

# THE WEAK EXPECTATION PROPERTY AND RIESZ INTERPOLATION

ALI S. KAVRUK

**ABSTRACT.** We show that Lance's weak expectation property is connected to tight Riesz interpolations in lattice theory. More precisely we first prove that if  $\mathcal{A} \subset B(H)$  is a unital  $C^*$ -subalgebra, where  $B(H)$  is the bounded linear operators on a Hilbert space  $H$ , then  $\mathcal{A}$  has  $(2, 2)$  tight Riesz interpolation property in  $B(H)$  (defined below). An extension of this requires an additional assumption on  $\mathcal{A}$ :  $\mathcal{A}$  has  $(2, 3)$  tight Riesz interpolation property in  $B(H)$  at every matricial level if and only if  $\mathcal{A}$  has the weak expectation property.

Let  $J = \text{span}\{(1, 1, -1, -1, -1)\}$  in  $\mathbb{C}^5$ . We show that a unital  $C^*$ -algebra  $\mathcal{A}$  has WEP if and only if  $\mathcal{A} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{A} \otimes_{\max} (\mathbb{C}^5/J)$  (here  $\otimes_{\min}$  and  $\otimes_{\max}$  are the minimal and the maximal operator system tensor products, respectively, and  $\mathbb{C}^5/J$  is the operator system quotient of  $\mathbb{C}^5$  by  $J$ ).

We express the Kirchberg conjecture (KC) in terms of a four dimensional operator system problem. We prove that KC has an affirmative answer if and only if  $\mathbb{C}^5/J$  has the double commutant expectation property if and only if  $\mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J = \mathbb{C}^5/J \otimes_c \mathbb{C}^5/J$  (here  $\otimes_c$  represents the commuting operator system tensor product).

We continue our research on finite dimensional operator systems by means of recently developed quotient, tensor and nuclearity theory [16], [15], [14]. The main purpose of the present paper can be divided into three parts. Letting

$$J = \text{span}\{(1, 1, -1, -1, -1)\} \subset \mathbb{C}^5,$$

the operator system quotient  $\mathbb{C}^5/J$  (which is different than the operator space quotient) can be identified (unittally and completely order isomorphically) with an operator subsystem of the full group  $C^*$ -algebra  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . By using this identification we first obtain a new weak expectation property (WEP) criteria:

**Theorem 0.1.** *A unital  $C^*$ -algebra  $\mathcal{A}$  has WEP if and only if we have the unital and complete order isomorphism*

$$\mathcal{A} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{A} \otimes_{\max} (\mathbb{C}^5/J).$$

This WEP criteria also allows us to re-express Kirchberg's conjecture [18] in terms of a problem about this four dimensional operator system. Recall that Kirchberg's conjecture asserts that every separable  $C^*$ -algebra that has the local lifting property (LLP) has WEP. In [15] this problem is approached in the operator system setting and the following reformulation is given: Every finite dimensional operator system that has the lifting property (LP) has the double commutant expectation property (DCEP). (DCEP is one extension of WEP from unital  $C^*$ -algebras to general operator systems and discussed briefly in Subsection 3.3 below). In [14] it was also proven that this is equivalent to the universal operator system generated by two contractions,  $\mathcal{S}_2$ , (which has LP) has DCEP. One of our main results in Section 5 is the following four dimensional reduction.

**Theorem 0.2.** *The following are equivalent:*

- (1) *Kirchberg's conjecture has an affirmative answer.*
- (2)  *$\mathbb{C}^5/J$  has DCEP.*

...

(3) *We have the complete order isomorphism*

$$(\mathbb{C}^5/J) \otimes_{\min} (\mathbb{C}^5/J) = (\mathbb{C}^5/J) \otimes_c (\mathbb{C}^5/J).$$

Here  $\otimes_c$  denotes the (maximal) commuting tensor product and we briefly summarized its main properties in Section 2.

In the final section we examine the role of WEP in tight Riesz separation properties and we prove, for larger arguments, that these two concepts are identical. Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of  $B(H)$ . We say that  $\mathcal{A}$  has the **(k,m) tight Riesz interpolation property in  $B(H)$** ,  $\text{TR}(k,m)$ -property in short, if for every self-adjoint elements  $x_1, \dots, x_k$  and  $y_1, \dots, y_m$  in  $\mathcal{A}$  whenever there is an element  $b$  in  $B(H)$  with

$$x_1, \dots, x_k < b < y_1, \dots, y_m$$

then there is an element  $a$  in  $\mathcal{A}$  such that

$$x_1, \dots, x_k < a < y_1, \dots, y_m.$$

Here  $x < y$  stands for  $\delta I \leq y - x$  for some positive  $\delta$  where  $I$  denotes the unit of  $B(H)$ . Likewise, we say that  $\mathcal{A}$  has the **complete  $\text{TR}(k,m)$ -property in  $B(H)$**  if  $M_n(\mathcal{A})$  has  $\text{TR}(k,m)$ -property in  $M_n(B(H))$  for every  $n$ . We first prove the following:

**Theorem 0.3.**  *$\mathcal{A} \subset B(H)$  has the complete  $\text{TR}(2,2)$ -property in  $B(H)$ .*

While (2,2)-interpolation is automatically satisfied, higher interpolations require an additional hypothesis on  $\mathcal{A}$ . The following is our main result in Section 7:

**Theorem 0.4.** *Let  $\mathcal{A} \subset B(H)$  is a unital  $C^*$ -algebra. Then the following are equivalent:*

- (1)  *$\mathcal{A}$  has the weak expectation property;*
- (2)  *$\mathcal{A}$  has the complete  $\text{TR}(2,3)$ -property in  $B(H)$ ;*
- (3)  *$\mathcal{A}$  has the complete  $\text{TR}(k,m)$ -property in  $B(H)$  for some  $k \geq 2$ ,  $m \geq 3$ ;*
- (4)  *$\mathcal{A}$  has the complete  $\text{TR}(k,m)$ -property in  $B(H)$  for all positive integers  $k$  and  $m$ .*

This characterization of WEP is independent of the particular faithful representation of the  $C^*$ -algebra on a Hilbert space. (In fact a careful reading of our proof indicates that we can even consider an (operator systematic) representation.) In this respect it differs from Lance's original definition of WEP [21], which requires that every faithful representation of the given  $C^*$ -algebra has a conditional expectation into its double commutant. This requirement in the original definition is essential since every  $C^*$ -algebra has at least one representation which has a conditional expectation into its double commutant.

Our proofs makes use of recently developed tensor, quotient and nuclearity theory of operator systems. In this regard we start with a brief overview on the operator systems. The preliminary section includes basic facts on duality, quotients,  $C^*$ -covers etc. We devote Section 2 for the tensor products in the category of operator systems. Here, after the axiomatic definition of tensor products we briefly summarized main properties of the minimal (min), maximal (max), commuting (c) and two asymmetric tensor products, enveloping left (el) and enveloping right (er). The set of all tensor products admits a natural (partial) order and the primary tensor products we consider have the following pattern:

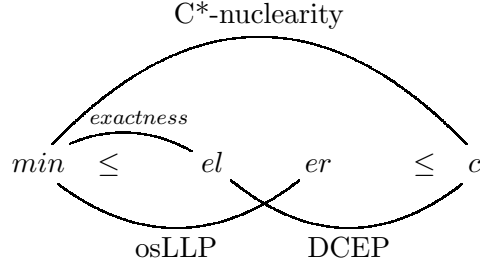
$$\otimes_{\min} \leq \otimes_{el} , \otimes_{er} \leq \otimes_c \leq \otimes_{\max}.$$

Section 3 includes several nuclearity related properties including the (operator system) local lifting property (osLLP), double commutant expectation property (DCEP), weak expectation property and exactness. These operator system notions, along with tensor characterizations,

studied in [15]. The term “nuclearity related” perhaps best seen in the operator system setting: Given tensor products  $\alpha \leq \beta$  we call an operator system  $(\alpha, \beta)$ -nuclear if

$$\mathcal{S} \otimes_{\alpha} \mathcal{T} = \mathcal{S} \otimes_{\beta} \mathcal{T} \text{ for every operator system } \mathcal{T}.$$

Our main purpose in Section 3 is to exhibit the following “nuclearity diagram”:



For example, an operator system  $\mathcal{S}$  is exact if and only if it is  $(\min, \text{el})$ -nuclear. The remaining of the Section 3 includes several interesting examples on the stability of these properties under (operator system) quotients, duality etc. For example, in contrast to  $C^*$ -algebra ideal quotients, exactness is not preserved under operator system quotients. Though, in the finite dimensional case, the lifting property is preserved under quotients by null subspaces, etc.

In Section 4 we recall basic facts on the coproducts of operator systems introduced by Kerr and Li [17]. This should be considered as operator system variant of the unital free products of  $C^*$ -algebras. The main purpose of this section is to obtain the following identification: Consider  $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b : a^2 = b^3 = e \rangle$ . Let  $\lambda$  be the universal representation of  $\mathbb{Z}_2 * \mathbb{Z}_3$  in its full group  $C^*$ -algebra  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . Let

$$\mathcal{S} = \text{span}\{\lambda(e), \lambda(a), \lambda(b), \lambda(b)^*\} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_3).$$

Then the last theorem of the section indicates that we have the unital complete order isomorphism

$$\mathbb{C}^5/J \cong \mathcal{S}.$$

Section 5 contains two of our main application that we pointed out at the beginning of introduction, namely the four dimensional version of WEP criteria and four dimensional operator system variant of Kirchberg’s conjecture.

Fixing the basis  $\{\acute{e}_1, \acute{e}_2, \acute{e}_3, \acute{e}_4\}$  of  $\mathbb{C}^5/J$  every element in the algebraic tensor  $\mathcal{S} \otimes (\mathbb{C}^5/J)$  can be uniquely written as

$$(0.1) \quad s_1 \otimes \acute{e}_1 + s_2 \otimes \acute{e}_2 + s_3 \otimes \acute{e}_3 + s_4 \otimes \acute{e}_4.$$

Section 6 is devoted to understand the positivity criteria in  $-\otimes_{\min} (\mathbb{C}^5/J)$ ,  $-\otimes_c (\mathbb{C}^5/J)$ , and  $-\otimes_{\max} (\mathbb{C}^5/J)$ . So simply put, when the above mentioned element is positive when the algebraic tensor  $\mathcal{S} \otimes (\mathbb{C}^5/J)$  is equipped with the minimal, commuting or the maximal tensor product. As a rehearsal we would like give the following part of Proposition 6.2. Let  $\mathcal{S} \subset B(H)$  be an operator subsystem and let  $u$  denote the expression in (0.1). Then

- (1)  $u > 0$  in  $\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J)$  if and only if there is an element  $b \in B(H)^+$  such that

$$s_1, s_2 > b \text{ and } s_3, s_4 \geq -b.$$

- (2)  $u > 0$  in  $\mathcal{S} \otimes_{\max} (\mathbb{C}^5/J)$  if and only if there is an element  $s \in \mathcal{S}^+$  such that

$$s_1, s_2 > s \text{ and } s_3, s_4 \geq -s.$$

Several positivity criteria that we obtain in Section 6 together with the WEP characterization given in Section 5 form the basic part of Section 7 and allows us to prove the equivalence of WEP and  $(k, m)$  tight Riesz interpolations when  $k \geq 2$  and  $m \geq 3$ .

## 1. PRELIMINARIES

In this section we establish the terminology and state the definitions and basic results that shall be used throughout the paper. It is assumed throughout that all C\*-algebras are unital and all ideals are closed and two sided (and consequently \*-closed). The C\*-algebra of  $n \times n$  matrices is denoted by  $M_n$ . By an operator system  $\mathcal{S}$  we mean a unital, \*-closed subspace of  $B(H)$  together with the induced matricial order structure. We refer the reader to [23] for an introductory exposition of these objects along with their abstract characterization due to Choi and Effros. If  $\varphi : \mathcal{S} \rightarrow \mathcal{T}$  is a linear map, where  $\mathcal{T}$  and  $\mathcal{S}$  are operator systems, the  $n^{\text{th}}$ -amplification  $\varphi^{(n)}$  is defined by  $\varphi \otimes id_n : \mathcal{S} \otimes M_n \rightarrow \mathcal{T} \otimes M_n$ .  $\varphi$  is called  $n$ -positive if  $\varphi^{(n)}$  is positive and completely positive (cp) if  $\varphi^{(n)}$  is positive for all  $n$ . If in addition  $\varphi(e) = e$ , i.e. if it maps the unit to unit, then we say that  $\varphi$  is unital and completely positive (ucp).

A pair  $(i, \mathcal{A})$  is called a C\*-cover of an operator system  $\mathcal{S}$  if  $i : \mathcal{S} \rightarrow \mathcal{A}$  is a unital complete order embedding such that  $i(\mathcal{S})$  (topologically) generates  $\mathcal{A}$  as a C\*-algebra. We often identify  $\mathcal{S}$  with  $i(\mathcal{S})$  and consider it as an operator subsystem of  $\mathcal{A}$ . Every operator system  $\mathcal{S}$  admits two special C\*-covers, the universal and the enveloping C\*-algebras, denoted by  $C_u^*(\mathcal{S})$  and  $C_e^*(\mathcal{S})$ , respectively. The universal C\*-algebra has the following ‘‘maximality’’ property: For every ucp map  $\varphi : \mathcal{S} \rightarrow \mathcal{B}$ , where  $\mathcal{B}$  is a C\*-algebra, there is a uniquely determined unital \*-homomorphism  $\pi : C_u^*(\mathcal{S}) \rightarrow \mathcal{B}$  which extends  $\varphi$ . The enveloping C\*-algebra is the ‘‘minimal’’ C\*-cover in the sense that for any C\*-cover  $(i, \mathcal{A})$  of  $\mathcal{S}$  there is a uniquely determined unital \*-homomorphism  $\pi : \mathcal{A} \rightarrow C_e^*(\mathcal{S})$  such that  $\pi(i(s)) = s$  for every  $s$  in  $\mathcal{S}$ . The enveloping C\*-algebra of  $\mathcal{S}$  can be identified with the C\*-algebra generated by  $\mathcal{S}$  in its injective envelope  $I(\mathcal{S})$ . The reader may refer to [23, Chp. 15] for an excellent survey on the injectivity of operator systems. (However, for convenience, we remark that every injective operator system has the structure of a C\*-algebra.)

**1.1. Duality.** The duality in the operator system, especially in the finite dimensional case, has had a substantial role in the study of tensor products. Starting with an operator system  $\mathcal{S}$  the Banach dual  $\mathcal{S}$  has a natural matricial order structure. For  $f$  in  $\mathcal{S}^d$ , the involution is defined by  $f^*(s) = \overline{f(s^*)}$ . The matricial order structure is described as:

$(f_{ij}) \in M_n(\mathcal{S}^d)$  is positive if the map  $\mathcal{S} \ni s \mapsto (f_{ij}(s)) \in M_n$  is cp.

Throughout the paper  $\mathcal{S}^d$  will always represent this matrix ordered vector space. The bidual Banach space  $\mathcal{S}^{dd}$  has also a natural matricial order structure arising from the fact that it is the dual of  $\mathcal{S}^d$ . The following is perhaps well known, see [16], e.g.:

**Theorem 1.1.**  *$\mathcal{S}^{dd}$  is an operator system with unit  $\hat{e}$ , the canonical image of  $e$  in  $\mathcal{S}^{dd}$ . Moreover, the canonical embedding of  $\mathcal{S}$  into  $\mathcal{S}^{dd}$  is a complete order embedding.*

A state  $f$  on  $\mathcal{S}$  (i.e. a positive linear functional with  $f(e) = 1$ ) is called **faithful** if  $s \geq 0$  and  $f(s) = 0$  implies that  $s = 0$ , in other words  $f$  maps non-zero positive elements to non-zero positive scalars. Every finite dimensional operator system  $\mathcal{S}$  possesses a faithful state [5, Sec. 6], and consequently, the matricially ordered space  $\mathcal{S}^d$  is again an operator system with the (non-canonical) Archimedean matrix order unit  $f$ .

**1.2. Quotients.** A subspace  $J$  of an operator system  $\mathcal{S}$  is called a **kernel** if it is the kernel of a ucp map defined from  $\mathcal{S}$  into another operator system  $\mathcal{T}$ . Note that  $J$  has to be a \*-closed and non-unital subspace of  $\mathcal{S}$ , however, these properties, in general, do not characterize a kernel. When  $J \subset \mathcal{S}$  is a kernel the algebraic quotient  $\mathcal{S}/J$  has a natural operator system structure with unit  $\hat{e} = e + J$ . We first define

$$D_n = \{(\hat{s}_{ij}) : (s_{ij}) \text{ is positive in } M_n(\mathcal{S})\}.$$

$\mathcal{S}/J$  together with cones  $\{D_n\}_{n=1}^\infty$  form a matricially ordered space, moreover,  $\dot{e}$  can be shown to be a matrix order unit. However  $D_n$  may fail to be closed in the order topology induced by  $\dot{e}_n$  and therefore another step, namely the completion of the cones, (also known as the Archimedeanization process) is required:

$$C_n = \{(\dot{s}_{ij}) : (\dot{s}_{ij}) + \epsilon \dot{e}_n \text{ is in } D_n \text{ for all } \epsilon > 0\}.$$

Now  $\mathcal{S}/J$  with matricial order structure  $\{C_n\}_{n=1}^\infty$  and unit  $\dot{e}$  is an operator system and is called the quotient operator system of  $\mathcal{S}$  by  $J$ .  $J$  is said to be **proximal** if  $D_1 = C_1$  and **completely proximal** if  $D_n = C_n$ .

**Remark 1.2.** A finite dimensional subspace  $J$  of an operator system  $\mathcal{S}$  is called a **null-subspace** if it is closed under the involution  $*$  and does not contain any positive other than 0. In [13] it was shown that every null subspace is a completely proximal kernel.

