COMMUTATORS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS ON NON-HOMOGENEOUS METRIC MEASURE SPACES

RULONG XIE, HUAJUN GONG* AND XIAOYAO ZHOU

Abstract. Let (X,d,μ) be a metric measure space satisfying both the geometrically doubling and the upper doubling measure conditions, which is called non-homogeneous metric measure space. In this paper, via a sharp maximal operator, the boundedness of commutators generated by multilinear singular integral with $RBMO(\mu)$ function on non-homogeneous metric measure spaces in m-multiple Lebesgue spaces is obtained.

1. Introduction

It is well known that the standard singular integral theory is constructed with the assumption of spaces satisfying the doubling measures condition. We recall that μ is said to satisfy the doubling condition if there exists a constant $C>0$ such that $\mu(B(x,2r))\leq C\mu(B(x,r))$ for all $x\in \mathrm{supp}\mu$ and $r>0$. A metric measure space (X,d,μ) equipped with a non-negative doubling measure μ is called a space of homogeneous type. In case of non-doubling measures, a non-negative measure μ only need to satisfy the polynomial growth condition, i.e., for all $x\in\mathbb{R}^n$ and $r>0$, there exist a constant $C_0>0$ and $k\in(0,n]$ such that,

$$\mu(B(x,r))\leq C_0r^k, \tag{1.1}$$

where $B(x,r)=\{y\in\mathbb{R}^n : |y-x|<r\}$. This breakthrough brings rapid development in harmonic analysis (see [2,5,8-9,17-22]). And the analysis on non-doubling measures has important applications in solving the long-standing open Painlevé's problem (see [18]).

However, as stated by Hytönen in [11], the measure satisfying (1) does not include the doubling measure as special cases. To solve this problem, a kind of metric measure space (X,d,μ), which is called non-homogeneous metric measures space, satisfying geometrically doubling and the upper doubling measure condition (see Definition 1.1 and 1.2) is introduced by Hytönen in [11]. The highlight of this kind of spaces is that it includes both the homogeneous spaces and metric spaces with polynomial growth measures as special cases. From then on, some results paralled to homogeneous spaces and non-doubling measures space are obtained (see [1,5,11-16] and the references therein). Hytönen et al. in [14] and Bui and Duong in [1] independently introduced the atomic Hardy space $H^1(\mu)$ and proved that the dual space

^{0}* Corresponding author

1991 *Mathematics Subject Classification.* 42B20; 42B25.

Key words and phrases. multilinear singular integral; commutators; non-homogeneous metric measure spaces; $RBMO(\mu)$.

of $H^1(\mu)$ is $RBMO(\mu)$. In [1], the authors also proved that Calderón-Zygmund operator and commutators of Calderón-Zygmund operators and RBMO functions are bounded in $L^p(\mu)$ for $1 < p < \infty$. Recently, some equivalent characterizations are established by Liu et al. in [16] for the boundedness of Carderon-Zygmund operators on $L^p(\mu)$ for $1 < p < \infty$. In [4], Fu et al. established boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces on non-homogeneous spaces.

On the other hand, the theory on multilinear singular integral operators has been considered by some researchers. In [3], Coifman and Meyers firstly established the theory of bilinear Calderón-Zygmund operators. Later, Gorafakos and Torres [6-7] established the boundedness of multilinear singular integral on the product Lebesgue spaces and Hardy spaces. The properties of multilinear singular integrals and commutators on non-doubling measures spaces $(\mathbb{R}^n, \mu)$ were established by Xu in [21-22]. Weighted norm inequalities for multilinear Calderón-Zygmund operators on non-homogeneous metric measure spaces were also constructed in [10].

In the setting of non-homogeneous metric measure spaces, it is natural to ask whether commutators of multilinear singular integral operators is also bounded in m-multiple Lebesgue spaces. This paper will give an affirmative answer to this question. In this paper, commutators generated by multilinear singular integrals with $RBMO(\mu)$ function on non-homogeneous metric spaces is introduced firstly. And we will prove that it is bounded in m-multiple Lebesgue spaces on non-homogeneous metric spaces, provided that multilinear singular integrals is bounded from m-multiple $L^1(\mu)\times\ldots\times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, where $L^p(\mu)$ and $L^{p,\infty}(\mu)$ denote the Lebesgue space and weak Lebesgue space respectively. This result in this paper includes the corresponding results on both the homogeneous spaces and $(\mathbb{R}^n, \mu)$ with non-doubling measures space. A variant of sharp maximal operator $M^\sharp$, Kolmogorov's theorem and some good properties of the dominating function λ (see Definition 1.2) are the main tools for proving the results of this paper.

