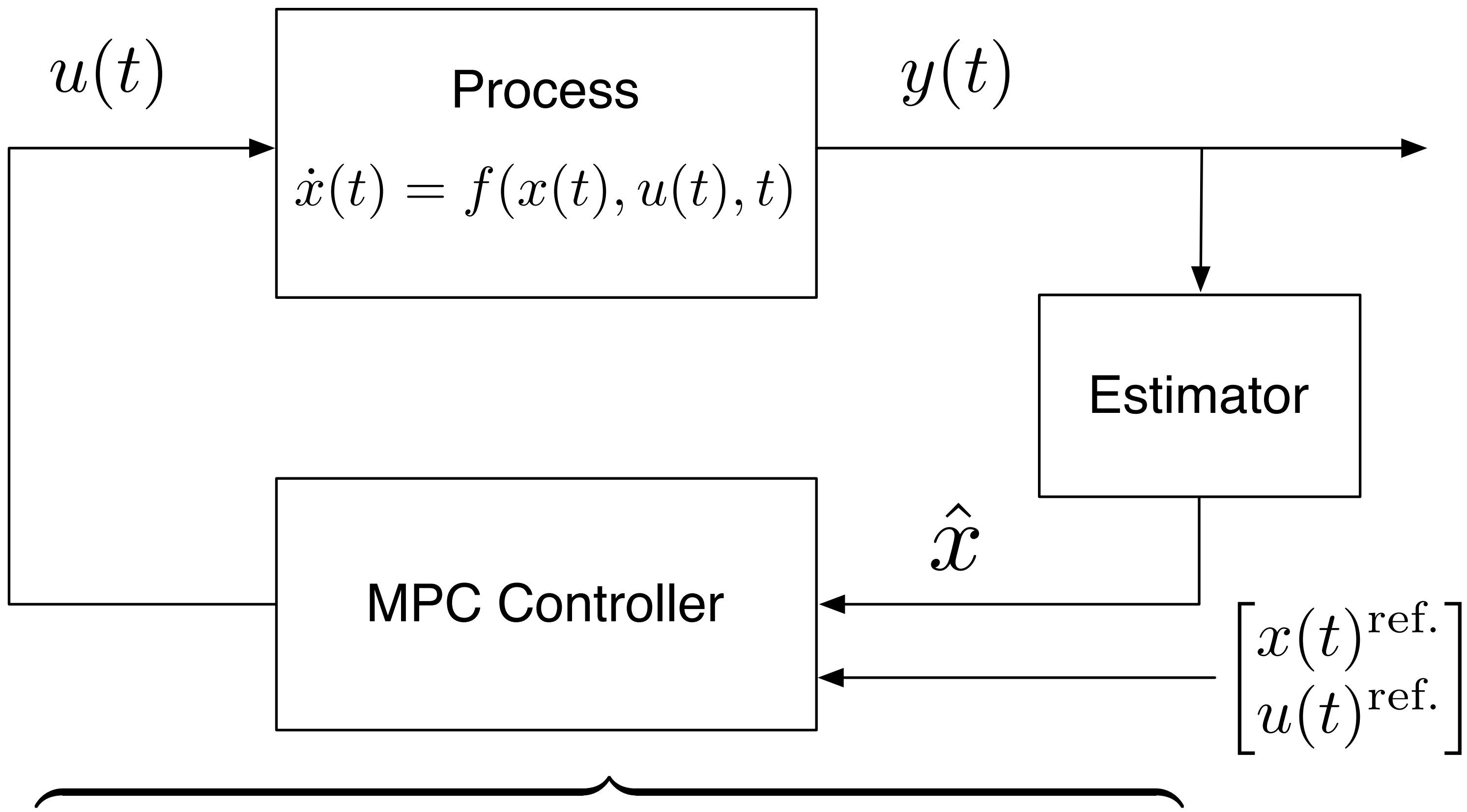
Outline

- Principle of Model Predictive Control
- Solving the underlying optimization problems
- Application: Control of a pendulum on a cart