*Example 1.3.* Let  $J_n \subset M_n$  be the set of diagonal operators with 0-trace. Clearly  $J_n$  is a null-subspace. So it is a completely proximal kernel.

*Example 1.4.* Let  $y \in \mathcal{S}$  be a self-adjoint element that is neither positive nor negative. Then  $J = \text{span}\{y\}$  is one dimensional null-subspace of  $\mathcal{S}$ .

*Example 1.5.* Let  $\mathbb{F}_n$  be the free group on  $n$ -generators. Let  $C^*(\mathbb{F}_n)$  be the full group  $C^*$ -algebra of  $\mathbb{F}_n$ . Consider  $J = \text{span}\{u_1, \dots, u_n, u_1^*, \dots, u_n^*\}$  where  $u_1, \dots, u_n$  are the universal unitaries. Then  $J$  is a null subspace and therefore a completely proximal kernel in  $C^*(\mathbb{F}_n)$ .

The operator system quotients have the following compatibility property with the morphisms: Letting  $J \subset \mathcal{S}$  be a kernel, if  $\varphi : \mathcal{S} \rightarrow \mathcal{T}$  is cp map with  $J \subset \ker(\varphi)$  then the induced map  $\bar{\varphi} : \mathcal{S}/J \rightarrow \mathcal{T}$  is still a cp map. Conversely if  $\psi : \mathcal{S}/J \rightarrow \mathcal{T}$  is a cp map then  $\psi \circ q : \mathcal{S} \rightarrow \mathcal{T}$ , where  $q : \mathcal{S} \rightarrow \mathcal{S}/J$  is the quotient map, is a cp map with kernel including  $J$ .

A surjective cp map  $\varphi : \mathcal{S} \rightarrow \mathcal{T}$  is called a **complete quotient map** if the induced map  $\bar{\varphi} : \mathcal{S}/\ker(\varphi) \rightarrow \mathcal{T}$ , which is cp, is a complete order isomorphism. These maps are the dual notions of the complete order embeddings. Following is from [14, Sec. 2].

**Theorem 1.6.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be finite dimensional operator systems. If  $i : \mathcal{S} \rightarrow \mathcal{T}$  is a complete order embedding then the adjoint map  $i^d : \mathcal{T}^d \rightarrow \mathcal{S}^d$  is a complete quotient map. Moreover by special selection of faithful states on  $\mathcal{S}$  and  $\mathcal{T}$  one may suppose that  $i^d$  is also unital and the kernel of  $i^d$  is a null-subspace of  $\mathcal{T}^d$ .*

So roughly speaking if  $\mathcal{S} \subset \mathcal{T}$  then we get  $\mathcal{S}^d = \mathcal{T}^d/J$  for some null-subspace  $J \subset \mathcal{T}^d$ . A moment of thought shows that  $J$  has to be the collection of linear functional that vanish on  $\mathcal{S}$ . The converse of this also true [7].

**Theorem 1.7** (Farenick, Paulsen). *Let  $q : \mathcal{S} \rightarrow \mathcal{T}$  be a complete quotient map. Then  $q^d : \mathcal{T}^d \rightarrow \mathcal{S}^d$  is a complete order embedding.*

A *unitality* problem of  $q^d$  may occur in this case. We need the kernel of  $q$  to be a null-subspace to be able to assume  $q^d$  is unital (by proper selections of faithful states on  $\mathcal{S}$  and  $\mathcal{T}$ ). This remark together with the above exhibit how null-subspaces occur naturally.

## 2. TENSOR PRODUCTS OF OPERATOR SYSTEMS

In this section we recall the axiomatic definition of tensor products in the category of operator systems and review properties of several tensor products established in [16]. Suppose  $\mathcal{S}$  and  $\mathcal{T}$  are two operator systems. A matricial cone structure  $\tau = \{C_n\}$  on  $\mathcal{S} \otimes \mathcal{T}$  where  $C_n \subset M_n(\mathcal{S} \otimes \mathcal{T})_{sa}$ , is called a tensor product structure if

- (1)  $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}, e_{\mathcal{S}} \otimes e_{\mathcal{T}})$  is an operator system,

- (2) for any  $(s_{ij}) \in M_n(\mathcal{S})^+$  and  $(t_{rs}) \in M_k(\mathcal{T})^+$ ,  $(s_{ij} \otimes t_{rs})$  is in  $C_{nk}$  for all  $n, k$ ,
- (3) if  $\phi : \mathcal{S} \rightarrow M_n$  and  $\psi : \mathcal{T} \rightarrow M_k$  are ucp maps then  $\phi \otimes \psi : \mathcal{S} \otimes \mathcal{T} \rightarrow M_{nk}$  is a ucp map for every  $n$  and  $k$ .

A mapping  $\tau : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$  is said to be an *operator system tensor product* (or simply a *tensor product*) provided  $\tau$  maps each pair  $(\mathcal{S}, \mathcal{T})$  to a tensor product structure on  $\mathcal{S} \otimes \mathcal{T}$ , denoted by  $\mathcal{S} \otimes_\tau \mathcal{T}$ . A tensor product  $\tau$  is said to be **functorial** if for every operator systems  $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$  and  $\mathcal{T}_2$  and every ucp maps  $\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  and  $\psi : \mathcal{T}_1 \rightarrow \mathcal{T}_2$  the associated map  $\phi \otimes \psi : \mathcal{S}_1 \otimes_\tau \mathcal{T}_1 \rightarrow \mathcal{S}_2 \otimes_\tau \mathcal{T}_2$  is ucp. A tensor product  $\tau$  is called *symmetric* if  $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{T} \otimes_\tau \mathcal{S}$  and *associative* if  $(\mathcal{S} \otimes_\tau \mathcal{T}) \otimes_\tau \mathcal{R} = \mathcal{S} \otimes_\tau (\mathcal{T} \otimes_\tau \mathcal{R})$  for every  $\mathcal{S}, \mathcal{T}$  and  $\mathcal{R}$ .

There is a natural partial order on the operator system tensor products: If  $\tau_1$  and  $\tau_2$  are two tensor products then we say that  $\tau_1 \leq \tau_2$  if for every operator systems  $\mathcal{S}$  and  $\mathcal{T}$  the identity  $id : \mathcal{S} \otimes_{\tau_2} \mathcal{T} \rightarrow \mathcal{S} \otimes_{\tau_1} \mathcal{T}$  is completely positive. In other words  $\tau_1$  is smaller with respect to  $\tau_2$  if the cones it generates are larger. (Recall that larger matricial cones generate smaller canonical operator space structure.) The partial order on operator system tensor products forms a lattice as pointed out in [16, Sec. 7] and raises fundamental nuclearity properties as we shall discuss in the next section.

In the remaining of this section we discuss several important tensor products, namely the minimal (min), maximal (max), maximal commuting (c), enveloping left (el) and enveloping right (er) tensor products. With respect to the partial order relation given in the previous paragraph we have the following schema [16] :

$$min \leq el, er \leq c \leq max.$$

**2.1. Minimal Tensor Product.** Let  $\mathcal{S}$  and  $\mathcal{T}$  be two operator systems. We define the matricial cone structure on the tensor product  $\mathcal{S} \otimes \mathcal{T}$  as follows:

$$C_n^{min}(\mathcal{S}, \mathcal{T}) = \{(u_{ij}) \in M_n(\mathcal{S} \otimes \mathcal{T}) : ((\phi \otimes \psi)(u_{ij}))_{ij} \in M_{nkm}^+ \text{ for every ucp maps } \phi : \mathcal{S} \rightarrow M_k \text{ and } \psi : \mathcal{T} \rightarrow M_m \text{ for all } k, m.\}.$$

The matricial cone structure  $\{C_n^{min}\}$  satisfies the axioms (1), (2), and (3) and the resulting operator system is denoted by  $\mathcal{S} \otimes_{min} \mathcal{T}$ . For the proofs of the following we refer the reader to [16, Sec. 4].

- (1) If  $\tau$  is another operator system structure on  $\mathcal{S} \otimes \mathcal{T}$  then we have that  $min \leq \tau$ . In other words  $\{C_n^{min}\}$  forms the largest cone structure.
- (2) The minimal tensor product, when considered as a map  $min : \mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}$ , is symmetric, associative and functorial.
- (3)  $min$  is injective in the sense that if  $\mathcal{S}_1 \subset \mathcal{S}_2$  and  $\mathcal{T}_1 \subset \mathcal{T}_2$  then  $\mathcal{S}_1 \otimes_{min} \mathcal{T}_1 \subset \mathcal{S}_2 \otimes_{min} \mathcal{T}_2$  completely order isomorphically.
- (4)  $min$  is *spatial*, that is, if  $\mathcal{S} \subset B(H)$  and  $\mathcal{T} \subset B(K)$  then the concrete operator system structure on  $\mathcal{S} \otimes \mathcal{T}$  arising from the inclusion  $B(H \otimes K)$  coincides with the minimal tensor product. From this one easily derives that  $min$  coincides with the the C\*-algebraic minimal tensor products when restricted to unital C\*-algebras (except for completion).

**2.2. Maximal Tensor Product.** The construction of the maximal tensor product of two operator systems  $\mathcal{S}$  and  $\mathcal{T}$  requires two steps. We first define

$$D_n^{max}(\mathcal{S}, \mathcal{T}) = \{A^*(P \otimes Q)A : P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+, A \in M_{km,n}, k, m \in \mathbb{N}\}.$$

Although the matricial order structure  $\{D_n^{max}\}$  is strict and compatible (for the definitions see [23, Chp. 13] e.g.) it might not be closed with respect to the order topology and hence another step, namely the completion of the cones, is required. Since after this step  $e_n$  is an Archimedean order unit this process is also known as the Archimedeanization process (see [24] e.g). We define

$$C_n^{max}(\mathcal{S}, \mathcal{T}) = \{P \in M_n(\mathcal{S} \otimes \mathcal{T}) : r(e_1 \otimes e_2)_n + P \in D_n^{max}(\mathcal{S}, \mathcal{T}) \forall r > 0\}.$$

Now the matrix order structure  $\{C_n^{max}\}$  satisfies all the axioms and the resulting operator system is denoted by  $\mathcal{S} \otimes_{max} \mathcal{T}$ . Below we listed the main properties of this tensor product:

- (1) Let  $\tau$  is another operator system structure on  $\mathcal{S} \otimes \mathcal{T}$  then  $\tau \leq max$ , that is,  $\{C_n^{max}\}$  is the smallest cone structure.
- (2)  $max$ , as  $min$ , has all properties symmetry, associativity and functoriality. Moreover, like  $min$ , it has the strong functoriality in the sense that if  $\varphi_i : \mathcal{S}_i \rightarrow \mathcal{T}_i$  are cp maps then the associative tensor map  $\varphi_1 \otimes \varphi_2 : \mathcal{S}_1 \otimes_{max} \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes_{max} \mathcal{T}_2$  is again cp.
- (3)  $max$  coincides with the C\*-algebraic maximal tensor product when restricted to unital C\*-algebras (again, except for completion).
- (4) As it is well known from C\*-algebras,  $max$  does not have the injectivity property that  $min$  possesses. However it is projective as shown by Han [9]: if  $q_1 : \mathcal{S}_1 \rightarrow \mathcal{T}_1$  and  $q_2 : \mathcal{S}_2 \rightarrow \mathcal{T}_2$  are complete quotient maps then the tensor map

$$q_1 \otimes q_2 : \mathcal{S}_1 \otimes_{max} \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes_{max} \mathcal{T}_2$$

is again a complete quotient map.

- (5) Lance's duality result regarding the maximal tensor products for C\*-algebras in [20] can be extended to general operator systems: A linear map  $f : \mathcal{S} \otimes_{max} \mathcal{T} \rightarrow \mathbb{C}$  is positive if and only if the corresponding map  $\varphi_f : \mathcal{S} \rightarrow \mathcal{T}^d$  is completely positive. Here  $\varphi_f(s)$  is the linear functional on  $T$  given by  $\varphi_f(s)(t) = f(s \otimes t)$ . (See also [16, Lem. 5.7 and Thm. 5.8].) Consequently we obtain the following representation of the maximal tensor product:

$$(\mathcal{S} \otimes_{max} \mathcal{T})^{d,+} = CP(\mathcal{S}, \mathcal{T}^d).$$

**2.3. (Maximal) Commuting Tensor Product.** Another important tensor product we want to discuss is the commuting (or maximal commuting) tensor product which is denoted by  $c$ . It agrees with the C\*-algebraic maximal tensor products on the category of unital C\*-algebras however it is different than  $max$  for general operator systems. We define the matricial order structure by using the ucp maps with commuting ranges. More precisely, if  $\mathcal{S}$  and  $\mathcal{T}$  are two operator systems then  $C_n^{com}$  consist of all  $(u_{ij}) \in M_n(\mathcal{S} \otimes \mathcal{T})$  with the property that for any Hilbert space  $H$ , any ucp  $\phi : \mathcal{S} \rightarrow B(H)$  and  $\psi : \mathcal{T} \rightarrow B(H)$  with commuting ranges

$$(\phi \cdot \psi)^{(n)}(u_{ij}) \geq 0$$

where  $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \rightarrow B(H)$  is the map defined by  $\phi \cdot \psi(s \otimes t) = \phi(s)\psi(t)$ . The matricial cone structure  $\{C_n^{com}\}$  satisfies the axioms (1), (2) and (3), and the resulting operator system is denoted by  $\mathcal{S} \otimes_c \mathcal{T}$ . We again list the main properties of this tensor product:

- (1) The commuting tensor product  $c$  is functorial and symmetric however we don't know whether is it associative or not. Though for every  $n$  we have

$$M_n(\mathcal{S} \otimes_c \mathcal{T}) = M_n(\mathcal{S}) \otimes_c \mathcal{T} = \mathcal{S} \otimes_c M_n(\mathcal{T}).$$

- (2) If  $\tau$  is an operator system structure on  $\mathcal{S} \otimes \mathcal{T}$  such that  $\mathcal{S} \otimes_\tau \mathcal{T}$  attains a representation in a  $B(H)$  with " $\mathcal{S}$ " and " $\mathcal{T}$ " portions are commuting then  $\tau \leq c$ . This directly follows from the definition of  $c$  and justifies the name "maximal commuting".

- (3) As we pointed out  $c$  and  $max$  coincides on the unital  $C^*$ -algebras. This result can be extended even further: If  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $\mathcal{S}$  is an operator system, then  $\mathcal{A} \otimes_c \mathcal{S} = \mathcal{A} \otimes_{max} \mathcal{S}$ .
- (4) For every  $\mathcal{S}$  and  $\mathcal{T}$  we have the unital complete order embedding

$$\mathcal{S} \otimes_c \mathcal{T} \subset C_u^*(\mathcal{S}) \otimes_{max} C_u^*(\mathcal{T}).$$

- (5) The ucp maps defined by the commuting tensor product are the compression of the ucp maps with commuting ranges, that is, if  $\varphi : \mathcal{S} \otimes_c \mathcal{T} \rightarrow B(H)$  is a ucp map Then there is Hilbert space  $K$  containing  $H$  as a Hilbert subspace and ucp maps  $\phi : \mathcal{S} \rightarrow B(K)$  and  $\psi : \mathcal{T} \rightarrow B(K)$  with commuting ranges such that  $\varphi = P_H \phi \cdot \psi|_H$ . Conversely, every such map is ucp.

**2.4. Some Asymmetric Tensor Products.** In this subsection we discuss the enveloping left ( $el$ ) and enveloping right ( $er$ ) tensor products. Given operator systems  $\mathcal{S}$  and  $\mathcal{T}$  we define

$$\mathcal{S} \otimes_{el} \mathcal{T} : \subseteq I(\mathcal{S}) \otimes_{max} \mathcal{T} \text{ and } \mathcal{S} \otimes_{er} \mathcal{T} : \subseteq \mathcal{S} \otimes_{max} I(\mathcal{T})$$

where  $I(\cdot)$  is the injective envelope of an operator system. We remark that  $I(\mathcal{S}) \otimes_c \mathcal{T} = I(\mathcal{S}) \otimes_{max} \mathcal{T}$  as the  $I(\mathcal{S})$  has the structure of a  $C^*$ -algebra. Here are main properties:

- (1)  $el$  and  $er$  are functorial tensor products. We don't know whether these tensor products are associative or not. However for every  $n$  we have

$$M_n(\mathcal{S} \otimes_{el} \mathcal{T}) = M_n(\mathcal{S}) \otimes_{el} \mathcal{T} = \mathcal{S} \otimes_{el} M_n(\mathcal{T}).$$

A similar ‘‘associativity with matrix algebras’’ holds for  $er$  too.

- (2) Both  $el$  and  $er$  are not symmetric but they are asymmetric in the sense that

$$\mathcal{S} \otimes_{el} \mathcal{T} = \mathcal{T} \otimes_{er} \mathcal{S} \quad \text{via the map} \quad s \otimes t \mapsto t \otimes s.$$

- (3) The tensor product  $el$  is the maximal left injective functorial tensor product, that is, for any  $\mathcal{S} \subset \mathcal{S}_1$  and  $\mathcal{T}$  we have

$$\mathcal{S} \otimes_{el} \mathcal{T} \subseteq \mathcal{S}_1 \otimes_{el} \mathcal{T}$$

and it is the maximal functorial tensor product with this property. Likewise,  $er$  is the maximal right injective tensor product.

- (4) Tensor product  $el$  is independent of the injective operator system that containing  $\mathcal{S}$  as an operator subsystem. For example if  $\mathcal{S} \subset B(H)$  then we have the complete order embedding  $\mathcal{S} \otimes_{el} \mathcal{T} \subseteq B(H) \otimes_{max} \mathcal{T}$ . This simply follows from the left injectivity of  $el$  and the fact that  $el$  and  $max$  coincides if the left tensorant is an injective operator system which directly follows from the definition.
- (5)  $el$  and  $er$  are in general not comparable but they both lie between  $min$  and  $c$ .