Before stating the main results of this paper, we firstly recall some notations and definitions.

Definition 1.1. [11]A metric space (X, d) is called geometrically doubling if there exists some $N_0 \in \mathbf{N}$ such that, for any ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Definition 1.2. [11]A metric measure space (X, d, μ) is said to be upper doubling if μ is a Borel measure on X and there exists a dominating function $\lambda : X\times(0, +\infty) \to (0, +\infty)$ and a constant $C_\lambda > 0$ such that for each $x \in X, r \longmapsto (x, r)$ is non-decreasing, and for all $x \in X, r > 0$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2) \tag{1.2}$$

Remark 1.3. (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x, r) = \mu(B(x, r))$. On the other hand, a metric space (X, d, μ) satisfying the polynomial growth condition (1)(in particular, $(X, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ with μ satisfying (1) for some $k \in (0, n]$)) is also an upper doubling measure space if we take $\lambda(x, r) = Cr^k$.

(ii) Let(X, d, μ) be an upper doubling space and λ be a dominating function on $X \times (0, +\infty)$ as in Definition 1.2. In [14], it was showed that there exists another dominating function $\tilde{\lambda}$ such that for all $x, y \in X$ with $d(x,y) \le r$,

$$\tilde{\lambda}(x,r) \le \tilde{C}\tilde{\lambda}(y,r). \tag{1.3}$$

Thus, we suppose that λ always satisfies (1.3) in this paper.

Definition 1.4. Let $\alpha, \beta \in (1, +\infty)$. A ball $B \subset X$ is called (α,β)-doubling if $\mu(\alpha B) \le \beta\mu(B)$.

As pointed in Lemma 2.3 of [1], there exist plenties of doubling balls with small radii and with large radii. In the rest of this paper, unless α and β are specified otherwise, by an (α,β) doubling ball we mean a $(6,\beta_0)$-doubling with a fixed number $\beta_0 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n = \log_2 N_0$ is viewed as a geometric dimension of the space.

Definition 1.5. [1]For any two balls $B \subset Q$, define

$$K_{B,Q} = 1 + \int_{r_B \le d(x,x_B) \le r_Q} \frac{d\mu(x)}{\lambda(x_B, d(x,x_B))}. \tag{1.4}$$

And, for two balls $B \subset Q$, one define the coefficient $K'_{B,Q}$ as follows. Let $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}} r_B \ge r_Q$, then we set

$$K'_{B,Q} = 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)}. \tag{1.5}$$

Remark 1.6. In the case that $\lambda(x, ar) = a^t\lambda(x,r)$ for $0 < t < \infty$, $x \in X$, and $a, r > 0$, one know that $K_{B,Q} \approx K'_{B,Q}$. However, in general, we only have $K_{B,Q} \le CK'_{B,Q}$. In this paper, we always suppose that $\lambda(x, ar) = a^t\lambda(x,r)$ for $0 < t < \infty$, $x \in X$, and $a, r > 0$. So we don't differentiate $K_{B,Q}$ with $K'_{B,Q}$ and always write $K_{B,Q}$ for simplicity in this paper.

Definition 1.7. A kernel $K(\cdot, \cdots, \cdot) \in L^1_{loc}((X)^{m+1} \backslash \{(x, y_1 \cdots, y_j, \cdots, y_m) : x = y_1 = \cdots = y_j = \cdots = y_m\})$ is called an m-linear Calderón-Zygmund kernel if it satisfies:

(i)

$$|K(x, y_1, \cdots, y_j, \cdots, y_m)| \le C\left[\sum_{j=1}^m \lambda(x, d(x,y_j))\right]^{-m} \tag{1.6}$$

for all $(x, y_1 \cdots, y_j, \cdots, y_m) \in (X)^{m+1}$ with $x \ne y_j$ for some j.