Principle of Model Predictive Control



$$\begin{aligned}
\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad & \int_{t_0}^{t_0+T} \frac{1}{2} \left(||x(t) - x(t)^{\text{ref.}}||_Q^2 + ||u(t) - u(t)^{\text{ref.}}||_R^2 \right) \\
\text{subject to:} \quad & x(t_0) = \hat{x} \\
& \dot{x}(t) = f(x(t), u(t), t) && \text{for all } t \in [t_0, t_0 + T) \\
& u(t)^l \le u(t) \le u(t)^u && \text{for all } t \in [t_0, t_0 + T) \\
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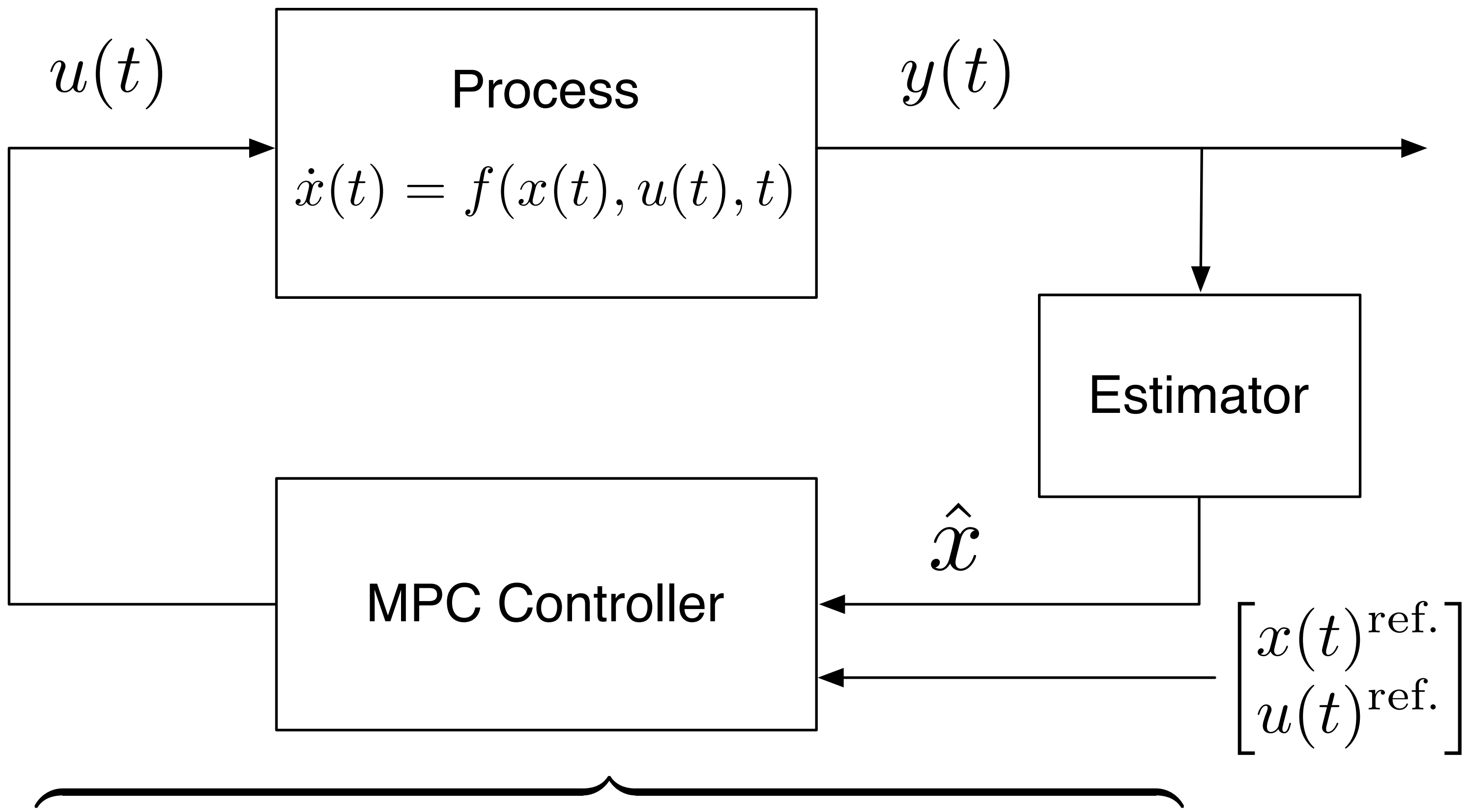
Principle of Model Predictive Control



- Multiple shooting discretization
- Real-Time Iteration scheme
- Code generated integrators with sensitivity propagation
- Interfaces to efficient Quadratic Programming (QP) solvers