### 3. CHARACTERIZATION OF VARIOUS NUCLEARITIES

In the previous section we have reviewed the tensor products in the category of operator systems. In this section we will overview the behavior of the operator systems under tensor products. More precisely, we will see several characterizations of the operator systems that fix a pair of tensor products.

Given two tensor products  $\tau_1 \leq \tau_2$ , an operator systems  $\mathcal{S}$  is said to be  $(\tau_1, \tau_2)$ -*nuclear* provided  $\mathcal{S} \otimes_{\tau_1} \mathcal{T} = \mathcal{S} \otimes_{\tau_2} \mathcal{T}$  for every operator system  $\mathcal{T}$ . We remark that the *place* of the operator system  $\mathcal{S}$  is important as not all the tensor products are symmetric.



**3.1. Operator System Local Lifting Property (osLLP).** We want to start with a discussion of (operator system) local lifting property (osLLP) which characterizes the operator systems having (min,er)-nuclearity.

**Definition 3.1.** An operator system  $\mathcal{S}$  is said to have **osLLP** if for every unital C\*-algebra  $\mathcal{A}$  and ideal  $I$  in  $\mathcal{A}$  and for every ucp map  $\varphi : \mathcal{S} \rightarrow \mathcal{A}/I$  the following holds: For every finite dimensional operator subsystem  $\mathcal{S}_0$  of  $\mathcal{S}$ , the restriction of  $\varphi$  on  $\mathcal{S}_0$ , say  $\varphi_0$ , lifts to a completely positive map on  $\mathcal{A}$  so that the following diagram commutes (where  $q : \mathcal{A} \rightarrow \mathcal{A}/I$  is the quotient map).

$$\begin{array}{ccc}
 & & \mathcal{A} \\
 & \nearrow \varphi_0 & \downarrow q \\
 \mathcal{S}_0 \subset \mathcal{S} & \xrightarrow{\text{ucp } \varphi} & \mathcal{A}/I
 \end{array}$$

Of course,  $\mathcal{S}$  may possess osLLP without a global lifting. We also remark that the completely positive local liftings can also be chosen to be ucp in the definition of osLLP (see the discussion in [15, Sec. 8]). The LLP definition for a C\*-algebra given in [18] is the same. So it follows that a unital C\*-algebra has LLP (in the sense of Kirchberg) if and only if it has osLLP. We can now state the connection of osLLP and tensor products given in [15]:

**Theorem 3.2.** *The following are equivalent for an operator system  $\mathcal{S}$ :*

- (1)  $\mathcal{S}$  has osLLP.
- (2)  $\mathcal{S} \otimes_{\min} B(H) = \mathcal{S} \otimes_{\max} B(H)$  for every Hilbert space  $H$  (or for  $H = l^2(\mathbb{N})$ ).
- (3)  $\mathcal{S}$  is (min,er)-nuclear, that is,  $\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_{\text{er}} \mathcal{T}$  for every  $\mathcal{T}$ .

Note that if  $\mathcal{A}$  is a C\*-algebra then the equivalence of (1) and (2) recovers a well known result of Kirchberg [18]. If we let  $\mathbb{B}$  denote  $B(l^2(\mathbb{N}))$ , the above equivalent conditions, in some similar context, is also called  $\mathbb{B}$ -nuclearity. (See [3], e.g.) Consequently for operator systems osLLP,  $\mathbb{B}$ -nuclearity and (min,er)-nuclearity are all equivalent.

**Remark 3.3.** The definition of LLP of a C\*-algebra in [25, Chp. 16 ] is different, it requires completely contractive liftings from finite dimensional operator subspaces. However, as it can be seen in [25, Thm. 16.2], all the approaches coincide for C\*-algebras.

**Note:** When we work with the finite dimensional operator systems we remove the extra word “local”, we even remove “os” and simply say “lifting property”.

**3.2. Weak Expectation Property (WEP).** If  $\mathcal{A}$  is a unital C\*-algebra then the bidual C\*-algebra  $\mathcal{A}^{**}$  is unital and completely order isomorphic to the bidual operator system  $\mathcal{A}^{dd}$ . This allows us to extend the notion of weak expectation to a more general setting. We say that an operator system  $\mathcal{S}$  has *WEP* if the canonical inclusion  $i : \mathcal{S} \hookrightarrow \mathcal{S}^{dd}$  extends to a ucp map on the injective envelope  $I(\mathcal{S})$ .

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{i} & \mathcal{S}^{dd} \\
 \cap & & \nearrow \\
 I(\mathcal{S}) & & 
 \end{array}$$

In [15] it was shown that WEP implies (el,max)-nuclearity and the difficult converse is shown in [9]. Consequently we have that

**Theorem 3.4.** *An operator system has WEP if and only if it is (el,max)-nuclear.*

If  $\mathcal{S}$  is a finite dimensional operator system then  $\mathcal{S}$  has WEP if and only if  $\mathcal{S}$  has the structure of a  $C^*$ -algebra (so it is the direct sum of matrix algebras). This follows by the fact that the canonical injection  $\mathcal{S} \hookrightarrow \mathcal{S}^{dd}$  is also surjective. Consequently, the *expectation* (the extended ucp map from  $I(\mathcal{S})$  into  $\mathcal{S}^{dd}$ ) can be used to define a multiplication on  $\mathcal{S}^{dd} \cong \mathcal{S}$  (see [23, Theorem 15.2], e.g.).

**3.3. Double Commutant Expectation Property (DCEP).** For a unital  $C^*$ -algebra  $\mathcal{A}$  the following are equivalent:

- (1) The canonical inclusion  $\mathcal{A}$  into  $\mathcal{A}^{**}$  extends to cp map on  $I(\mathcal{A})$ .
- (2) For all (unital,  $C^*$ -algebraic) inclusion  $\mathcal{A} \subset B(H)$  there is a ucp map  $\gamma : B(H) \rightarrow \mathcal{A}''$ , where  $\mathcal{A}''$  is the double commutant of  $\mathcal{A}$ , extending the inclusion  $\mathcal{A} \subset B(H)$ .

In either case we say that  $\mathcal{A}$  has the WEP. As pointed out by Vern Paulsen it can be shown, by using Arveson's commutant lifting theorem ([1] or [23, Thm. 12.7]) e.g., that in (2) the  $C^*$ -algebraic inclusion can be replaced by unital complete order embedding. Moreover, by using straightforward injectivity techniques, one can replace the domain of  $\gamma$  by the injective envelope  $I(\mathcal{A})$  of  $\mathcal{A}$ . These two equivalent notions differ in operator system setting. While (1) leads to the concept of WEP for general operator systems (above), (2) extends as follows:

**Definition 3.5.** We say that  $\mathcal{S}$  has **DCEP** if every (operator system) inclusion  $\mathcal{S} \subset B(H)$  extends to a ucp map from  $I(\mathcal{S})$  into  $\mathcal{S}''$ , the double commutant of  $\mathcal{S}$  in  $B(H)$ .

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & B(H) \supseteq \mathcal{S}'' \\ \cap & & \nearrow \text{dotted arrow} \\ I(\mathcal{S}) & & \end{array}$$

Many fundamental results and problems regarding WEP for unital  $C^*$ -algebras reduce to DCEP for general operator systems. The following is a direct consequence of Theorem 7.1 and 7.6 in [15]:

**Theorem 3.6.** *The following are equivalent for an operator system  $\mathcal{S}$ :*

- (1)  $\mathcal{S}$  is (el,c)-nuclear, that is,  $\mathcal{S} \otimes_{el} \mathcal{T} = \mathcal{S} \otimes_c \mathcal{T}$  for every  $\mathcal{T}$ .
- (2)  $\mathcal{S}$  has DCEP.
- (3)  $\mathcal{S} \otimes_{min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{max} C^*(\mathbb{F}_\infty)$ .
- (4) For any  $\mathcal{S} \subset \mathcal{A}$  and  $\mathcal{B}$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras, we have the (operator system) embedding  $\mathcal{S} \otimes_{max} \mathcal{B} \subset \mathcal{A} \otimes_{max} \mathcal{B}$ .

Here  $C^*(\mathbb{F}_\infty)$  is the full group  $C^*$ -algebra of the free group on countably infinite number of generators  $\mathbb{F}_\infty$ . Note that (3) is Kirchberg's WEP characterization in [18] and (4) is Lance's seminuclearity in [21] for unital  $C^*$ -algebras.

Note that the equivalence WEP and DCEP for unital  $C^*$ -algebras can be inferred from the fact that *c* and *max* coincide when one of the tensorant is a  $C^*$ -algebra. So (el,max)-nuclearity (WEP) and (el,c)-nuclearity (DCEP) coincide in this case. For general operator systems  $\text{WEP} \implies \text{DCEP}$  which simply follows from (el,max)-nuclearity implies (el,c)-nuclearity. The converse fails even on two dimensional operator systems: In [14] it was shown that every two dimensional operator system is (min,c)-nuclear hence they have DCEP. But, by the last paragraph in the above subsection, only  $\mathbb{C}^2$  (and its isomorphism class) has WEP.

**3.4. Exactness.** The importance of exactness and its connection to the tensor theory of  $C^*$ -algebras ensued by Kirchberg [19], [18]. Exactness is a categorical concept and requires a correct notion of quotient theory. The operator system quotients established in [15] that we reviewed in preliminaries has been used to extend the exactness to general operator systems.

Before starting the definition we recall a couple of results from [15]: Let  $\mathcal{S}$  be an operator system,  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $I$  be an ideal in  $\mathcal{A}$ . Then  $\mathcal{S} \bar{\otimes} I$  is a kernel in  $\mathcal{S} \hat{\otimes}_{min} \mathcal{A}$  where  $\hat{\otimes}_{min}$  represents the completed minimal tensor product and  $\bar{\otimes}$  denotes the closure of the algebraic tensor product in the larger space. By using the functoriality of the minimal tensor product it is easy to see that the map

$$\mathcal{S} \hat{\otimes}_{min} \mathcal{A} \xrightarrow{id \otimes q} \mathcal{S} \hat{\otimes}_{min} (\mathcal{A}/I),$$

where  $id$  is the identity on  $\mathcal{S}$  and  $q$  is the quotient map from  $\mathcal{A}$  onto  $\mathcal{A}/I$ , is ucp and its kernel contains  $\mathcal{S} \bar{\otimes} I$ . Consequently the induced map

$$(\mathcal{S} \hat{\otimes}_{min} \mathcal{A}) / (\mathcal{S} \bar{\otimes} I) \longrightarrow \mathcal{S} \hat{\otimes}_{min} (\mathcal{A}/I)$$

is still unital and completely positive. An operator system is said to be **exact** if this induced map is a bijective and a complete order isomorphism for every  $C^*$ -algebra  $\mathcal{A}$  and ideal  $I$  in  $\mathcal{A}$ . In other words we have the equality

$$(\mathcal{S} \hat{\otimes}_{min} \mathcal{A}) / (\mathcal{S} \bar{\otimes} I) = \mathcal{S} \hat{\otimes}_{min} (\mathcal{A}/I).$$

We remark that the induced map may fail to be surjective or injective, moreover even if it has these properties it may fail to be a complete order isomorphism.

**Remark 3.7.** If  $\mathcal{S}$  is finite dimensional then  $\mathcal{S} \otimes_{min} \mathcal{A} = \mathcal{S} \hat{\otimes}_{min} \mathcal{A}$  and  $\mathcal{S} \bar{\otimes} I = \mathcal{S} \otimes I$ . Moreover the induced (ucp) map

$$(\mathcal{S} \otimes_{min} \mathcal{A}) / (\mathcal{S} \otimes I) \longrightarrow \mathcal{S} \otimes_{min} (\mathcal{A}/I)$$

is always bijective. Thus, for this case, exactness is equivalent to the inverse of induced map being a complete order isomorphism for every  $\mathcal{A}$  and  $I \subset \mathcal{A}$  (see [14]).

**Note:** The term exactness in this paper coincides with 1-exactness in [15].

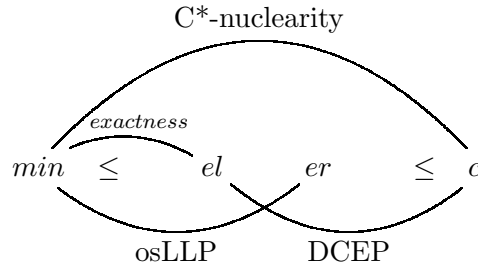
A unital  $C^*$ -algebra is exact (in the sense of Kirchberg) if and only if it is an exact operator system which follows from the fact that the unital  $C^*$ -algebra ideal quotient coincides with the operator system kernel quotient. The following is Theorem 5.7 of [15]:

**Theorem 3.8.** *An operator system is exact if and only if it is (min,el)-nuclear.*

In Theorem 3.13 we will see that exactness and the lifting property are dual pairs. We want to finish this subsection with the following stability property [15]:

**Proposition 3.9.** *Exactness passes to operator subsystems. That is, if  $\mathcal{S}$  is exact then every operator subsystem of  $\mathcal{S}$  is exact. Conversely, if every finite dimensional operator subsystem of  $\mathcal{S}$  is exact then  $\mathcal{S}$  is exact.*

**3.5. Final Remarks, Stability, and Examples on Nuclearity.** In this subsection we review the behavior of nuclearity related properties under basic algebraic operations such as quotients and duality. We start with the following nuclearity schema which summarizes the tensorial characterizations of several properties we have discussed:



An operator system  $\mathcal{S}$  is said to be **C\*-nuclear** if  $\mathcal{S} \otimes_{\min} \mathcal{A} = \mathcal{S} \otimes_{\max} \mathcal{A}$  for every C\*-algebra  $\mathcal{A}$ . It is elementary to show that ([14]) C\*-nuclearity and (min,c)-nuclearity coincide.

A C\*-algebra  $\mathcal{A}$  is said to be **nuclear** if  $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$  for every C\*-algebra  $\mathcal{B}$ . It follows that a unital C\*-algebra  $\mathcal{A}$  is nuclear if and only if it is (min,max)-nuclear operator systems. So in this case  $\mathcal{A}$  has the all the properties in the above schema. We also remark that Han and Paulsen [10] prove that an operator system  $\mathcal{S}$  is (min,max)-nuclear if and only if it has *completely positive factorization property* (in the sense of [19]). This extends a well known result of Choi and Effros on nuclear C\*-algebras.

**Remark 3.10.** Let  $\mathcal{S}$  be a finite dimensional operator system. Then  $\mathcal{S}$  is (c,max)-nuclear if and only if it has the structure of a C\*-algebra ([14]). Consequently, if  $\mathcal{S}$  is a non-C\*-algebra then C\*-nuclearity is the highest nuclearity that one should expect (of course among  $\min \leq el, er \leq c \leq \max$ ). This, on the finite dimensional case, puts the importance of above mentioned properties.

*Example 3.11.* It is evident from the above table that exactness and DCEP together are equivalent to C\*-nuclearity. It was shown in [14] that the operator subsystem

$$\mathcal{R} = \text{span}\{I, E_{12}, E_{21}, E_{34}, E_{43}\} \subset M_4$$

does not have the lifting property. Clearly it is exact (since  $M_4$  is nuclear, in particular,  $M_4$  is exact so any operator subsystem is exact). Note that  $\mathcal{R}$  cannot have DCEP as DCEP + exactness = C\*-nuclearity  $\Rightarrow$  LP.

*Example 3.12.* Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are unital C\*-algebras such that  $\mathcal{A}$  has LLP (eq. osLLP) and  $\mathcal{B}$  has WEP (eq. DCEP). Then, by using the tensor characterizations in the above table, we have

$$\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{er} \mathcal{B}.$$

Since  $\mathcal{B}$  has WEP, (and taking into account the asymmetry of  $el$  and  $er$ ) we have

$$\mathcal{A} \otimes_{er} \mathcal{B} = \mathcal{A} \otimes_{c=\max} \mathcal{B}.$$

Thus  $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ , so we recover a well known result of Kirchberg [18].

The relation of exactness and the lifting property perhaps best seen in the finite dimensional case. Following is from [14].

**Theorem 3.13.** *Let  $\mathcal{S}$  be a finite dimensional operator system. Then  $\mathcal{S}$  is exact if and only if  $\mathcal{S}^d$  has the lifting property (and vice versa). In other words,  $\mathcal{S}$  is (min,el)-nuclear if and only if  $\mathcal{S}^d$  is (min,er)-nuclear.*

*Example 3.14.* Let  $\mathcal{R}$  be the above operator system. Then the dual operator system  $\mathcal{R}^d$  is not exact but has the lifting property. We don't know whether  $\mathcal{R}^d$  has DCEP. In [14] it was shown that the Kirchberg conjecture has an affirmative answer if and only if  $\mathcal{R}^d$  has DCEP.

In contrast to C\*-algebra ideal quotients the lifting property is stable quotients by null-subspaces [14]:

**Theorem 3.15.** *Let  $\mathcal{S}$  be a finite dimensional operator system and  $J$  be a null-subspace in  $\mathcal{S}$ . If  $\mathcal{S}$  has the lifting property then  $\mathcal{S}/J$  has the same property.*

*Example 3.16.* Let  $J_3 \subset M_3$  be the diagonal operators with 0-trace. Then  $M_3/J_3$  has the lifting property. This follows by the fact that  $M_3$  is nuclear so has the lifting property.

*Example 3.17.* Unlike to (separable) C\*-algebra/ideal quotients exactness is not preserved under operator system/kernel quotients. It was shown in [14] that  $M_3/J_3$  is not exact (depending heavily on a result of Wassermann [26]). We don't know whether  $M_3/J_3$  has DCEP or not. This problem is again equivalent to the Kirchberg conjecture.

We will also need the following fact from [9] and [14]. First we remark that given operator systems  $\mathcal{S}$  and  $\mathcal{T}$  if  $\dim(\mathcal{T})$  is finite then the completion  $\mathcal{S} \hat{\otimes}_\tau \mathcal{T}$  of any tensor product  $\tau$  is same as the algebraic tensor product.