(ii) There exists $0 < \delta \le 1$ such that

$$\begin{aligned} &|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x', y_1, \cdots, y_j, \cdots, y_m)| \\ \le& \frac{Cd(x,x')^\delta}{\left[\sum_{j=1}^m d(x,y_j)\right]^\delta \left[\sum_{j=1}^m \lambda(x, d(x,y_j))\right]^m}, \end{aligned} \tag{1.7}$$

provided that $Cd(x,x') \le \max_{1\le j\le m} d(x,y_j)$ and for each j,

$$\begin{aligned} &|K(x,y_1,\cdots,y_j,\cdots,y_m) - K(x,y_1,\cdots,y_j',\cdots,y_m)| \\ &\le \frac{Cd(y_j,y_j')^\delta}{\left[\sum_{j=1}^m d(x,y_j)\right]^\delta \left[\sum_{j=1}^m \lambda(x,d(x,y_j))\right]^m}, \end{aligned} \tag{1.8}$$

provided that $Cd(y_j,y_j') \le \max_{1\le j\le m} d(x,y_j)$.

A multilinear operator T is called a multilinear Calderón-Zygmund singular integral operator with the above kernel K satisfying (1.6), (1.7) and (1.8) if, for $f_1,\cdots f_m$ are L^∞ functions with compact support and $x \notin \bigcap_{j=1}^m \mathrm{supp} f_j$,

$$T(f_1,\cdots f_m)(x) = \int_{X^m} K(x,y_1,\cdots y_m) f_1(y_1)\cdots f_m(y_m) d\mu(y_1)\cdots d\mu(y_m). \tag{1.9}$$

Remark 1.8. Because $\max_{1\le j\le m} d(x,y_j) \le \sum_{j=1}^m d(x,y_j) \le m \max_{1\le j\le m} d(x,y_j)$, (ii) in Definition 1.7 is equivalent to (ii') in the following statement.

(ii') There exists $0<\delta\le 1$ such that

$$\begin{aligned} &|K(x,y_1,\cdots,y_j,\cdots,y_m) - K(x',y_1,\cdots,y_j,\cdots,y_m)| \\ \le &\frac{Cd(x,x')^\delta}{\left[\max_{1\le j\le m} d(x,y_j)\right]^\delta \left[\sum_{j=1}^m \lambda(x,d(x,y_j))\right]^m}, \end{aligned} \tag{1.10}$$

provided that $Cd(x,x') \le \max_{1\le j\le m} d(x,y_j)$ and for each j,

$$\begin{aligned} &|K(x,y_1,\cdots,y_j,\cdots,y_m) - K(x,y_1,\cdots,y_j',\cdots,y_m)| \\ \le &\frac{Cd(y_j,y_j')^\delta}{\left[\max_{1\le j\le m} d(x,y_j)\right]^\delta \left[\sum_{j=1}^m \lambda(x,d(x,y_j))\right]^m}, \end{aligned} \tag{1.11}$$

provided that $Cd(y_j,y_j') \le \max_{1\le j\le m} d(x,y_j)$.

Definition 1.9. [1] Let $\rho>1$ be some fixed constant. A function $b \in L^1_{loc}(\mu)$ is said to belong to $RBMO(\mu)$ if there exists a constant $C>0$ such that for any ball B

$$\frac{1}{\mu(\rho B)} \int_B |b(x) - m_{\widetilde{B}} b| d\mu(x) \le C, \tag{1.12}$$

and for any two doubling balls $B \subset Q$,

$$|m_B(b) - m_Q(b)| \le CK_{B,Q}, \tag{1.13}$$

where $\widetilde{B}$ is the smallest (α,β)-doubling ball of the form $6^k B$ with $k \in \mathbf{N}\bigcup\{0\}$, and $m_{\widetilde{B}}(b)$ is the mean value of b on $\widetilde{B}$, namely,

$$m_{\widetilde{B}}(b) = \frac{1}{\mu(\widetilde{B})} \int_{\widetilde{B}} b(x) d\mu(x).$$

The minimal constant C appearing in (1.12) and (1.13) is defined to be the $RBMO(\mu)$ norm of b and denoted by $||b||_*$.