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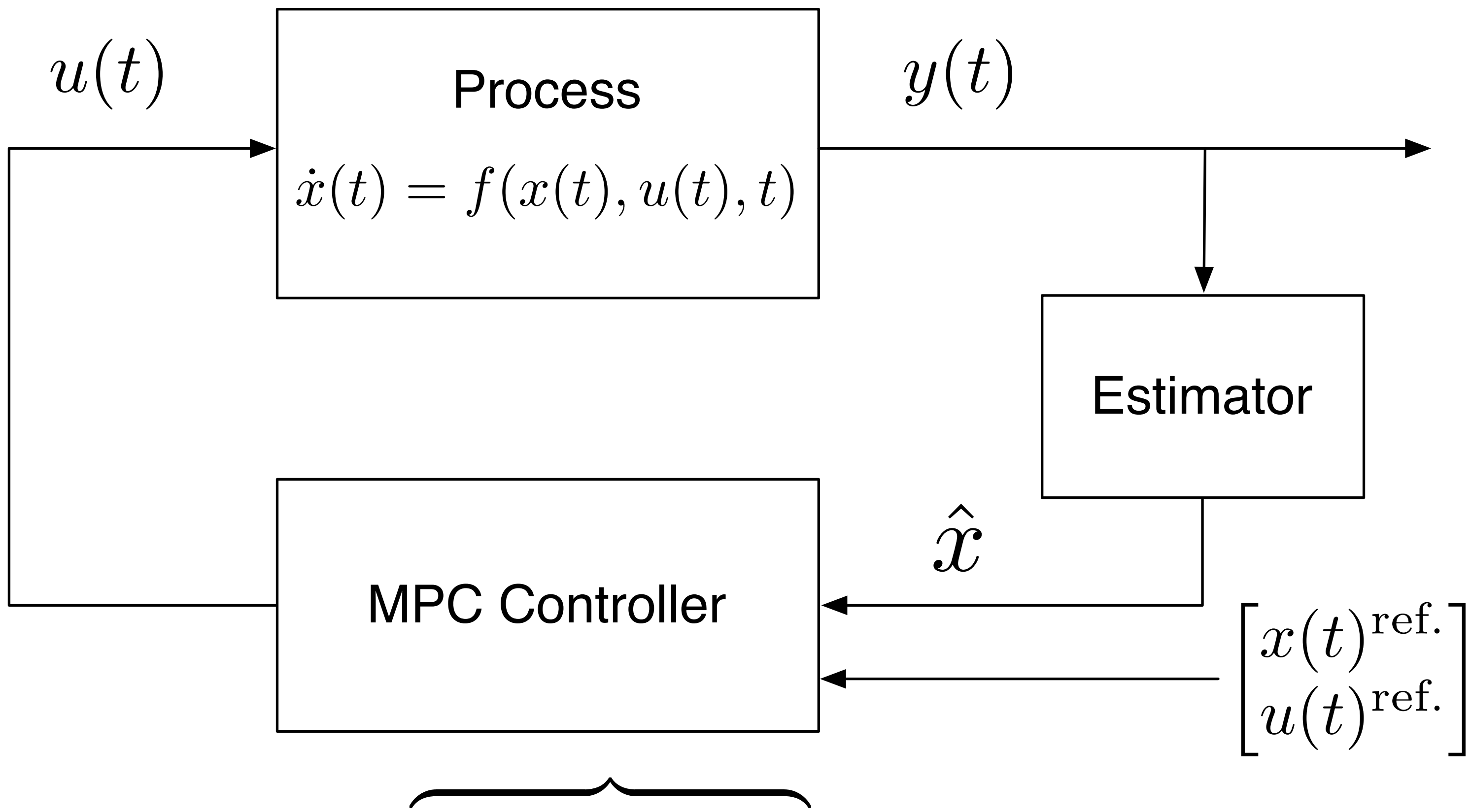

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$$\underset{Z}{\text{minimize}} \ \frac{1}{2} Z^T H Z + h^T Z$$
$$\text{subject to: } AZ = b$$
$$\underline{Z} \leq Z \leq \bar{Z}$$

- x_k value of states on grid points
- u_k value of controls on intervals
- u_0 applied control

ACADO
code generation

Structured QP

$$Z = [x_0^T \ u_0^T \ldots u_{N-1}^T \ x_N^T]^T$$

Solving the underlying QP subproblems

Many approaches for solving the arising QPs:

- Interior-point methods
- Active-set methods
- First-order methods
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Generalized Dual Fast Gradient Method

- Relaxation of equality constraints, unconstrained dual.
- Matrix M allows different steps in different directions.
- First step has an analytical solution when H is diagonal.

$$Z_{k+1} = \underset{\underline{Z} \leq Z \leq \bar{Z}}{\operatorname{argmin}} \frac{1}{2} Z^T H Z + h^T Z + Y_k^T (AZ - b)$$

$$\Lambda_{k+1} = Y_k + M^{-1}(AZ_{k+1} - b)$$

$$Y_{k+1} = \Lambda_{k+1} + \beta_k(\Lambda_{k+1} - \Lambda_k)$$

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M : block tri-diagonal structure

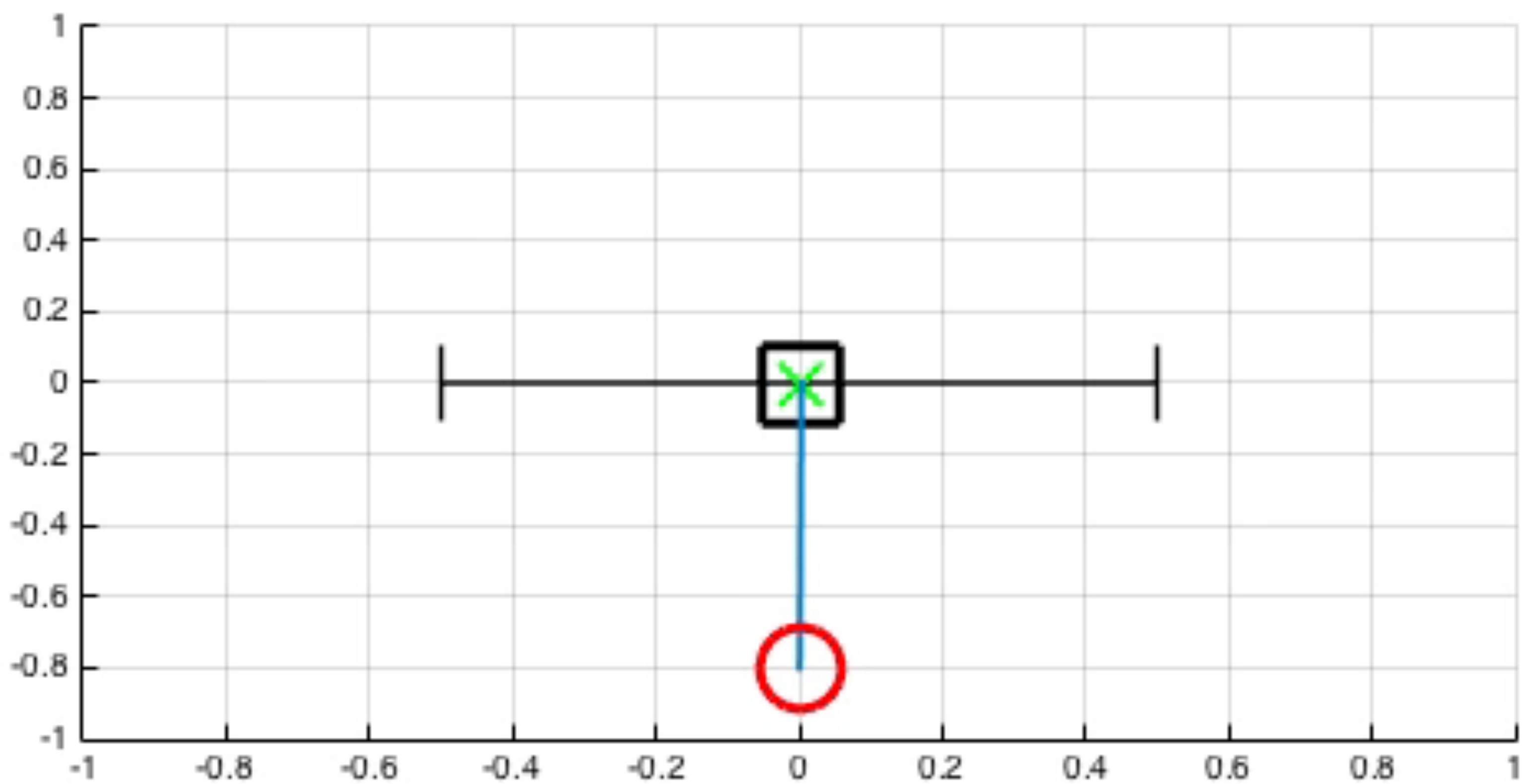
$M = AH^{-1}A^T = R^T R$

$M^{-1}v = R \backslash (R^T \backslash v)$

Control of pendulum on a cart

QP:

- T_s = 50 ms
- 605 primal vars
- 505 dual vars



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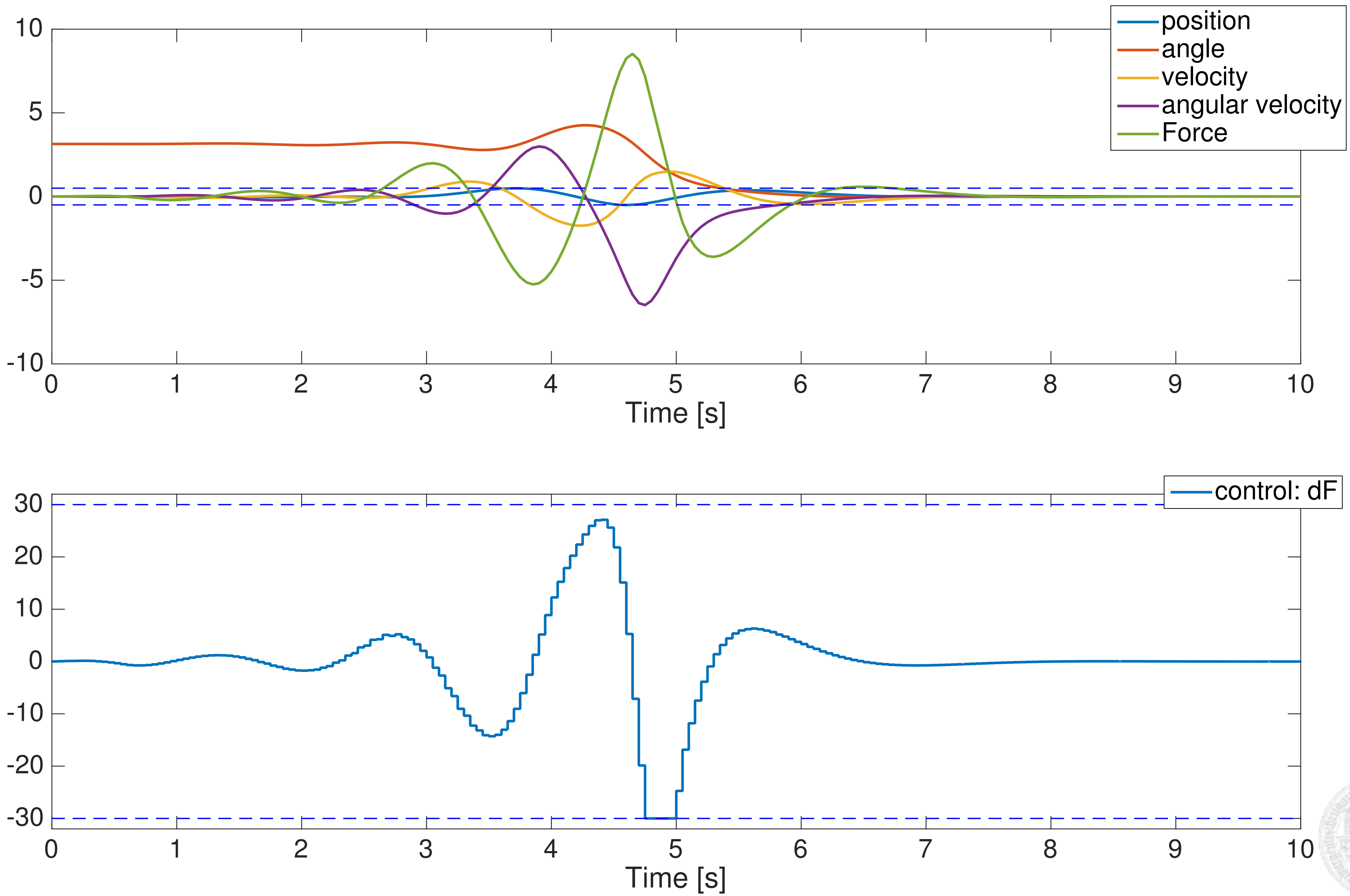