**Theorem 3.18.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be operator systems with  $\dim(\mathcal{T}) < \infty$  and let  $J$  be a null-subspace of  $\mathcal{T}$ . Then we have the unital complete order isomorphism*

$$(\mathcal{S} \otimes_{\max} \mathcal{T})/(\mathcal{S} \otimes J) = \mathcal{S} \otimes_{\max} (\mathcal{T}/J).$$

*If, in addition,  $\dim(\mathcal{S}) < \infty$ , then  $\mathcal{S} \otimes J$  is a null-subspace of  $\mathcal{S} \otimes_{\max} \mathcal{T}$  (and so a completely proximal kernel).*

#### 4. COPRODUCTS OF OPERATOR SYSTEMS

In this section we review the the amalgamated sum of two operator systems over their unit introduced in [17] (or coproduct of two operator systems with the language of [8]). Given two operator systems  $\mathcal{S}$  and  $\mathcal{T}$  there is an operator system  $\mathcal{U}$  with unital and completely order embeddings  $i : \mathcal{S} \hookrightarrow \mathcal{U}$  and  $j : \mathcal{T} \hookrightarrow \mathcal{U}$  such that the following holds: Whenever  $\phi : \mathcal{S} \rightarrow \mathcal{R}$  and  $\psi : \mathcal{T} \rightarrow \mathcal{R}$  are ucp maps, where  $\mathcal{R}$  is any operator system, then there is a unique ucp map  $\varphi : \mathcal{U} \rightarrow \mathcal{R}$  such that  $\phi(s) = \varphi(i(s))$  and  $\psi(t) = \varphi(j(t))$  for all  $s \in \mathcal{S}$  and  $t \in \mathcal{T}$ . We will call  $\mathcal{U}$  together with embeddings  $i$  and  $j$  the *coproduct* of  $\mathcal{S}$  and  $\mathcal{T}$  and we will denote it by  $\mathcal{S} \oplus_1 \mathcal{T}$ . The following commuting diagram summarizes the universal property of the coproduct.

$$\begin{array}{ccc}
 \mathcal{S} & & \\
 \downarrow i & \searrow \text{ucp } \phi & \\
 \mathcal{S} \oplus_1 \mathcal{T} & \xrightarrow{! \text{ ucp } \varphi} & \mathcal{R} \\
 \uparrow j & \nearrow \text{ucp } \psi & \\
 \mathcal{T} & & 
 \end{array}$$

Once such an object is proven to exist it is easy to see that it has to be unique up to a unital complete order isomorphism. We leave the verification of this to the reader which is based on the fact that  $\mathcal{U}$  must be spanned by the elements of  $i(\mathcal{S})$  and  $j(\mathcal{T})$  as  $\varphi$  is uniquely determined.

There are several ways to construct the coproduct of two operator systems. We first recall the free product of C\*-algebras. Given unital C\*-algebras  $\mathcal{A}$  and  $\mathcal{B}$  the *unital free product*  $\mathcal{A} *_1 \mathcal{B}$  is a C\*-algebra with C\*-algebraic inclusions  $i : \mathcal{A} \rightarrow \mathcal{A} *_1 \mathcal{B}$  and  $j : \mathcal{B} \rightarrow \mathcal{A} *_1 \mathcal{B}$  such that whenever  $\pi : \mathcal{A} \rightarrow \mathcal{C}$  and  $\rho : \mathcal{B} \rightarrow \mathcal{C}$  are unital \*-homomorphisms, where  $\mathcal{C}$  is any C\*-algebra, then there is a unique unital \*-homomorphism  $\gamma : \mathcal{A} *_1 \mathcal{B} \rightarrow \mathcal{C}$  such that  $\gamma(i(a)) = \pi(a)$  and  $\gamma(j(b)) = \rho(b)$  for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . We often identify  $\mathcal{A}$  and  $\mathcal{B}$  with their canonical images in  $\mathcal{A} *_1 \mathcal{B}$ . First and the easiest way to see the existence of the coproduct is the following.

**Proposition 4.1.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be two operator systems. Then*

$$\mathcal{S} \oplus_1 \mathcal{T} = \{s + t : s \in \mathcal{S}, t \in \mathcal{T}\} \subset C_u^*(\mathcal{S}) *_1 C_u^*(\mathcal{T}).$$

*Proof.* Suppose  $\phi : \mathcal{S} \rightarrow \mathcal{R}$  and  $\psi : \mathcal{T} \rightarrow \mathcal{R}$  are ucp maps, where  $\mathcal{R}$  is any operator system. Let  $\mathcal{R} \subset B(H)$ . Both  $\phi$  and  $\psi$  extends to unital \*-homomorphisms  $\pi : C_u^*(\mathcal{S}) \rightarrow B(H)$  and  $\rho : C_u^*(\mathcal{T}) \rightarrow B(H)$ , respectively. Let  $\gamma$  be the unital \*-homomorphism from  $C_u^*(\mathcal{S}) *_1 C_u^*(\mathcal{T})$  into  $B(H)$  extending  $\pi$  and  $\rho$ . Now by restricting  $\gamma$  the operator subsystem  $\{s + t : s \in \mathcal{S}, t \in \mathcal{T}\}$  we obtain a ucp map. Note that the image of the restricted  $\gamma$  still lies in  $\mathcal{R}$ .  $\square$

A more general form of this will be more useful. We will need the following:

**Lemma 4.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be unital  $C^*$ -algebras such that they both have a one dimensional representation, that is, there are unital  $*$ -homomorphisms  $w_1 : \mathcal{A} \rightarrow \mathbb{C}$  and  $w_2 : \mathcal{B} \rightarrow \mathbb{C}$ . If  $\phi : \mathcal{A} \rightarrow B(H)$  and  $\psi : \mathcal{B} \rightarrow B(H)$  are ucp maps then there is a ucp map  $\varphi : \mathcal{A} *_1 \mathcal{B} \rightarrow B(H)$  such that  $\varphi(a) = \phi(a)$  and  $\varphi(b) = \psi(b)$  for every  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .*

*Proof.* By using the Stinespring representation theorem we can find a Hilbert space  $H_1$  and a unital  $*$ -homomorphism  $\pi : \mathcal{A} \rightarrow B(H \oplus H_1)$  such that  $\phi = V^*\pi(\cdot)V$  where  $V^* = (1 \ 0)$ . Similarly let  $(\rho, H \oplus H_2, W)$  be the Stinespring representation of  $\psi$ . Set  $K = H \oplus H_1 \oplus H_2$ . Let  $\tilde{\pi} = \pi \oplus w_1(\cdot)I_2$  and similarly let  $\tilde{\rho} = \rho \oplus w_2(\cdot)I_1$ . More precisely, if

$$\pi(a) = \begin{pmatrix} \phi(a) & x \\ y & z \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} \psi(b) & X \\ Y & Z \end{pmatrix}$$

for some  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  written in the matrix form w.r.t.  $H \oplus H_1$  and  $H \oplus H_2$ , respectively, then

$$\tilde{\pi}(a) = \begin{pmatrix} \phi(a) & x & 0 \\ y & z & 0 \\ 0 & 0 & w_1(a)I_2 \end{pmatrix} \text{ and } \tilde{\rho}(b) = \begin{pmatrix} \psi(b) & 0 & X \\ 0 & w_2(b)I_1 & 0 \\ Y & 0 & Z \end{pmatrix}.$$

Clearly  $\tilde{\pi}$  and  $\tilde{\rho}$  are still unital  $*$ -homomorphisms such that  $\phi = U^*\tilde{\pi}(\cdot)U$  and  $\psi = U^*\tilde{\rho}(\cdot)U$  where  $U^* = (1 \ 0 \ 0)$ . Let  $\gamma : \mathcal{A} *_1 \mathcal{B} \rightarrow B(K)$  be the unital  $*$ -homomorphism extending  $\tilde{\pi}$  and  $\tilde{\rho}$ . Clearly  $\varphi = U^*\gamma(\cdot)U$  is a ucp map from  $\mathcal{A} *_1 \mathcal{B}$  into  $B(H)$  with desired properties.  $\square$

Following [15], we will say that an operator subsystem  $\mathcal{S}$  of a unital  $C^*$ -algebra  $\mathcal{A}$  *contains enough unitaries* if there is a collection of unitaries that belongs to  $\mathcal{S}$  and generates  $\mathcal{A}$  as a  $C^*$ -algebra, that is,  $\mathcal{A}$  is the smallest  $C^*$ -algebra that contains these unitaries.

**Proposition 4.3.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be unital  $C^*$ -algebras such that they both have a one dimensional representation. Let  $\mathcal{S} \subset \mathcal{A}$  and  $\mathcal{T} \subset \mathcal{B}$  be operator subsystems. Then we have*

$$\mathcal{S} \oplus_1 \mathcal{T} = \{s + t : s \in \mathcal{S}, t \in \mathcal{T}\} \subset \mathcal{A} *_1 \mathcal{B}.$$

*Moreover, if  $\mathcal{S}$  and  $\mathcal{T}$  contains enough unitaries in  $\mathcal{A}$  and  $\mathcal{B}$ , resp., then  $\{s + t : s \in \mathcal{S}, t \in \mathcal{T}\}$  contains enough unitaries in  $\mathcal{A} *_1 \mathcal{B}$ .*

*Proof.* We simply show that the operator system  $\{s + t : s \in \mathcal{S}, t \in \mathcal{T}\} \subset \mathcal{A} *_1 \mathcal{B}$  satisfies the universal property of the coproduct of  $\mathcal{S}$  and  $\mathcal{T}$ . Let  $\phi : \mathcal{S} \rightarrow \mathcal{R}$  and  $\psi : \mathcal{T} \rightarrow \mathcal{R}$  be two ucp maps, where  $\mathcal{R}$  is any operator system. Let  $\mathcal{R} \subset B(H)$ . By using Arveson's extension theorem, let  $\tilde{\phi} : \mathcal{A} \rightarrow B(H)$  and  $\tilde{\psi} : \mathcal{B} \rightarrow B(H)$  be the ucp extensions of  $\phi$  and  $\psi$ . By the above lemma there is a ucp map  $\varphi : \mathcal{A} *_1 \mathcal{B} \rightarrow B(H)$  such that  $\varphi(a) = \tilde{\phi}(a)$  and  $\varphi(b) = \tilde{\psi}(b)$  for every  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . Now the restriction of this map on  $\{s + t : s \in \mathcal{S}, t \in \mathcal{T}\}$  is the desired extension of  $\phi$  and  $\psi$ . To see the final part note that  $\mathcal{A} \subset \mathcal{A} *_1 \mathcal{B}$  as a  $C^*$ -subalgebra. So unitaries in  $\mathcal{S}$  already generates  $\mathcal{A} \subset \mathcal{A} *_1 \mathcal{B}$ . Likewise, unitaries in  $\mathcal{T}$  generates  $\mathcal{B} \subset \mathcal{A} *_1 \mathcal{B}$ . Since  $\mathcal{A} *_1 \mathcal{B}$  is the smallest  $C^*$ -algebra that contains  $\mathcal{A}$  and  $\mathcal{B}$  the result follows.  $\square$

**Corollary 4.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two unital  $C^*$ -algebras. Suppose both  $\mathcal{A}$  and  $\mathcal{B}$  have one dimensional representations. Then*

$$\mathcal{A} \oplus_1 \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\} \subset \mathcal{A} *_1 \mathcal{B}.$$

*Moreover, the operator subsystem  $\{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$  contains enough unitaries in  $\mathcal{A} *_1 \mathcal{B}$ .*

**Remark 4.5.** For any operator system  $\mathcal{S}$ ,  $C_u^*(\mathcal{S})$  possesses a one dimensional representation. More precisely, if  $f : \mathcal{S} \rightarrow \mathbb{C}$  is a state then it extends to unital  $*$ -homomorphism on  $C_u^*(\mathcal{S})$ . Thus, our first construction of coproduct is a special case of the above proposition.

**Remark 4.6.** If  $G$  is a discrete group then the full  $C^*$ -algebra  $C^*(G)$  has a one dimensional representation. In fact,  $\rho : G \rightarrow \mathbb{C}$  given by  $\rho(g) = 1$  is a unitary representation so the universal property of  $C^*(G)$  ensures that  $\rho$  extends to a unital  $*$ -homomorphism on  $C^*(G)$ .

Now, we will discuss a more concrete construction given in [14]. This also justifies the notation of the coproduct. If  $\mathcal{S}$  and  $\mathcal{T}$  be operator systems then  $span\{(e, -e)\}$  is a one dimensional null subspace of  $\mathcal{S} \oplus \mathcal{T}$ . We will show that:

**Proposition 4.7.**  $\mathcal{S} \oplus_1 \mathcal{T} = (\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\}$ .

*Proof.* Consider  $i : \mathcal{S} \rightarrow (\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\}$  given by  $s \mapsto (2s, 0) + J$ . We claim that  $i$  is a unital complete order embedding. First note that

$$i(e) = (2e, 0) + J = (2e, 0) + (-e, e) + J = (e, e) + J$$

so  $i$  is unital. It is also easy to see that  $i$  is a cp map as it can be written as the composition of two cp maps:  $\mathcal{S} \rightarrow \mathcal{S} \oplus \mathcal{T}$ ,  $s \mapsto (2s, 0)$  and the quotient map. To see that it is a complete order embedding let  $((2s_{ij}, 0) + J)$  be positive in  $M_k((\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\})$ . We will show that  $(s_{ij})$  is in  $M_k(\mathcal{S})^+$ . Since quotient by null subspaces are completely proximal  $((2s_{ij}, 0) + J)$  has positive representative in  $\mathcal{S} \oplus \mathcal{T}$ , say

$$(2s_{ij}, 0) + (\alpha_{ij}e, -\alpha_{ij}e) = (2s_{ij} + \alpha_{ij}e, -\alpha_{ij}e).$$

This clearly forces  $(-\alpha_{ij}e)$  to be positive in  $M_k(\mathcal{T})$ , equivalently, we have that  $(-\alpha_{ij}) \in M_k^+$ . Finally, Since  $(2s_{ij} + \alpha_{ij}e)$  and  $(-\alpha_{ij}e)$  are two positive elements of  $M_k(\mathcal{S})$  it follows that their addition is also positive. This proves that  $i$  is a unital complete order embedding.

Likewise  $j : \mathcal{T} \rightarrow (\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\}$ ;  $t \mapsto (0, 2t) + J$  is a unital complete order embedding. It is also well establish that

Finally suppose  $\phi : \mathcal{S} \rightarrow \mathcal{R}$  and  $\psi : \mathcal{T} \rightarrow \mathcal{R}$  are two ucp maps, where  $\mathcal{R}$  is an operator system. Define  $\varphi : (\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\} \rightarrow \mathcal{R}$  by  $\varphi((s, t) + J) = \phi(s)/2 + \psi(t)/2$ . Note that  $\varphi$  is well defined as both  $\phi$  and  $\psi$  are unital. Moreover  $\varphi(i(s)) = \varphi((2s, 0) + J) = \phi(s)$ . Likewise  $\varphi(j(t)) = \psi(t)$ . Thus we only need to show that  $\varphi$  is a cp map. Consider  $\gamma : \mathcal{S} \oplus \mathcal{T} \rightarrow \mathcal{R}$  given by  $\gamma((s, t)) = \phi(s)/2 + \psi(t)/2$ . Clearly  $\gamma$  is a ucp map such that  $(e, -e)$  belongs to its kernel. Consequently the induced map  $\bar{\gamma} : (\mathcal{S} \oplus \mathcal{T})/span\{(e, -e)\} \rightarrow \mathcal{R}$  is still a ucp map by the universal property of the operator system quotients. Clearly  $\bar{\gamma}$  coincides with  $\varphi$ . This finishes the proof.  $\square$

Supposing  $G$  and  $H$  discrete groups then we have the  $C^*$ -algebraic isomorphism

$$C^*(G) *_1 C^*(H) \cong C^*(G * H)$$

in a natural way. We refer the reader to [25, pg. 149] on a discussion on this topic. Letting  $\mathbb{Z}_k$  be the cyclic group of order  $k$ , it is well known that the group  $C^*$ -algebra  $C^*(\mathbb{Z}_k)$  can be identified with  $\mathbb{C}^k$  (see [2, pg. 60]). Consequently we have that

$$(4.1) \quad \mathbb{C}^k *_1 \mathbb{C}^m \cong C^*(\mathbb{Z}_k) *_1 C^*(\mathbb{Z}_m) \cong C^*(\mathbb{Z}_k * \mathbb{Z}_m).$$

Note that the group  $\mathbb{Z}_k * \mathbb{Z}_m$  can be given as  $\mathbb{Z}_k * \mathbb{Z}_m = \langle a, b : a^k = b^m = e \rangle$ . By identifying  $\mathbb{Z}_k * \mathbb{Z}_m$  with its canonical image in  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  we have that

$$(4.2) \quad \mathcal{S} = span\{e, a, a^2, \dots, a^{k-1}, b, b^2, \dots, b^{m-1}\} \subset C^*(\mathbb{Z}_k * \mathbb{Z}_m)$$

is closed under the involution and consequently an operator subsystem. Under the natural identification between the  $C^*$ -algebras given in 4.1 the following operator subsystems

$$\{x + y : x \in \mathbb{C}^k, x \in \mathbb{C}^k\} \subset \mathbb{C}^k *_1 \mathbb{C}^m \text{ and } \mathcal{S} \subset C^*(\mathbb{Z}_k * \mathbb{Z}_m)$$

are invariant and consequently they are unital completely order isomorphic. This allows us to draw the following conclusion which is the main purpose of this section.