For $1 \leq i \leq k$, we denote by C_i^k the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(i)\}$ of $\{1, 2, \cdots, k\}$ with i different elements. For any $\sigma \in C_i^k$, the complementary sequence σ' is given by $\sigma' = \{1, 2, \cdots, k\} \backslash \sigma$. Moreover, for $b_i \in RBMO(\mu), i = 1, \cdots, k$, let $\vec{b} = (b_1, b_2, \cdots, b_k)$ be a finite family of locally integrable function. For all $1 \leq i \leq k$ and $\sigma = \{\sigma(1), \cdots, \sigma(i)\} \in C_i^k$, we set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(i)})$ and the product $b_\sigma(x) = b_{\sigma(1)}(x) \cdots b_{\sigma(i)}(x)$. Also, we denote $\vec{f} = (f_1, \cdots, f_k)$, $\vec{f}_\sigma = (f_{\sigma(1)}, \cdots, f_{\sigma(i)})$ and $\vec{b}_{\sigma'}\vec{f}_{\sigma'} = (b_{\sigma'(i+1)}f_{\sigma'(i+1)}, \cdots, b_{\sigma'(k)}f_{\sigma'(k)})$.

Definition 1.10. A kind of commutators generated by multilinear singular integral operator T with $b_i \in RBMO(\mu), i = 1, \cdots, k$ is defined as follows:

$$[\vec{b}, T](\vec{f})(x) = \sum_{i=0}^{k} \sum_{\sigma \in C_i^k} (-1)^{k-i} b_\sigma(x) T(\vec{f}_\sigma, \vec{b}_{\sigma'}\vec{f}_{\sigma'})(x). \tag{1.14}$$

In particular, when $k = 2$, we can obtain

$$\begin{aligned} [b_1, b_2, T](f_1, f_2)(x) =& b_1(x)b_2(x)T(f_1, f_2)(x) - b_1(x)T(f_1, b_2f_2)(x) \\ &- b_2(x)T(b_1f_1, f_2)(x) + T(b_1f_1, b_2f_2)(x). \end{aligned} \tag{1.15}$$

Also, we define $[b_1, T]$ and $[b_2, T]$ as follows respectively.

$$[b_1, T](f_1, f_2)(x) = b_1(x)T(f_1, f_2)(x) - T(b_1f_1, f_2)(x), \tag{1.16}$$

$$[b_2, T](f_1, f_2)(x) = b_2(x)T(f_1, f_2)(x) - T(f_1, b_2f_2)(x). \tag{1.17}$$

For the sake of simplicity and without loss of generality, we only consider the case of $k = 2$ in this paper. Let us state the main result as follows.

Theorem 1.11. *Suppose that μ is a Radon measure with $||\mu|| = \infty$. Let $[b_1, b_2, T]$ defined by (1.15). Let $1 < p_1, p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1 \in RBMO(\mu)$ and $b_2 \in RBMO(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$||[b_1, b_2, T](f_1, f_2)||_{L^q(\mu)} \leq C||f_1||_{L^{p_1}(\mu)}||f_2||_{L^{p_2}(\mu)}, \tag{1.18}$$

where $\dfrac{1}{q} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$.

Throughout this paper, C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. And p' is the conjugate index of p, namely, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

2. Proof of Main Result

To prove the main theorem, we firstly give some notations and lemmas.

Let $f \in L^1_{loc}(\mu)$, the sharp maximal operator is defined by

$$M^\sharp f(x) = \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_{\widetilde{B}}(f)| d\mu(y) + \sup_{(B,Q) \in \Delta_x} \frac{|m_B(f) - m_Q(f)|}{K_{B,Q}}, \tag{2.1}$$

where $\Delta_x := \{(B,Q) : x \in B \subset Q$ and $B,\ Q$ are doubling balls$\}$ and the non centered doubling maximal operator is denoted by

$$Nf(x) = \sup_{\substack{B \ni x, \\ B \text{ doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y).$$

For any $0 < \delta < 1$, we also define that

$$M^\sharp_\delta f(x) = \{M^\sharp(|f|^\delta)(x)\}^{1/\delta} \tag{2.2}$$

and

$$N_\delta f(x) = \{N(|f|^\delta)(x)\}^{1/\delta}. \tag{2.3}$$

We can obtain that for any $f \in L^1_{loc}(\mu)$,

$$|f(x)| \le N_\delta f(x) \tag{2.4}$$

for $\mu - a.e.\ x \in X$. Let us give an explanation for inequality (2.4). By the Lebesgue differential theorem, we obtain that $|f(x)| \le Nf(x)$. Hence

$$|f(x)| = [|f(x)|^\delta]^{1/\delta} \le \{N(|f|^\delta)(x)\}^{1/\delta} = N_\delta f(x).$$