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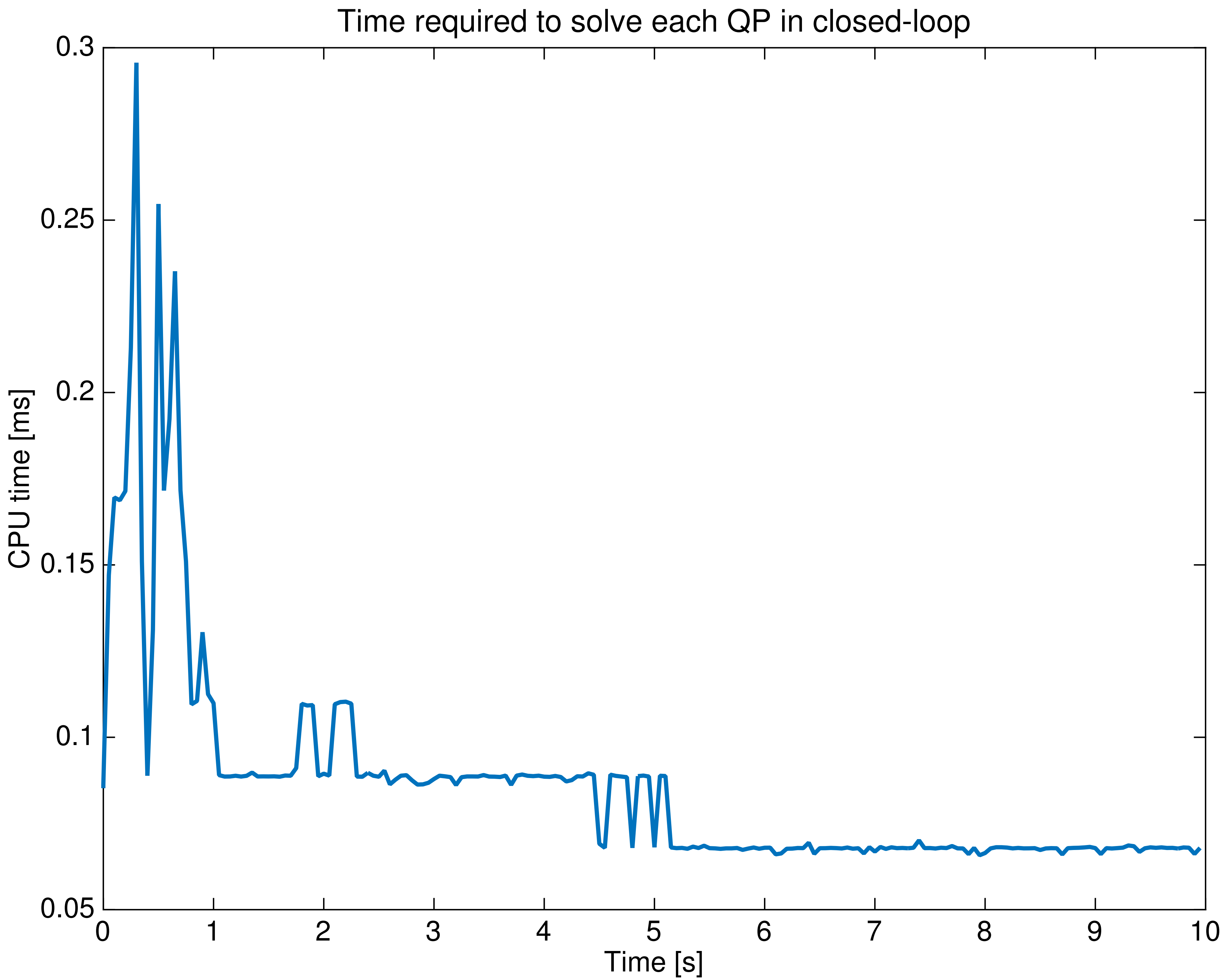

Conclusions

- Fast, embedded MPC requires the solution of optimization problems with several hundred variables in a few milliseconds.
- QP solvers based on first-order methods (e.g., fast gradient method, ADMM, …) are suitable candidates for real-time nonlinear MPC:
    - Simple, highly parallelizable algorithmic schemes.
    - Low cost per iteration.
    - Shifted optimal solution of previous QP provides a good initial guess that accelerates convergence.

References:

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Thank you for your attention.

Questions?

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