**Theorem 4.8.** *Let  $k$  and  $m$  be positive integers. The following operator systems are unitally completely order isomorphic:*

- (1)  $\mathbb{C}^{k+m} / \text{span}\{(\underbrace{1, \dots, 1}_{k\text{-terms}}, \underbrace{-1, \dots, -1}_{m\text{-terms}})\}$ .
- (2)  $\mathbb{C}^k \oplus_1 \mathbb{C}^k$ .
- (3)  $\{x + y : x \in \mathbb{C}^k, y \in \mathbb{C}^m\} \subset \mathbb{C}^k *_1 \mathbb{C}^m$ .
- (4) *The operator subsystem  $\mathcal{S} \subset C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  given in 4.2.*

Moreover, each of these operator systems contain enough unitaries in  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$ .

*Proof.* (1)=(2) can be seen by Proposition 4.7 and (2)=(3) follows from Corollary 4.4. By the above discussion we have (3)=(4). Finally, clearly the operator system in (4) contains enough unitaries in  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$ .  $\square$

**Remark 4.9.** In the next section we are particularly interested in the four dimensional operator system  $\mathbb{C}^5 / \text{span}\{(1, 1, -1, -1, -1)\}$  and exhibit several universal properties of this operator system. By the using the above theorem for  $k = 2$ ,  $m = 3$  we see that

- (1)  $\mathbb{C}^5 / \text{span}\{(1, 1, -1, -1, -1)\}$ ,
- (2)  $\mathbb{C}^2 \oplus_1 \mathbb{C}^3$ ,
- (3)  $\{x + y : x \in \mathbb{C}^2, y \in \mathbb{C}^3\} \subset \mathbb{C}^2 *_1 \mathbb{C}^3$ ,
- (4)  $\text{span}\{e, a, b, b^*\} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  where  $\mathbb{Z}_2 * \mathbb{Z}_3$  is given by  $\langle a, b : a^2 = b^3 = e \rangle$

are all unitally and completely order isomorphic. In [13], the operator system in (4) is given as an example of a four dimensional operator system which is not exact. We will recover this in later sections.

## 5. WEAK EXPECTATION PROPERTY (WEP)

In this section we exhibit a new WEP criteria and express the Kirchberg Conjecture in terms of a problem about four dimensional operator system problem. As we pointed out in the introductory section WEP is one of the fundamental nuclearity property ensued by Lance [21]. It characterizes the *semi-nuclear*  $C^*$ -algebras in the sense that the maximal tensor product which is projective behaves injectively for this class of  $C^*$ -algebras. More precisely  $\mathcal{A}$  has WEP if and only if for every  $C^*$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  with  $\mathcal{A} \subset \mathcal{B}$  one has

$$\mathcal{A} \otimes_{\max} \mathcal{C} \subset \mathcal{B} \otimes_{\max} \mathcal{C}.$$

Recall from last section that an operator subsystem  $\mathcal{S}$  of a  $C^*$ -algebra  $\mathcal{A}$  is said to *contain enough unitaries* if there is a collection of unitaries in  $\mathcal{S}$  that generates  $\mathcal{A}$  as a  $C^*$ -algebra. Such an operator system contains great deal of information about the nuclearity properties of  $\mathcal{A}$  which we shall state below. In many instances this allows us to retrieve the properties of a  $C^*$ -algebra by using a very low dimensional operator system. The following proposition is the key point and it was inspired by a work of Pisier [25]. The proof can be found in [15].

**Proposition 5.1.** *Suppose  $\mathcal{S} \subset \mathcal{A}$  and  $\mathcal{T} \subset \mathcal{B}$  contain enough unitaries. Then*

$$\mathcal{S} \otimes_{\min} \mathcal{T} = \mathcal{S} \otimes_{\mathcal{C}} \mathcal{T} \implies \mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}.$$

**Proposition 5.2.** *Let  $\mathcal{S} \subset \mathcal{A}$  contain enough unitaries. Then:*

- (1)  $\mathcal{S}$  is exact  $\implies \mathcal{A}$  is exact.
- (2)  $\mathcal{S}$  has DCEP  $\implies \mathcal{A}$  has WEP.
- (3)  $\mathcal{S}$  is  $C^*$ -nuclear  $\implies \mathcal{A}$  nuclear.
- (4)  $\mathcal{S}$  has osLLP  $\implies \mathcal{A}$  has LLP.



*Proof.* The reader may refer to [15] for the proofs of (1), (2), and (3). We only prove (4). By Theorem 3.2, the condition on  $\mathcal{S}$  requires that  $\mathcal{S} \otimes_{\min} B(H) = \mathcal{S} \otimes_{\max} B(H)$  for every Hilbert space  $H$ . So by the above proposition (with  $\mathcal{T} = B(H) = \mathcal{B}$  and using the fact that  $c$  and  $\max$  coincides when one of the tensorant is a  $C^*$ -algebra) we get  $\mathcal{A} \otimes_{\min} B(H) = \mathcal{A} \otimes_{\max} B(H)$  for every  $H$ , equivalently,  $\mathcal{A}$  has LLP.  $\square$

In [4], Boca proves that LLP is preserved under unital free products. Consequently:

**Corollary 5.3** (Boca). *The group  $C^*$ -algebra  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  has LLP.*

*Proof.* This is a consequence of the identification

$$\mathbb{C}^k *_1 \mathbb{C}^m \cong C^*(\mathbb{Z}_k) *_1 C^*(\mathbb{Z}_m) \cong C^*(\mathbb{Z}_k * \mathbb{Z}_m).$$

Since  $\mathbb{C}^k$  and  $\mathbb{C}^m$  has LLP it follows that  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  has LLP.  $\square$

*Remark.* This can be alternately proved as follows: Since  $\mathbb{C}^{k+m}/J$ , where

$$J = \text{span}\left\{\underbrace{(1, \dots, 1)}_{k\text{-terms}}, \underbrace{(-1, \dots, -1)}_{m\text{-terms}}\right\},$$

can be identified with an operator subsystem of  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  which contain enough unitaries, by the above proposition, it is enough to prove that  $\mathbb{C}^{k+m}/J$  has the lifting property. But this a simple consequence of Theorem 3.15.

Let  $\mathbb{F}_\infty$  be the free group on the countably infinite number of generators and let  $C^*(\mathbb{F}_\infty)$  be the full group  $C^*$ -algebra of  $\mathbb{F}_\infty$ . It is well establish that the free group  $\mathbb{F}_\infty$  embeds in  $\mathbb{Z}_2 * \mathbb{Z}_3$  (see [11, Pg. 24] e.g.). In the following we identify the groups with their canonical images in the their (full) group  $C^*$ -algebras. It is essentially [25, Prop. 8.8].

**Proposition 5.4.** *Let  $H$  be subgroup of  $G$ . Then  $C^*(H)$  embeds in  $C^*(G)$ . More precisely, the unitary representation  $\rho : H \rightarrow C^*(G)$  given by  $h \mapsto h$  extends to bijective unital  $*$ -homomorphism  $\pi$ . Moreover, this embedding has a ucp inverse.*

So roughly speaking if  $H$  is a subgroup of  $G$  then the identity on  $C^*(H)$  decomposes via ucp maps on  $C^*(G)$ . The following is a direct consequence of [14, Lem. 5.2].

**Proposition 5.5.** *Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras such that the identity decomposes via ucp map on  $\mathcal{B}$ , that is, there are ucp maps  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  and  $\psi : \mathcal{B} \rightarrow \mathcal{A}$  such that  $\psi(\phi(a)) = a$  for all  $a$  in  $\mathcal{A}$ . Then any nuclearity property of  $\mathcal{B}$  passes to  $\mathcal{A}$ . More precisely,*

- (1) *if  $\mathcal{B}$  is nuclear then  $\mathcal{A}$  is nuclear;*
- (2) *if  $\mathcal{B}$  is exact then  $\mathcal{A}$  is exact;*
- (3) *if  $\mathcal{B}$  has WEP then  $\mathcal{A}$  has WEP;*
- (4) *if  $\mathcal{B}$  has LLP then  $\mathcal{A}$  has LLP.*

Since  $\mathbb{F}_\infty$  embeds in  $\mathbb{Z}_2 * \mathbb{Z}_3$ , the full  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  embeds in  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  with a ucp inverse. So, by the above proposition, any nuclearity property of  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  passes to  $C^*(\mathbb{F}_\infty)$ . We are now ready to state Kirchberg's WEP characterization [18] and its slight modification which will be more useful for us:

**Theorem 5.6.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra. Then the following are equivalent:*

- (1)  *$\mathcal{A}$  has WEP.*
- (2)  *$\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$ .*
- (3)  *$\mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) = \mathcal{A} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ .*

*Proof.* The equivalence of (1) and (2) is Kirchberg's WEP characterization. (1) implies (3) follows from the fact that  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  has LLP. Since  $\mathcal{A}$  has WEP, by Example 3.12, we obtain (3). To see (3)  $\Rightarrow$  (2), let  $\phi : C^*(\mathbb{F}_\infty) \rightarrow C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  and  $\psi : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \rightarrow C^*(\mathbb{F}_\infty)$  be ucp maps such that their composition is the identity on  $C^*(\mathbb{F}_\infty)$ . In the following we will use the fact that the *min* and the *max* are functorial tensor product. We will also use the fact that if the composition of two ucp maps is a complete order embedding then the first map has the same property. We have

$$\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \xrightarrow{id \otimes \phi} \mathcal{A} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \xrightarrow{id \otimes \psi} \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$$

is a sequence of ucp maps such that the composition is the identity on  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$ . This means that  $id \otimes \phi$  is a complete order embedding. If we consider the same ucp maps with *max* is replaced by *min* we again see that  $id \otimes \phi$  is a complete order embedding. Thus we have the embeddings

$$\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \xrightarrow{id \otimes \phi} \mathcal{A} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \text{ and } \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) \xrightarrow{id \otimes \phi} \mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3).$$

Since the tensor products on the right hand side coincides it follows that (3) implies (2).  $\square$

The following is one of our main results in this section. It is a four dimensional operator system version of the above theorem. In the remaining of this and the next section we will have several application of this theorem. As usual  $J$  stands for the one dimensional null-subspace  $span\{(1, 1, -1, -1, -1)\} \subset \mathbb{C}^5$ .

**Theorem 5.7.** *A unital  $C^*$ -algebra  $\mathcal{A}$  has WEP if and only if*

$$\mathcal{A} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{A} \otimes_{\max} (\mathbb{C}^5/J).$$

*Proof.* If  $\mathcal{A}$  has WEP then it is  $(el, \max)$ -nuclear. So we get  $\mathcal{A} \otimes_{el} \mathbb{C}^5/J = \mathcal{A} \otimes_{\max} \mathbb{C}^5/J$ . Also, by Theorem 3.15,  $\mathbb{C}^5/J$  has the lifting property or  $(\min, er)$ -nuclearity. Since it is written to right hand side, by using the asymmetry of *el* and *er*, we have  $\mathcal{A} \otimes_{\min} \mathbb{C}^5/J = \mathcal{A} \otimes_{el} \mathbb{C}^5/J$ . This proves one direction. Now suppose the converse. From the last section we have that  $\mathbb{C}^5/J$  unittally completely order isomorphic to an operator subsystem of  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  which contains enough unitaries. Now, by using Proposition 5.1,  $\mathcal{A} \otimes_{\min} \mathbb{C}^5/J = \mathcal{A} \otimes_{c=\max} \mathbb{C}^5/J$  implies that  $\mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) = \mathcal{A} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . Thus, by the above theorem,  $\mathcal{A}$  has WEP.  $\square$

The above theorem can also be extended to general operator systems. We need a preliminary lemma. The following is [14, Lem. 5.4].

**Lemma 5.8.** *Let  $\mathcal{T} \subset \mathcal{B}$  contains enough unitaries, say  $\{u_\alpha\}$ , and  $\varphi : \mathcal{B} \rightarrow \mathcal{C}$ , where  $\mathcal{C}$  is a  $C^*$ -algebra, such that  $\varphi(u_\alpha)$  is a unitary in  $\mathcal{C}$  for all  $\alpha$  then  $\varphi$  must be a unital  $*$ -homomorphism.*

**Theorem 5.9.** *An operator system  $\mathcal{S}$  has DCEP if and only if  $\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{S} \otimes_c (\mathbb{C}^5/J)$ .*

*Proof.* First assume that  $\mathcal{S}$  has DCEP (eq.  $(el, c)$ -nuclearity). Since  $(\mathbb{C}^5/J)$  has the lifting property (or  $(\min, er)$ -nuclearity) we obtain

$$\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{S} \otimes_{el} (\mathbb{C}^5/J) = \mathcal{S} \otimes_c (\mathbb{C}^5/J).$$

This proves one direction. Conversely let  $\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{S} \otimes_c (\mathbb{C}^5/J)$ . Recall that if we let  $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b : a^2 = b^3 = e \rangle$  then  $\mathcal{R} = span\{e, a, b, b^*\}$  is an operator subsystem of  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  which contains enough unitaries. Moreover  $\mathbb{C}^5/J$  and  $\mathcal{R}$  are unittally completely order isomorphic. So our assumption is equivalent to  $\mathcal{S} \otimes_{\min} \mathcal{R} = \mathcal{S} \otimes_c \mathcal{R}$ . As a first step we claim that  $\mathcal{S} \otimes_{\min} \mathcal{R} \subset \mathcal{S} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  (and let  $i$  denote this inclusion). In fact letting  $\tau$  be the operator system structure on  $\mathcal{S} \otimes \mathcal{R}$  arising from the inclusion  $\mathcal{S} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$

we see that  $\min \leq \tau \leq c$ . Since  $\min = c$  by our assumption our claim follows. Secondly we wish to show that

$$(5.1) \quad \mathcal{S} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) = \mathcal{S} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3).$$

To see this we first represent  $\mathcal{S} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  into a  $B(H)$  such a way that the portions “ $\mathcal{S}$ ” and “ $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ ” commute and “ $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ ” is a  $C^*$ -subalgebra of  $B(H)$ . Let  $\mathcal{A}$  be any  $C^*$ -algebra containing  $\mathcal{S}$  as an operator subsystem. By the injectivity of  $\min$  we have  $\mathcal{S} \otimes_{\min} \mathcal{R} \subset \mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . By using Arveson’s lifting theorem we obtain a ucp map  $\gamma : \mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \rightarrow B(H)$  extending  $i$ .

$$\begin{array}{ccc} \mathcal{S} \otimes_{\min} \mathcal{R} & \xrightarrow{i} & \mathcal{S} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \subset \dots \rightarrow B(H) \\ \cap & & \nearrow \gamma \\ \mathcal{A} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) & & \end{array}$$

When  $\gamma$  is restricted to “ $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ ” it must be the identity. In fact the above lemma ensures that it has to be a unital  $*$ -homomorphism as it maps  $a, b, b^*$  to  $a, b, b^*$ , respectively. From this it is easy to see that it is the identity on “ $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ ” as  $a, b, b^*$  generates “ $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ ”. This means that  $\gamma$  has to be  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ -module map in the sense that for  $x \in \mathcal{A}$  and  $y \in C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$   $\gamma(x \otimes y) = \gamma(x \otimes e)\gamma(e \otimes y)$ . This follows from the theory of Choi on multiplicative domains [23, Thm. 3.18]. Now if we restrict  $\gamma$  on  $\mathcal{S} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  it is the identity since for  $s \in \mathcal{S}$  and  $y \in C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  we have  $\gamma(s \otimes y) = \gamma(s \otimes e)\gamma(e \otimes y) = (s \otimes e)(e \otimes y) = s \otimes y$ . This proves our claim, that is, the equality in Equation 5.1 is satisfied. As a final step we want to show that

$$\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty).$$

This again follows from the fact that the identity on  $C^*(\mathbb{F}_\infty)$  factors via ucp maps on  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . The proof is same as that of Theorem 5.6, (3) $\Rightarrow$ (2). Now the DCEP criteria given in Subsection 3.3 implies that  $\mathcal{S}$  has DCEP.  $\square$

**Remark 5.10.** Clearly Theorem 5.7 can be viewed as a corollary of Theorem 5.9 as DCEP and WEP coincide on  $C^*$ -algebras and  $c$  and  $\max$  are the same when one of the tensorant is a  $C^*$ -algebra. We proved Theorem 5.7 separately as simpler arguments are used.

**Corollary 5.11.** *Let  $\mathcal{A} \subset B(H)$  be a unital  $C^*$ -subalgebra. Then  $\mathcal{A}$  has WEP if and only if we have the complete order embedding*

$$(5.2) \quad \mathcal{A} \otimes_{\max} \mathbb{C}^5/J \subset B(H) \otimes_{\max} \mathbb{C}^5/J.$$

*Proof.* By the injectivity of the minimal tensor product we readily have the complete order embedding

$$(5.3) \quad \mathcal{A} \otimes_{\min} \mathbb{C}^5/J \subset B(H) \otimes_{\min} \mathbb{C}^5/J.$$

Moreover, since  $\mathbb{C}^5/J$  has the lifting property we also have  $B(H) \otimes_{\min} \mathbb{C}^5/J = B(H) \otimes_{\max} \mathbb{C}^5/J$ . Now supposing  $\mathcal{A}$  has WEP, by the above theorem, we can simply replace  $\min$  by  $\max$  in 5.3 and obtain 5.2. Conversely assuming 5.2 holds, combining with 5.3 we get  $\mathcal{A} \otimes_{\min} \mathbb{C}^5/J = \mathcal{A} \otimes_{\max} \mathbb{C}^5/J$ . Thus, by the above theorem, we conclude that  $\mathcal{A}$  has WEP.  $\square$

**Remark 5.12.** The above corollary can be extended as follows: Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of a  $C^*$ -algebra  $\mathcal{B}$ . Suppose  $\mathcal{B}$  has WEP. Then  $\mathcal{A}$  has WEP if and only if we have the completely order embedding

$$\mathcal{A} \otimes_{\max} \mathbb{C}^5/J \subset \mathcal{B} \otimes_{\max} \mathbb{C}^5/J.$$

In fact we already have that  $\mathcal{A} \otimes_{\min} \mathbb{C}^5/J \subset \mathcal{B} \otimes_{\min} \mathbb{C}^5/J$ . Also, as  $\mathcal{B}$  has WEP, by Theorem 5.7,  $\mathcal{B} \otimes_{\min} \mathbb{C}^5/J = \mathcal{B} \otimes_{\max} \mathbb{C}^5/J$ . Following the same argument in the proof of the above corollary we obtain the desired result.