Let $\rho > 1$, $p \in (1,\infty)$ and $r \in (1,p)$, the non-centered maximal operator $M_{r,(\rho)}f$ is defined by

$$M_{r,(\rho)}f(x) = \sup_{B \ni x} \left\{ \frac{1}{\mu(\rho B)} \int_B |f(y)|^r d\mu(y) \right\}^{1/r}. \tag{2.5}$$

When $r = 1$, we simply write $M_{1,(\rho)}f(x)$ as $M_{(\rho)}f$. If $\rho \ge 5$, then the operator $M_{(\rho)}f$ is bounded on $L^p(\mu)$ for $p > 1$ and $M_{r,(\rho)}$ is bounded on $L^p(\mu)$ for $p > r$ (see [1]).

From Theorem 4.2 in [1], it is easy to obtain that

Lemma 2.1. *Let $f \in L^1_{loc}(\mu)$ with $\int_X f(x)d\mu(x) = 0$ if $||\mu|| < \infty$. For $1 < p < \infty$ and $0 < \delta < 1$, if $\inf(1, N_\delta f) \in L^p(\mu)$, then there exists a constant $C > 0$ such that*

$$||N_\delta(f)||_{L^p(\mu)} \le C||M^\sharp_\delta(f)||_{L^p(\mu)}. \tag{2.6}$$

Lemma 2.2. [4,19] *Let $1 \le p < \infty$ and $1 < \rho < \infty$. Then $b \in RBMO(\mu)$ if and only if for any ball $B \subset X$,*

$$\left\{ \frac{1}{\mu(\rho B)} \int_B |b_B - m_{\tilde{B}}(b)|^p d\mu(x) \right\}^{1/p} \le C||b||_*, \tag{2.7}$$

and for any two doubling balls $B \subset Q$,

$$|m_B(b) - m_Q(b)| \le CK_{B,Q}||b||_*. \tag{2.8}$$

Lemma 2.3. [4]

$$|m_{\widetilde{6^j \frac{6}{5} B}}(b) - m_{\tilde{B}}(b)| \le Cj||b||_*. \tag{2.9}$$

Lemma 2.4. [10] *Suppose that μ is a Radon measure with $||\mu|| = \infty$. Let T be defined by (1.9) with $m = 2$. Let $1 < p_1, p_2 < +\infty$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$||T(f_1, f_2)||_{L^q(\mu)} \le C||f_1||_{L^{p_1}(\mu)}||f_2||_{L^{p_2}(\mu)}, \tag{2.10}$$

where $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.

Lemma 2.5. *Suppose that $[b_1, b_2, T]$ is defined by (1.15), $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$, $1 < r < q$ and $b_1, b_2 \in RBMO(\mu)$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that for any $x \in X$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$,*

$$\begin{aligned} M^{\sharp}_{\delta}[b_1, b_2, T](f_1, f_2)(x) &\leq C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) \\ &+ C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) + C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) \\ &+ C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \end{aligned} \tag{2.11}$$

$$M^{\sharp}_{\delta}[b_1, T](f_1, f_2)(x) \leq C||b_1||_* M_{r,(6)}(T(f_1, f_2))(x) + C||b_1||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \tag{2.12}$$

and

$$M^{\sharp}_{\delta}[b_2, T](f_1, f_2)(x) \leq C||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + C||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \tag{2.13}$$

Proof. Because $L^\infty(\mu)$ with compact support is dense in $L^p(\mu)$ for $1 < p < \infty$, we only consider the situation of $f_1, f_2 \in L^\infty(\mu)$ with compact support. Also, by Corollary 3.11 in [4], without loss of generality, we can assume that $b_1, b_2 \in L^\infty(\mu)$.