**Remark 5.13.** Any injective operator system has WEP. (Also recall that every injective operator system has a structure of a C\*-algebra.) Thus, a unital C\*-algebra has WEP if and only if

$$\mathcal{A} \otimes_{\max} \mathbb{C}^5/J \subset I(\mathcal{A}) \otimes_{\max} \mathbb{C}^5/J$$

completely order isomorphically where  $I(\mathcal{A})$  is the injective envelope of  $\mathcal{A}$ .

The following is the four dimensional operator system variant of the Kirchberg Conjecture.

**Theorem 5.14.** *The following are equivalent:*

- (1) *The Kirchberg conjecture has an affirmative answer.*
- (2)  *$\mathbb{C}^5/J$  has DCEP.*
- (3)  *$(\mathbb{C}^5/J) \otimes_{\min} (\mathbb{C}^5/J) = (\mathbb{C}^5/J) \otimes_c (\mathbb{C}^5/J)$ .*

For its proof we will need:

**Proposition 5.15.** *The following are equivalent:*

- (1) *The Kirchberg conjecture has an affirmative answer.*
- (2)  *$C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  has WEP.*
- (3)  *$C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) = C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ .*

*Proof.* (3) $\Rightarrow$ (2): Follows from Theorem 5.6. (1) $\Rightarrow$ (3): So every C\*-algebra that has LLP has WEP. In particular  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  has WEP as it has LLP. Now (3) follows from Example 3.12. Finally we will prove (2) $\Rightarrow$ (1). Since the identity on  $C^*(\mathbb{F}_\infty)$  decomposes via ucp maps through  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  it follows that  $C^*(\mathbb{F}_\infty)$  has WEP, in other words, the Kirchberg conjecture has an affirmative answer.  $\square$

*proof of Theorem 5.14.* Recall from last section that we can identify  $\mathbb{C}^5/J$  with an operator subsystem of  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  that contains enough unitaries.

(3) $\Rightarrow$ (1): This is a result of Proposition 5.1. So we have that

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3) = C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_3),$$

in other words, the Kirchberg conjecture is true by the above proposition.

(1) $\Rightarrow$ (2): Recall from the preliminaries that the Kirchberg conjecture is equivalent to statement that every finite dimensional operator system that has the lifting property has DCEP. By Theorem 3.15,  $\mathbb{C}^5/J$  has the lifting property. So it must have DCEP.

(2) $\Rightarrow$ (3): Since  $\mathbb{C}^5/J$  has the lifting property it is (min,er)-nuclear. This readily shows that  $\mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J = \mathbb{C}^5/J \otimes_{er} \mathbb{C}^5/J$ . Assuming (2) it is also (el,c)-nuclear. (Applied to  $\mathbb{C}^5/J$  on the right hand side) we get  $\mathbb{C}^5/J \otimes_{er} \mathbb{C}^5/J = \mathbb{C}^5/J \otimes_c \mathbb{C}^5/J$ . So (3) follows.  $\square$

Several questions are in the order:

**Question 5.16.** Is  $\mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J = \mathbb{C}^5/J \otimes_{\max} \mathbb{C}^5/J$ ?

**Question 5.17.** Is  $(\mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J)^+ = (\mathbb{C}^5/J \otimes_{\max} \mathbb{C}^5/J)^+$ ?

**Question 5.18.** Is  $(\mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J)^+ = (\mathbb{C}^5/J \otimes_c \mathbb{C}^5/J)^+$ ?

*Remark.* Clearly if the first question is true then we have that the Kirchberg conjecture has an affirmative answer. We put the second and the third questions just to emphasize the difficulty of this problem.

6. FURTHER PROPERTIES OF  $\mathbb{C}^5/J$  AND EXAMPLES

Letting  $J = \text{span}\{(1, 1, -1, -1, -1)\}$ , the quotient operator system  $\mathbb{C}^5/J$  has two important properties: A unital  $C^*$ -algebra  $\mathcal{A}$  has WEP if and only if we have the complete order isomorphism

$$\mathcal{A} \otimes_{\min} (\mathbb{C}^5/J) = \mathcal{A} \otimes_{\max} (\mathbb{C}^5/J).$$

Moreover, the Kirchberg conjecture has an affirmative answer if and only if we have

$$(\mathbb{C}^5/J) \otimes_{\min} (\mathbb{C}^5/J) = (\mathbb{C}^5/J) \otimes_c (\mathbb{C}^5/J)$$

(equivalently  $\mathbb{C}^5/J$  has DCEP). Keeping these observation in mind it is essential to understand further nuclearity properties of the four dimensional operator system  $\mathbb{C}^5/J$ . While Theorem 3.15 ensures that  $\mathbb{C}^5/J$  has the lifting property we have:

**Proposition 6.1.**  $\mathbb{C}^5/J$  is not exact.

*Proof.* By identifying  $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b : a^2 = b^3 = e \rangle$  with its canonical representation in  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  we have that  $\mathcal{S} = \text{span}\{e, a, b, b^*\}$  is an operator subsystem of  $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ . In [14, Rem. 11.6] it was shown that  $\mathcal{S}$  is not exact. Since  $\mathbb{C}^5/J$  and  $\mathcal{S}$  are unitaly and completely order isomorphic (Remark 4.9),  $\mathbb{C}^5/J$  is not exact.  $\square$

We start with the following positivity criteria. An element  $x$  of an operator system  $\mathcal{S}$  will be written  $x > 0$  if  $x \geq \epsilon e$  for some  $\epsilon > 0$ . Also, a positive element  $y$  in an operator system quotient  $\mathcal{S}/J$  may not have a positive representation as it may obtain through the Archimedeanization process. However, if  $y > 0$  in  $\mathcal{S}/J$  then  $y - \epsilon e$  has a positive representation in  $\mathcal{S}$  for some small  $\epsilon > 0$ .

**Proposition 6.2.** Let  $\mathcal{S}$  be an operator system. Take the basis  $\{\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4\}$  for  $\mathbb{C}^5/J$ . Let

$$u = s_1 \otimes \dot{e}_1 + s_2 \otimes \dot{e}_2 + s_3 \otimes \dot{e}_3 + s_4 \otimes \dot{e}_4 \in \mathcal{S} \otimes \mathbb{C}^5/J.$$

Then:

- (1)  $u > 0$  in  $\mathcal{S} \otimes_{\min} \mathbb{C}^5/J \iff \mathcal{S}$  embeds in a larger operator system  $\tilde{\mathcal{S}}$  ( $\mathcal{S} \subset \tilde{\mathcal{S}}$ ) such that there is an  $s \in \tilde{\mathcal{S}}^+$  with  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$ .
- (2)  $u > 0$  in  $\mathcal{S} \otimes_c \mathbb{C}^5/J \iff$  there is an  $s \in C_u^*(\mathcal{S})^+$  with  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$ .
- (3)  $u > 0$  in  $\mathcal{S} \otimes_{\max} \mathbb{C}^5/J \iff$  there is an  $s \in \mathcal{S}^+$  with  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$ .  
Moreover, if  $\mathcal{S}$  is a finite dimensional then the strict inequalities  $>$  can be taken  $\geq$ .

*Proof.* (3): This is really based on the projectivity of the maximal tensor product:

$$(\mathcal{S} \otimes_{\max} \mathbb{C}^5)/(\mathcal{S} \otimes J) = \mathcal{S} \otimes_{\max} \mathbb{C}^5/J.$$

Also, in  $\mathbb{C}^5/J$  we have  $\dot{e}_1 + \dot{e}_2 - \dot{e}_3 - \dot{e}_4 - \dot{e}_5 = 0$ . Thus,  $\dot{e} = \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 + \dot{e}_5 = 2\dot{e}_1 + 2\dot{e}_2$ . Now  $u > 0$  if and only if  $u - \epsilon(e \otimes \dot{e})$  has a positive representative in  $\mathcal{S} \otimes_{\max} \mathbb{C}^5$ , say  $x_1 \otimes e_1 + x_2 \otimes e_2 + x_3 \otimes e_3 + x_4 \otimes e_4 + x_5 \otimes e_5$ , for some small  $\epsilon > 0$ . Note that each of  $x_i$  has to be a positive element in  $\mathcal{S}$ . This means that

$$s_1 \otimes \dot{e}_1 + s_2 \otimes \dot{e}_2 + s_3 \otimes \dot{e}_3 + s_4 \otimes \dot{e}_4 - \epsilon(e \otimes \dot{e}) = x_1 \otimes \dot{e}_1 + x_2 \otimes \dot{e}_2 + x_3 \otimes \dot{e}_3 + x_4 \otimes \dot{e}_4 + x_5 \otimes \dot{e}_5.$$

Now using the fact that  $\dot{e}_5 = \dot{e}_1 + \dot{e}_2 - \dot{e}_3 - \dot{e}_4$  and  $\dot{e} = 2\dot{e}_1 + 2\dot{e}_2$  we obtain following equalities:

$$\begin{aligned} s_1 &= x_1 + x_5 + 2\epsilon e \\ s_2 &= x_2 + x_5 + 2\epsilon e \\ s_3 &= x_3 - x_5 \\ s_4 &= x_4 - x_5. \end{aligned}$$

Finally putting  $s = x_5$  we clearly get  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$  with  $s \in \mathcal{S}^+$ . Note that given such elements we can reconstruct positive elements  $x_1, \dots, x_5$  such that the reverse direction

follows. This proves the first part of (3). The additional part follows from the fact that  $\mathcal{S} \otimes J \subset \mathcal{S} \otimes_{\max} \mathbb{C}^5$  is a completely proximal kernel (Theorem 3.18). So every positive element in the quotient has a positive representative in  $\mathcal{S} \otimes_{\max} \mathbb{C}^5$ .

(1): It is enough to take  $\tilde{\mathcal{S}} = B(H)$ . Recall that  $\mathbb{C}^5/J$  has the lifting property thus we have that

$$B(H) \otimes_{\min} (\mathbb{C}^5/J) = B(H) \otimes_{\max} (\mathbb{C}^5/J).$$

Consequently, by the injectivity of the minimal tensor product, we have the embedding

$$\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J) \subset B(H) \otimes_{\min} (\mathbb{C}^5/J) = B(H) \otimes_{\max} (\mathbb{C}^5/J).$$

So  $u > 0$  in  $\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J)$  if and only if  $u > 0$  in  $B(H) \otimes_{\max} (\mathbb{C}^5/J)$ . Now by part (3), this is equivalent to existence of an element  $s \in B(H)^+$  with  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$ . This proves (1).

(2): From the preliminary section we have that  $\mathcal{S} \otimes_c (\mathbb{C}^5/J) \subset C_u^*(\mathcal{S}) \otimes_{\max} (\mathbb{C}^5/J)$ . Thus,  $u > 0$  in  $\mathcal{S} \otimes_c (\mathbb{C}^5/J)$  if and only if  $u > 0$  in  $C^*(\mathcal{S}) \otimes_{\max} (\mathbb{C}^5/J)$ . Again by using (3), the latter statement is equivalent to existence of a positive element  $s$  being in  $C_u^*(\mathcal{S})$  such that  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$ .  $\square$

The sharp contrast between (1) and (3) allows us to construct low dimensional operator systems such that the minimal and the maximal tensor product with  $\mathbb{C}^5/J$  don't coincide:

*Example 6.3.*  $\mathbb{C}^5$  has a three dimensional operator subsystem, say  $\mathcal{S}$ , such that

$$\mathcal{S} \otimes_{\min} (\mathbb{C}^5/J) \neq \mathcal{S} \otimes_{\max} (\mathbb{C}^5/J).$$

In fact for small  $\epsilon > 0$  consider

$$\mathcal{S} = \text{span}\left\{ \underbrace{(1, 1, 1, 1, 1)}_e, \underbrace{(0, \epsilon, 1/2, 1 - \epsilon, 1)}_a, \underbrace{(1, 0, -1/2, 0, 1)}_b \right\} \subset \mathbb{C}^5.$$

We claim that  $e \otimes \dot{e}_1 + a \otimes \dot{e}_2 + b \otimes \dot{e}_3 + b \otimes \dot{e}_4$  is positive in  $\mathcal{S} \otimes_{\min} \mathbb{C}^5/J$  but not positive in  $\mathcal{S} \otimes_{\max} \mathbb{C}^5/J$ . First note that  $c = (0, \epsilon, 1/2, \epsilon, 0) \in \mathbb{C}^5$  is positive such that

$$e, a \geq c \text{ and } b \geq -c.$$

So by the above positivity criteria ((3), additional part) we get  $e \otimes \dot{e}_1 + a \otimes \dot{e}_2 + b \otimes \dot{e}_3 + b \otimes \dot{e}_4$  is positive in  $\mathbb{C}^5 \otimes_{\max} \mathbb{C}^5/J$ . Since  $\mathcal{S} \otimes_{\min} \mathbb{C}^5/J \subset \mathbb{C}^5 \otimes_{\min} \mathbb{C}^5/J = \mathbb{C}^5 \otimes_{\max} \mathbb{C}^5/J$ , it is a positive element in  $\mathcal{S} \otimes_{\min} \mathbb{C}^5/J$ . On the other hand there is no element  $h \in \mathcal{S}^+$  such that  $e, a \geq h$  and  $b \geq -h$ . In fact we necessarily have that  $h = \alpha e + \beta a + \theta b$ . We leave the verification of this fact to the reader.

*Example 6.4.* Let  $\mathcal{S}$  be the operator system in the above example. Let  $y = (1, 1, -1, -1, -1)$ . Consider the four dimensional operator subsystem  $\mathcal{T} = \text{span}\{e, a, b, y\} \subset \mathbb{C}^5$ . We still have

$$\mathcal{T} \otimes_{\min} \mathbb{C}^5/J \neq \mathcal{T} \otimes_{\max} \mathbb{C}^5/J.$$

In a similar fashion we can show that  $e \otimes \dot{e}_1 + a \otimes \dot{e}_2 + b \otimes \dot{e}_3 + b \otimes \dot{e}_4$  is positive in  $\mathcal{T} \otimes_{\min} \mathbb{C}^5/J$  but not positive in  $\mathcal{T} \otimes_{\max} \mathbb{C}^5/J$ . It is elementary to see that  $\mathcal{T}/J \subset \mathbb{C}^5/J$ . (In fact, in general, whenever  $J \subset \mathcal{S} \subset \tilde{\mathcal{S}}$ , where  $J$  is a kernel in  $\tilde{\mathcal{S}}$ , then the induced map  $\mathcal{S}/J \rightarrow \tilde{\mathcal{S}}/J$  is a complete order embedding.) Thus we get

$$\mathcal{T}/J \otimes_{\min} \mathbb{C}^5/J \subset \mathbb{C}^5/J \otimes_{\min} \mathbb{C}^5/J.$$

So we direct the following question:

$$\text{Is } \mathcal{T}/J \otimes_{\min} \mathbb{C}^5/J = \mathcal{T}/J \otimes_{\max} \mathbb{C}^5/J?$$

*Example 6.5.* There are self-adjoint elements  $s_1, s_2, s_3, s_4$  in the Calkin algebra  $\mathbb{B}/\mathbb{K}$  such that for every representation  $\mathbb{B}/\mathbb{K} \subset B(K)$ , where  $K$  is a Hilbert space, there is a positive element  $s \in B(K)$  with  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$  but there is no positive element in  $\mathbb{B}/\mathbb{K}$  with these properties. This is based on the fact that  $\mathbb{B}/\mathbb{K}$  does not have WEP. (The reader may refer to [14] for a proof of this well-known fact.) Thus we have

$$\mathbb{B}/\mathbb{K} \otimes_{\min} (\mathbb{C}^5/J) \neq \mathbb{B}/\mathbb{K} \otimes_{\max} (\mathbb{C}^5/J).$$

By using the  $C^*$ -algebraic identification  $M_n(\mathbb{B}/\mathbb{K}) \cong \mathbb{B}/\mathbb{K}$ , this inequality fails through an element at the ground level. Now the positivity criteria (1) and (3) imply that such elements exists in  $\mathbb{B}/\mathbb{K}$ . (Also note that if  $s_1, s_2 > s$  and  $s_3, s_4 \geq -s$  for some element  $s \in B(K)^+$  then for every representation  $\mathbb{B}/\mathbb{K} \subset B(\tilde{K})$  there is a positive element in  $\tilde{s}$  in  $B(\tilde{K})$  with these properties. This follows from Arveson's extension theorem.)

We turn back to the positivity characterization given in Proposition 6.2. Another variant can be given as follows which will have a prominent role when we study separation properties in the next section:

**Proposition 6.6.** *Let  $\mathcal{S}$  be an operator system and  $s_1, \dots, s_5$  be self-adjoint elements of  $\mathcal{S}$ . Then there is a self-adjoint element  $s \in \mathcal{S}$  such that  $s_3, s_4, s_5 < s < s_1, s_2$  if and only if*

$$u = s_1 \otimes \dot{e}_1 + s_2 \otimes \dot{e}_2 - s_3 \otimes \dot{e}_3 - s_4 \otimes \dot{e}_4 - s_5 \otimes \dot{e}_5 > 0$$

in  $\mathcal{S} \otimes_{\max} (\mathbb{C}^5/J)$ .

*Proof.* In  $\mathbb{C}^5$  we have  $\dot{e}_5 = \dot{e}_1 + \dot{e}_2 - \dot{e}_3 - \dot{e}_4$ . Therefore we can rewrite  $u$  as

$$u = (s_1 - s_5) \otimes \dot{e}_1 + (s_2 - s_5) \otimes \dot{e}_2 + (-s_3 + s_5) \otimes \dot{e}_3 + (-s_4 + s_5) \otimes \dot{e}_4.$$

Recall from Proposition 6.2 that  $u > 0$  if and only if there is an element  $t \in \mathcal{S}^+$  such that

$$s_1 - s_5 > t, \quad s_2 - s_5 > t, \quad -s_3 + s_5 \geq -t, \quad -s_4 + s_5 \geq -t.$$

Setting  $s = s_5 + t$ , we obtain a self-adjoint element such that

$$s_1 > s, \quad s_2 > s, \quad -s_3 \geq -s, \quad -s_4 \geq -s, \quad -s_5 \geq -s,$$

equivalently  $s_3, s_4, s_5 \leq s < s_1, s_2$ . Consequently we obtain that  $u > 0$  if and only if the latter condition holds. It is easy to see that the latter condition is equivalent to existence of a self-adjoint where we can take the inequalities strict (by a small  $\epsilon$ -perturbation of  $s$  if necessary). This finishes the proof.  $\square$

So far we have worked with the four dimensional operator system  $\mathbb{C}^5/J$ . Similar results and positivity criteria can be extended to a more general setting. For positive integers  $k$  and  $m$  we define

$$J_{k,m} = \text{span}\left\{ \underbrace{(1, \dots, 1)}_{k\text{-terms}}, \underbrace{(-1, \dots, -1)}_{m\text{-terms}} \right\} \subset \mathbb{C}^{k+m}.$$

Before going into details we simply recall a couple of properties.  $J_{k,m}$  is a one-dimensional null-subspace of  $\mathbb{C}^{k+m}$  so it is a completely proximal kernel. Moreover the quotient  $\mathbb{C}^{k+m}/J_{k,m}$  has the lifting property as the lifting property is stable under quotients by null-subspaces. Also by Theorem 4.8 the following operator systems

- (1)  $\mathbb{C}^{k+m}/J_{k,m}$ ;
- (2)  $\mathbb{C}^k \oplus_1 \mathbb{C}^m$ ;
- (3)  $\text{span}\{e, a, a^2, \dots, a^{k-1}, b, b^2, \dots, b^{m-1}\} \subset C^*(\mathbb{Z}_k * \mathbb{Z}_m)$

are unittally and completely order isomorphic.

**Lemma 6.7.** *Suppose  $k \leq k_1$  and  $m \leq m_1$ . Then  $\mathbb{C}^{k+m}/J_{k,m}$  can be identified with an operator subsystem of  $\mathbb{C}^{k_1+m_1}/J_{k_1,m_1}$ , moreover, this inclusion has a ucp inverse.*

*Proof.* Let  $i$  is the embedding of  $\mathbb{C}^{k+m}$  into  $\mathbb{C}^{k_1+m_1}$  given by

$$(a_1, \dots, a_k, b_1, \dots, b_m) \mapsto (\underbrace{a_1, \dots, a_1}_{k_1-k \text{ terms}}, a_1, a_2, \dots, a_k, b_1, \dots, b_m, \underbrace{b_m, \dots, b_m}_{m_1-m \text{ terms}})$$

then the composition  $\mathbb{C}^{k+m} \rightarrow \mathbb{C}^{k_1+m_1} \rightarrow \mathbb{C}^{k_1+m_1}/J_{k_1,m_1}$  has the kernel  $J_{m,k}$ . So the induced map  $\bar{i} : \mathbb{C}^{k+m}/J_{k,m} \rightarrow \mathbb{C}^{k_1+m_1}/J_{k_1,m_1}$  is ucp. Likewise consider the projection  $q$  from  $\mathbb{C}^{k_1+m_1}$  onto  $\mathbb{C}^{k+m}$  given by

$$(a_1, \dots, a_{k_1}, b_1, \dots, b_{m_1}) \mapsto (a_{k_1-k+1}, \dots, a_{k_1}, b_1, \dots, b_m).$$

Since the composition of the ucp maps  $\mathbb{C}^{k_1+m_1} \rightarrow \mathbb{C}^{k+m} \rightarrow \mathbb{C}^{k+m}/J_{k,m}$  contains  $J_{k_1,m_1}$  in its kernel it follows that the induced map  $\bar{q} : \mathbb{C}^{k_1+m_1}/J_{k_1,m_1} \rightarrow \mathbb{C}^{k+m}/J_{k,m}$  is ucp. It is elementary to verify that the composition

$$\mathbb{C}^{k+m}/J_{k,m} \xrightarrow{\bar{i}} \mathbb{C}^{k_1+m_1}/J_{k_1,m_1} \xrightarrow{\bar{q}} \mathbb{C}^{k+m}/J_{k,m}$$

is the identity on  $\mathbb{C}^{k+m}/J_{k,m}$ . This shows that  $\bar{i}$  is an embedding with ucp inverse  $\bar{q}$ .  $\square$

**Corollary 6.8.** *If  $2 \leq k$  and  $3 \leq m$ ,  $\mathbb{C}^{k+m}/J_{k,m}$  is not exact. So  $C^*(\mathbb{Z}_k * \mathbb{Z}_m)$  is not exact.*

*Proof.* Otherwise  $\mathbb{C}^5/J_{2,3}$  is exact as it embeds in  $\mathbb{C}^{k+m}/J_{k,m}$  and exactness is stable when passing to operator subsystems. The second part follows from Proposition 5.1.  $\square$

Despite this  $\mathbb{C}^4/J_{2,2}$  is  $C^*$ -nuclear which we will prove soon. In the following we give a positivity criteria in  $-\otimes_{\max} \mathbb{C}^{k+m}/J_{k,m}$ .

**Proposition 6.9.** *Let  $\mathcal{S}$  be an operator system. We take the basis  $\dot{e}_1, \dots, \dot{e}_{k+m-1}$  for  $\mathbb{C}^{k+m}/J_{k,m}$ . Then an element*

$$u = s_1 \otimes \dot{e}_1 + \dots + s_{k+m-1} \otimes \dot{e}_{k+m-1} > 0$$

*in  $\mathcal{S} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$  if and only if there is an element  $s \in \mathcal{S}^+$  such that*

$$s < s_1, \dots, s_k \text{ and } -s \leq s_{k+1}, \dots, s_{k+m-1}.$$

*Proof.* The proof is similar to that of Proposition 6.2. We again use the projectivity of the maximal tensor product:

$$(\mathcal{S} \otimes_{\max} \mathbb{C}^{k+m})/(\mathcal{S} \otimes J_{k,m}) = \mathcal{S} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}).$$

In  $\mathbb{C}^{k+m}/J_{k,m}$  we have  $\dot{e}_{k+m} = \dot{e}_1 + \dots + \dot{e}_k - \dot{e}_{k+1} - \dots - \dot{e}_{k+m-1}$ . Consequently we obtain

$$\dot{e} = \dot{e}_1 + \dots + \dot{e}_{k+m} = 2\dot{e}_1 + \dots + 2\dot{e}_k.$$

Now  $u > 0$  if and only if  $u - \epsilon(e \otimes \dot{e})$  has a positive representative in  $\mathcal{S} \otimes_{\max} \mathbb{C}^{k+m}$ , say  $x_1 \otimes e_1 + \dots + x_{k+m} \otimes e_{k+m}$ , for some small  $\epsilon > 0$ . Clearly each of  $x_i$  belongs to  $\mathcal{S}^+$ . This means that

$$s_1 \otimes \dot{e}_1 + \dots + s_{k+m-1} \otimes \dot{e}_{k+m-1} - \epsilon(e \otimes \dot{e}) = x_1 \otimes \dot{e}_1 + \dots + x_{k+m} \otimes \dot{e}_{k+m}.$$

Now using the fact that  $\dot{e}_{k+m} = \dot{e}_1 + \dots + \dot{e}_k - \dot{e}_{k+1} - \dots - \dot{e}_{k+m-1}$  and  $\dot{e} = \dot{e}_1 + \dots + \dot{e}_{k+m} = 2\dot{e}_1 + \dots + 2\dot{e}_k$  we obtain following equalities:

$$\begin{array}{rcl} s_1 & = & x_1 + x_{k+m} + 2\epsilon e & s_{k+1} & = & x_{k+1} - x_{k+m} \\ \vdots & \vdots & \vdots & \text{and} & \vdots & \vdots \\ s_k & = & x_k + x_{k+m} + 2\epsilon e & s_{k+m-1} & = & x_{k+m-1} - x_{k+m} \end{array}$$



Finally putting  $s = x_{k+m}$  we clearly get

$$s < s_1, \dots, s_k \text{ and } -s \leq s_{k+1}, \dots, s_{k+m-1}.$$

Note that given such elements we can reconstruct positive elements  $x_1, \dots, x_{k+m}$  such that the reverse direction follows.  $\square$

**Proposition 6.10.** *Let  $\mathcal{S}$  be an operator system and  $s_1, \dots, s_k, t_1, \dots, t_m$  be self-adjoint elements of  $\mathcal{S}$ . Then there is a self-adjoint element  $s$  in  $\mathcal{S}$  such that*

$$t_1, \dots, t_m < s < s_1, \dots, s_k$$

if and only if the following element

$$u = s_1 \otimes \dot{e}_1 + \dots + s_k \otimes \dot{e}_k - t_1 \otimes \dot{e}_{k+1} - \dots - t_m \otimes \dot{e}_{k+m}$$

is strictly positive in  $\mathcal{S} \otimes_{\max} \mathbb{C}^{k+m} / J_{k,m}$ .

*Proof.* By using the fact that  $\dot{e}_{k+m} = \dot{e}_1 + \dots + \dot{e}_k - \dot{e}_{k+1} - \dots - \dot{e}_{k+m-1}$  we can re-write  $u$  as

$$\begin{aligned} u &= (s_1 - t_m) \otimes \dot{e}_1 + \dots + (s_k - t_m) \otimes \dot{e}_k \\ &\quad + (-t_1 + t_m) \otimes \dot{e}_{k+1} + \dots + (-t_{m-1} + t_m) \otimes \dot{e}_{k+m-1}. \end{aligned}$$

By using the above proposition we have that  $u > 0$  if and only if there is an element  $t \in \mathcal{S}^+$  such that

$$(6.1) \quad -t \leq -t_1 + t_m, \dots, -t_{m-1} + t_m \text{ and } t < s_1 - t_m, \dots, s_k - t_m.$$

By setting  $s = t + t_m$  we obtain a self-adjoint element and last condition becomes

$$(6.2) \quad t_1, \dots, t_m \leq s < s_1, \dots, s_k.$$

Note that the conditions in Equation 6.1 and 6.2 are equivalent (one can simply reconstruct  $t$  by setting  $t = s - t_m$ ). Now a simple perturbation argument shows that we can take the inequality in Equation 6.2 to be strict. This finishes the proof.  $\square$

We close this section by showing that  $\mathbb{C}^4 / J_{2,2}$  is  $C^*$ -nuclear. By identifying  $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b : a^2 = b^2 = e \rangle$  by its canonical image in  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  we have that both  $a$  and  $b$  are self-adjoint unitaries. We set

$$\mathcal{S} = \text{span}\{e, a, b\} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_2).$$

The following lemma can easily be verified by the reader.

**Lemma 6.11.** *Let  $a$  be a self-adjoint element of a  $C^*$ -algebra  $\mathcal{A}$  such that  $-e \leq a \leq e$ . Then*

$$U_a = \begin{pmatrix} a & \sqrt{e+a}\sqrt{e-a} \\ \sqrt{e+a}\sqrt{e-a} & -a \end{pmatrix}$$

is a self-adjoint unitary in  $M_2(\mathcal{A})$ .

**Lemma 6.12.** *Let  $\mathcal{S} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  be the above operator subsystem and let  $\mathcal{A}$  be a unital  $C^*$ -algebra. Then every ucp map  $\varphi : \mathcal{S} \rightarrow \mathcal{A}$  extends to a ucp map on  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ .*

*Proof.* First notice that  $a$  and  $b$  are self-adjoint unitaries in  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ . It is not hard to see that  $-e \leq a, b \leq e$ . Let  $\varphi(a) = A$  and  $\varphi(b) = B$ . Clearly  $-e \leq A, B \leq e$  in  $\mathcal{A}$ . Let  $U_A$  and  $U_B$  be the self-adjoint unitaries in  $M_2(\mathcal{A})$  as above. Consider the unitary representation

$$\rho : \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow M_2(\mathcal{A}) \text{ given by } a \mapsto U_A \text{ and } b \mapsto U_B.$$

Let  $\pi : C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \rightarrow M_2(\mathcal{A})$  be the unital  $*$ -homomorphism extending  $\rho$ . It is easy to see that  $\psi : M_2(\mathcal{A}) \rightarrow \mathcal{A}$ ,  $(A_{ij}) \mapsto A_{11}$  is a ucp map. Thus  $\psi \circ \pi$  is a ucp map from  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  into  $\mathcal{A}$ . Note that  $(\psi \circ \pi)(a) = A$  and  $(\psi \circ \pi)(b) = B$ . So it extends  $\varphi$ .  $\square$

By using the fact that  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  is nuclear we can deduce the following:

**Corollary 6.13.** *The above operator subsystem  $\mathcal{S} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  is  $C^*$ -nuclear.*

*Proof.* Let  $\mathcal{A}$  be a  $C^*$ -algebra. We first claim that every ucp map  $\varphi : \mathcal{S} \otimes_{max} \mathcal{A} \rightarrow B(H)$  extends to a ucp map on  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{max} \mathcal{A}$ . This is enough to conclude that we have the complete order embedding  $\mathcal{S} \otimes_{max} \mathcal{A} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{max} \mathcal{A}$ . Since  $c$  and  $max$  coincides when one of the tensorant is a  $C^*$ -algebra we can think of  $max$  as  $c$ . This means that there is a Hilbert space  $K \supset H$ , ucp maps  $\phi : \mathcal{S} \rightarrow B(K)$  and  $\psi : \mathcal{A} \rightarrow B(K)$  with commuting ranges such that  $\varphi = V^*(\phi \cdot \psi)V$ , where  $V$  is the inclusion of  $H$  in  $K$  and so  $V^*$  is the projection onto  $H$ . By using the above lemma, let  $\tilde{\phi} : C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \rightarrow B(K)$  be the ucp extension of  $\phi$  such that the image of  $\tilde{\phi}$  stays in the  $C^*$ -algebra generated by  $\phi(\mathcal{S})$ . Clearly  $\tilde{\phi}$  and  $\psi$  still have the commuting ranges. This means that  $\tilde{\phi} \cdot \psi$  is a ucp map from  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{max} \mathcal{A}$  into  $B(K)$ . The compression of this map, i.e.,  $V^*(\tilde{\phi} \cdot \psi)V$  is ucp and it extends  $\varphi$ . This proves our claim. Finally by the injectivity of the minimal tensor product we have that  $\mathcal{S} \otimes_{min} \mathcal{A} \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{min} \mathcal{A}$ . Now since  $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  is nuclear we obtain  $\mathcal{S} \otimes_{min} \mathcal{A} = \mathcal{S} \otimes_{max} \mathcal{A}$ . Since  $\mathcal{A}$  was arbitrary,  $\mathcal{S}$  is  $C^*$ -nuclear.  $\square$

**Corollary 6.14.**  $\mathbb{C}^4/J_{2,2}$ , i.e.,  $\mathbb{C}^4/\text{span}\{(1, 1, -1, -1)\}$ , is  $C^*$ -nuclear.

*Proof.* This is a consequence of the identification in Theorem 4.8.  $\mathbb{C}^4/J_{2,2}$  and the operator system  $\mathcal{S}$  in the above corollary are unittally completely order isomorphic.  $\square$

*Example 6.15.* Let  $\mathcal{A} \subset B(H)$  be a unital  $C^*$ -subalgebra. Let  $a, b, c \in \mathcal{A}$  such that there is an element  $x \in B(H)^+$  with  $a, b > x$  and  $c \geq -x$ . Then there is an element in  $\mathcal{A}^+$  with these properties, that is, there is  $y \in \mathcal{A}^+$  with  $a, b > y$  and  $c \geq -y$ . To see this first note that

$$\mathcal{A} \otimes_{max} (\mathbb{C}^4/J_{2,2}) \subset B(H) \otimes_{max} (\mathbb{C}^4/J_{2,2})$$

which follows from the injectivity of the minimal tensor product and  $C^*$ -nuclearity of  $(\mathbb{C}^4/J_{2,2})$ . Now if take the basis  $\{\dot{e}_1, \dot{e}_2, \dot{e}_3\}$  for  $(\mathbb{C}^4/J_{2,2})$  then the element  $u = a \otimes \dot{e}_1 + b \otimes \dot{e}_2 + c \otimes \dot{e}_3$  is strictly positive in  $B(H) \otimes_{max} (\mathbb{C}^4/J_{2,2})$  if and only if there is an element  $x \in B(H)^+$  with  $a, b > x$  and  $c \geq -x$  (Proposition 6.2). So the condition we assumed is equivalent to  $u > 0$ . The above unital complete order embedding ensures that  $u > 0$  in  $\mathcal{A} \otimes_{max} (\mathbb{C}^4/J_{2,2})$  too. So such an element exists in  $\mathcal{A}$ .

## 7. "RELATIVE" TIGHT RIESZ INTERPOLATION PROPERTY

In this section we present an equivalent formulation of Lance's weak expectation property in terms of a separation property in matricial order structure of a  $C^*$ -algebra  $\mathcal{A}$  which corresponds to a non-commutative Riesz interpolation property. Though our construction makes use of quotient and tensor theory of operator systems we restrict our application on  $C^*$ -algebras as the weak expectation property is well known characteristic in this context.

**Definition 7.1.** Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of a  $C^*$ -algebra  $\mathcal{B}$ . We say that  $\mathcal{A}$  has the  $(k, m)$  tight Riesz interpolation property in  $\mathcal{B}$ ,  $\text{TR}(k, m)$ -property in short, if for any  $x_1, \dots, x_k$  and  $y_1, \dots, y_m$  in  $\mathcal{A}_{sa}$  if there exists an element  $b \in \mathcal{B}_{sa}$  with

$$x_1, \dots, x_k < b < y_1, \dots, y_m$$

then there is an element  $a \in \mathcal{A}_{sa}$  such that

$$x_1, \dots, x_k < a < y_1, \dots, y_m.$$

We say that  $\mathcal{A}$  has the complete  $(k, m)$  tight Riesz interpolation property in  $\mathcal{B}$  if  $M_n(\mathcal{A})$  has  $\text{TR}(k, m)$ -property in  $M_n(\mathcal{B})$  for every  $n$  and we abbreviate the latter condition as complete  $\text{TR}(k, m)$ -property.

*Remark.* In lattice group theory a group  $G$  is said to have  $\text{TR}(k, m)$ -property if for any  $x_1, \dots, x_k$  and  $y_1, \dots, y_m$  in  $G$  with  $x_i < y_j$  for all  $i$  and  $j$  there is an element  $g \in G$  such that  $x_i < g < y_j$  for all  $i$  and  $j$ . We remark that in this sense even the additive abelian group of selfadjoints in  $M_2$  fails to have  $\text{TR}(2, 2)$ -property. The following example is pointed out by Vern Paulsen. Consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, c = \begin{pmatrix} 1.1 & 0.5 \\ 0.5 & 3.6 \end{pmatrix}, \text{ and } d = \begin{pmatrix} 3.6 & 0.5 \\ 0.5 & 1.1 \end{pmatrix}.$$

It follows that  $a, b < c, d$  but there is no element  $x$  in  $M_2$  with  $a, b < x < c, d$ . We leave the verification of this to the reader. Consequently, our understanding in the present paper is *relative*  $\text{TR}(k, m)$ -property, that is, if such pairs have a separation in a larger object then whether it exists in the smaller object.

We mostly interested in this phenomena when  $\mathcal{B} = B(H)$  for the following reason: Let  $\mathcal{A}$  be a  $C^*$ -subalgebra of  $B(H)$  and let  $x_1, \dots, x_k$  and  $y_1, \dots, y_m$  be self-adjoint elements of  $\mathcal{A}$ . Suppose that  $\mathcal{A}$  embeds into an operator system  $\mathcal{S}$  unitaly and completely order isomorphically (via  $i$ ) such that there exists an element  $s \in \mathcal{S}_{sa}$  with

$$i(x_1), \dots, i(x_k) < s < i(y_1), \dots, i(y_m).$$

Then there is an element  $b \in B(H)$  such that

$$x_1, \dots, x_k < b < y_1, \dots, y_m.$$

In other words, whenever interpolation exists in a representation of  $\mathcal{A}$  then it exists in  $B(H)$ . In fact, by Arveson's extension theorem [1], the inclusion  $\mathcal{A} \subset B(H)$  extends to a ucp map from  $\mathcal{S}$  into  $B(H)$ , say  $\varphi$ . It is elementary to see that  $b = \varphi(s)$  has the desired property.

**Theorem 7.2.** *Let  $\mathcal{A}$  be a unital  $C^*$ -subalgebra of  $\mathcal{B}$ . Then the following are equivalent:*

- (1)  $\mathcal{A}$  has the complete  $\text{TR}(k, m)$ -property in  $\mathcal{B}$ .
- (2) We have a unital and complete order embedding

$$\mathcal{A} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}) \subset \mathcal{B} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}).$$

*Proof.* Before starting the proof note that if we remove the word ‘‘complete’’ in (1) and ‘‘completely’’ in (2) and prove the result this way then the original statement automatically satisfied. This simply follows from the associativity of the maximal tensor product which yields:  $M_n(\mathcal{S} \otimes \mathcal{T}) = M_n(\mathcal{S}) \otimes_{\max} \mathcal{T}$  for every operator systems  $\mathcal{S}, \mathcal{T}$  and  $n$ . Secondly, for the compatibility with our previous results, we remark that  $\mathcal{A}$  has  $\text{TR}(k, m)$ -property in  $\mathcal{B}$  if and only if it has  $\text{TR}(m, k)$ -property in  $\mathcal{B}$ .

Turning back to proof first suppose (2). Let  $x_1, \dots, x_m$  and  $y_1, \dots, y_k$  be elements in  $\mathcal{A}_{sa}$  such that there is an element  $b$  in  $\mathcal{B}_{sa}$  with  $x_i < b < y_j$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, k$ . Py Proposition 6.10 we get

$$u = y_1 \otimes \dot{e}_1 + \dots + y_k \otimes \dot{e}_k - x_1 \otimes \dot{e}_{k+1} - \dots - x_m \otimes \dot{e}_{k+m} > 0$$

in  $\mathcal{B} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$ . Since we assumed (2),  $u > 0$  in  $\mathcal{A} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$  too. Again by using Proposition 6.10, there is an element  $a \in \mathcal{A}_{sa}$  with  $x_i < a < y_j$  for all  $i$  and  $j$ . Conversely suppose that  $\mathcal{A}$  has  $\text{TR}(m, k)$ -property in  $\mathcal{B}$ . We take the basis  $\{\dot{e}_1, \dots, \dot{e}_{k+m-1}\}$  for  $\mathbb{C}^{k+m}/J_{k,m}$ . To establish the order embedding between the tensor products we need to show that if

$$v = a_1 \otimes \dot{e}_1 + \dots + a_{k+m-1} \otimes \dot{e}_{k+m-1} \geq 0$$

in  $\mathcal{B} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$ , where  $a_1, \dots, a_{k+m-1} \in \mathcal{A}$ , then  $v \geq 0$  in  $\mathcal{A} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$ . A moment of thought shows that we can replace  $\geq$  by  $>$ . So suppose  $v > 0$ . Note that  $a_1, \dots, a_{k+m-1}$  must be self-adjoint elements of  $\mathcal{A}$ . By simply adding the term  $0 \otimes$

$\dot{e}_{k+m}$  to  $v$  and using Proposition 6.10 we see that there is an element  $b$  in  $\mathcal{B}_{sa}$  such that  $0, -a_{k+m-1}, \dots, -a_{k+1} < b < a_1, \dots, a_k$ . Since we assumed  $TR(m, k)$ -property there is an element  $a \in \mathcal{A}_{sa}$  such that  $0, -a_{k+m-1}, \dots, -a_{k+1} < a < a_1, \dots, a_k$ . Again by using Proposition 6.10  $v > 0$  in  $\mathcal{A} \otimes_{max} (\mathbb{C}^{k+m}/J_{k,m})$ .  $\square$

We are ready to state our main results.

**Theorem 7.3.** *Let  $\mathcal{A} \subset B(H)$  be a unital  $C^*$ -subalgebra. Then  $\mathcal{A}$  has the complete  $TR(2,2)$ -property in  $B(H)$ .*

*Proof.* By Theorem 7.2,  $\mathcal{A}$  has the complete  $TR(2,2)$ -property in  $B(H)$  if and only if we have

$$\mathcal{A} \otimes_{max} (\mathbb{C}^4/J_{2,2}) \subset B(H) \otimes_{max} (\mathbb{C}^4/J_{2,2}).$$

Since  $\mathbb{C}^4/J_{2,2}$  is  $C^*$ -nuclear we can replace  $max$  by  $min$  so the latter condition holds.  $\square$

Here is the extension of this result.

**Theorem 7.4.** *Let  $\mathcal{A} \subset B(H)$  be a unital  $C^*$ -algebra. Then the following are equivalent:*

- (1)  $\mathcal{A}$  has WEP.
- (2)  $\mathcal{A}$  has the complete  $TR(2,3)$ -property in  $B(H)$ .
- (3)  $\mathcal{A}$  has the complete  $TR(k,m)$ -property in  $B(H)$  for some  $k \geq 2$  and  $m \geq 3$ .
- (4)  $\mathcal{A}$  has the complete  $TR(k,m)$ -property in  $B(H)$  for all integers  $k, m \geq 1$ .

*Proof.* (4) $\Rightarrow$ (3) is clear. To prove (3) $\Rightarrow$ (2) one can simply use the definition of the tight Riesz interpolation. We will show (2) $\Rightarrow$ (1). By Theorem 7.2, (2) is equivalent to

$$\mathcal{A} \otimes_{max} (\mathbb{C}^5/J_{2,3}) \subset B(H) \otimes_{max} (\mathbb{C}^5/J_{2,3}).$$

Recall from Corollary 5.11 that this complete order embedding is equivalent to  $\mathcal{A}$  having WEP. Finally suppose (1). To obtain (4), by considering Theorem 7.2, we need to show that

$$(7.1) \quad \mathcal{A} \otimes_{max} (\mathbb{C}^{k+m}/J_{k,m}) \subset B(H) \otimes_{max} (\mathbb{C}^{k+m}/J_{k,m})$$

for every  $k, m \geq 1$ . Note that since  $\mathbb{C}^{k+m}/J_{k,m}$  has the lifting property we have that

$$B(H) \otimes_{min} (\mathbb{C}^{k+m}/J_{k,m}) = B(H) \otimes_{max} (\mathbb{C}^{k+m}/J_{k,m}).$$

Also using the fact that WEP is equivalent to (el,max)-nuclearity and lifting property is equivalent to (min,er)-nuclearity (and considering  $\mathbb{C}^{k+m}/J_{k,m}$  is on the right-hand side) we have

$$\mathcal{A} \otimes_{min} (\mathbb{C}^{k+m}/J_{k,m}) = \mathcal{A} \otimes_{el} (\mathbb{C}^{k+m}/J_{k,m}) = \mathcal{A} \otimes_{max} (\mathbb{C}^{k+m}/J_{k,m}).$$

So both of the maximal tensor products in Equation 7.1 can be replaced by  $min$ . Since the minimal tensor product is injective the result follows.  $\square$

Starting with a unital  $C^*$ -algebra with WEP we can characterize its  $C^*$ -subalgebras that have WEP.

**Corollary 7.5.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra of  $B$ . Suppose that  $\mathcal{B}$  has WEP. Then  $\mathcal{A}$  has WEP if and only if  $\mathcal{A}$  has the complete  $TR(2,3)$ -property in  $\mathcal{B}$ .*

*Proof.* By Theorem 5.7,  $\mathcal{A}$  has WEP if and only if

$$\mathcal{A} \otimes_{min} (\mathbb{C}^5/J) = \mathcal{A} \otimes_{max} (\mathbb{C}^5/J).$$

We readily have that

$$\mathcal{B} \otimes_{min} (\mathbb{C}^5/J) = \mathcal{B} \otimes_{max} (\mathbb{C}^5/J).$$

So  $\mathcal{A}$  has WEP if and only if we have the complete order embedding

$$\mathcal{A} \otimes_{max} (\mathbb{C}^5/J) \subset \mathcal{B} \otimes_{max} (\mathbb{C}^5/J).$$

By Theorem 7.2, this is equivalent to  $\mathcal{A}$  having complete TR(2,3)-property in  $\mathcal{B}$ .  $\square$

**Corollary 7.6.** *A unital  $C^*$ -algebra has WEP if and only if it has the complete TR(2,3)-property in its injective envelope  $I(\mathcal{A})$ .*

*Proof.* Follows from the fact that every injective operator system has WEP. (This is elementary to see by using the definition of WEP.)  $\square$

**Corollary 7.7.** *Let  $k, m$  be positive integers. Then every unital  $C^*$ -algebra  $\mathcal{A}$  has the complete TR( $k, m$ )-property in  $\mathcal{A}^{**}$ .*

*Proof.* By [15, Lem. 6.5.] we have the unital complete order embedding

$$\mathcal{A} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}) \subset \mathcal{A}^{**} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}).$$

So Theorem 7.2 yields the desired conclusion.  $\square$

**Corollary 7.8.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra with WEP. Then for every unital  $C^*$ -algebraic inclusion  $\mathcal{A} \subset \mathcal{B}$ ,  $\mathcal{A}$  has the complete TR( $k, m$ )-property in  $\mathcal{B}$ , for any  $k, m \geq 1$ .*

*Proof.* Given positive integers  $k$  and  $m$ , we need to show that

$$\mathcal{A} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}) \subset \mathcal{B} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m}).$$

Let  $\tau$  be the operator system structure on the algebraic tensor product  $\mathcal{A} \otimes (\mathbb{C}^{k+m}/J_{k,m})$  induced by  $\mathcal{B} \otimes_{\max} (\mathbb{C}^{k+m}/J_{k,m})$ . Clearly  $\min \leq \tau \leq \max$ . Since the minimal and the maximal tensor product of  $\mathcal{A}$  and  $\mathbb{C}^{k+m}/J_{k,m}$  coincide (Theorem 5.7), we get  $\min = \tau = \max$ . Simply replacing  $\tau$  with  $\max$  we obtain that the above embedding holds.  $\square$

**7.1. Ordered spaces.** In ordered function space theory (let us restrict ourself with Kadison's function spaces in real case [12] or AOU spaces in complex case -in the sense of Paulsen and Tomforde [24]) a space  $V$  is said to have TR( $k, m$ )-property if for every  $v_1, \dots, v_k$  and  $w_1, \dots, w_m$  with  $v_i < w_j$  for all  $i = 1, \dots, k$  and  $j = 1, \dots, m$  there is an element  $v$  such that

$$v_1, \dots, v_k < v < w_1, \dots, w_m.$$

We remark that a function space can be thought concretely as a unital real subspace of  $C_{\mathbb{R}}(X)$ . Likewise, an AOU space can be considered concretely as a unital subspaces of  $C(X)$  which is closed under the involution. Note that the conditions on  $v_1, \dots, v_k$  and  $w_1, \dots, w_m$ , i.e.,  $v_i < w_j$  for all  $i = 1, \dots, k$  and  $j = 1, \dots, m$ , is equivalent to a separation in a larger object, namely  $C_{\mathbb{R}}(X)$  or  $C(X)$ . Since the least upper bound of  $v_1, \dots, v_k$ , say  $v$ , trivially exists in  $C_{\mathbb{R}}(X)$  (or  $C(X)$ ) (which, indeed, makes  $C_{\mathbb{R}}(X)$  and  $C(X)$  a Riesz space or a vector lattice), a small perturbation  $\tilde{v}$  of  $v$  satisfies  $v_1, \dots, v_k < \tilde{v} < w_1, \dots, w_m$ . Therefore our definition of "relative" TR( $k, m$ )-property carries out the same character to  $C^*$ -algebra theory. We refer the reader to [22] for the tensorial aspects of Riesz interpolation properties in ordered function space theory.

## REFERENCES

- [1] W. B. Arveson, *Subalgebras of  $C^*$ -algebras I*, Acta Math. **123** (1969) 141-224.
- [2] B.V.R. Bhat, G.A. Elliott, P.A. Fillmore, Editors, *Lectures on Operator Theory*, Fields Institute Monographs **13**, American Mathematical Society, Providence (1999)
- [3] D. P. Blecher and B. L. Duncan, *Nuclearity-related properties for nonselfadjoint algebras*, J. Operator Theory **65** (2011) 47-70.
- [4] F. P. Boca, *A note on full free product  $C^*$ -algebras, lifting and quasidiagonality*, Operator theory, operator algebras and related topics (Timisoara, 1996), Theta Found., Bucharest, 1997, pp. 51-63.
- [5] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977) 156-209.
- [6] M. D. Choi and E. G. Effros, *Nuclear  $C^*$ -algebras and the approximation property*, Amer. J. Math. **100** (1978) 61-79.

- [7] D. Farenick and V. I. Paulsen, *Operator system quotients of matrix algebras and their tensor products*, arXiv:1101.0790.
- [8] T. Fritz, *Operator system structures on the unital direct sum of  $C^*$ -algebras*, 2010, arXiv:1011.1247.
- [9] K. H. Han, *On maximal tensor products and quotient maps of operator systems*, arXiv:1010.0380v2
- [10] K. H. Han and V. Paulsen, *An approximation theorem for nuclear operator systems*, J. Funct. Anal. **261** (2011), 999-1009.
- [11] P. de La Harpe, *Topics in geometric group theory*, The University of Chicago Press, (2000)
- [12] R.V. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., 1951, (1951). no. 7.
- [13] A.S. Kavruk, *Tensor products of operator systems and applications*, Ph.D. thesis, Univ. of Houston, 2011.
- [14] A.S. Kavruk, *Nuclearity Related Properties in Operator Systems*, arXiv:1107.2133
- [15] A. S. Kavruk, V. I. Paulsen, I. G. Todorov and M. Tomforde, *Quotients, exactness and nuclearity in the operator system category*, preprint.
- [16] A. S. Kavruk, V. I. Paulsen, I. G. Todorov and M. Tomforde, *Tensor products of operator systems*, J. Funct. Anal., **261** (2011) 267-299.
- [17] D. Kerr and H. Li *On Gromov-Hausdorff convergence for operator metric spaces*, J. Operator Theory, **62** (2009) 83-109.
- [18] E. Kirchberg, *On non-semisplit extensions, tensor products and exactness of group  $C^*$ -algebras*, Invent. Math. **112** (1993) 449 - 489.
- [19] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Functional Anal. **129** (1995), 35-63.
- [20] C. Lance, *On nuclear  $C^*$ -algebras*, J. Funct. Anal. **12** (1973) 157-176.
- [21] C. Lance, *Tensor products and nuclear  $C^*$ -algebras*, Proceedings of Symposia in Pure Mathematics, Vol. **38** (1982) 379-399.
- [22] I. Namioka and R.R. Phelps, *Tensor products of compact convex sets*, Pacific J. of Math. Vol. **31**, No. 2, (1969)
- [23] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, 2002.
- [24] V. I. Paulsen and M. Tomforde, *Vector spaces with an order unit*, Indiana Univ. Math. J., to appear.
- [25] G. Pisier, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series **294**, Cambridge University Press, 2003.
- [26] S. Wassermann, *The slice map problem for  $C^*$ -algebras*, Proc. London Math. Soc. **32** (1976) 537-559.