As in the proof of Theorem 9.1 in [19], to obtain (2.11), it suffices to show that

$$\begin{aligned} &\left(\frac{1}{\mu(6B)} \int_B \big||[b_1, b_2, T](f_1, f_2)(z)|^\delta - |h_B|^\delta\big| d\mu(z)\right)^{1/\delta} \\ \leq &C||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + C||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\ &+ C||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) + C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x), \end{aligned} \tag{2.14}$$

holds for any x and ball B with $x \in B$, and

$$\begin{aligned} |h_B - h_Q| \leq CK^2_{B,Q}\Big[&||b_1||_*||b_2||_* M_{r,(6)}(T(f_1, f_2))(x) + ||b_1||_* M_{r,(6)}([b_2, T](f_1, f_2))(x) \\ &+ ||b_2||_* M_{r,(6)}([b_1, T](f_1, f_2))(x) + ||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x)\Big]. \end{aligned} \tag{2.15}$$

for all balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball, Q is a doubling ball. For any ball B, we denote

$$h_B := m_B(T((b_1 - m_{\tilde{B}}(b_1)) f_1 \chi_{X \setminus \frac{6}{5}B}, (b_2 - m_{\tilde{B}}(b_2)) f_2 \chi_{X \setminus \frac{6}{5}B})),$$

and

$$h_Q := m_Q(T((b_1 - m_Q(b_1)) f_1 \chi_{X \setminus \frac{6}{5}Q}, (b_2 - m_Q(b_2)) f_2 \chi_{X \setminus \frac{6}{5}Q})).$$

Write

$$[b_1, b_2, T] = T((b_1 - b_1(z)) f_1, (b_2 - b_2(z)) f_2),$$

and

(2.16)
$$\begin{aligned}
&T((b_1-m_{\tilde{B}}(b_1))f_1,(b_2-m_{\tilde{B}}(b_2))f_2)\\
=&T((b_1-b_1(z)+b_1(z)-m_{\tilde{B}}(b_1))f_1,(b_2-b_2(z)+b_2(z)-m_{\tilde{B}}(b_2))f_2)\\
=&(b_1(z)-m_{\tilde{B}}(b_1))(b_2(z)-m_{\tilde{B}}(b_2))T(f_1,f_2)\\
&-(b_1(z)-m_{\tilde{B}}(b_1))T(f_1,(b_2-b_2(z))f_2)\\
&-(b_2(z)-m_{\tilde{B}}(b_2))T((b_1-b_1(z))f_1,f_2)+T((b_1-b_1(z))f_1,(b_2-b_2(z))f_2).
\end{aligned}$$

Then

(2.17)
$$\begin{aligned}
&\left(\frac{1}{\mu(6B)}\int_B ||[b_1,b_2,T](f_1,f_2)(z)|^\delta-|h_B|^\delta d\mu(z)\right)^{1/\delta}\\
\leq &C\left(\frac{1}{\mu(6B)}\int_B |[b_1,b_2,T](f_1,f_2)(z)-h_B|^\delta d\mu(z)\right)^{1/\delta}\\
\leq &C\left(\frac{1}{\mu(6B)}\int_B |(b_1(z)-m_{\tilde{B}}(b_1))(b_2(z)-m_{\tilde{B}}(b_2))T(f_1,f_2)(z)|^\delta d\mu(z)\right)^{1/\delta}\\
&+C\left(\frac{1}{\mu(6B)}\int_B |(b_1(z)-m_{\tilde{B}}(b_1))T(f_1,(b_2-b_2(z))f_2)(z)|^\delta d\mu(z)\right)^{1/\delta}\\
&+C\left(\frac{1}{\mu(6B)}\int_B |(b_2(z)-m_{\tilde{B}}(b_2))T((b_1-b_1(z))f_1,f_2)(z)|^\delta d\mu(z)\right)^{1/\delta}\\
&+C\left(\frac{1}{\mu(6B)}\int_B |T((b_1-m_{\tilde{B}}(b_1))f_1,(b_2-m_{\tilde{B}}(b_2))f_2)(z)-h_B|^\delta d\mu(z)\right)^{1/\delta}\\
=:&E_1+E_2+E_3+E_4.
\end{aligned}$$

We firstly estimate E_1. Let $r_1,r_2>1$ such that $\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{\delta}$. By Hölder's inequality and Lemma 2.2, we yields

$$\begin{aligned}
E_1\leq &C\left(\frac{1}{\mu(6B)}\int_B |b_1(z)-m_{\tilde{B}}b_1|^{r_1}d\mu(z)\right)^{1/r_1}\\
&\times\left(\frac{1}{\mu(6B)}\int_B |b_2(z)-m_{\tilde{B}}b_2|^{r_2}d\mu(z)\right)^{1/r_2}\\
&\times\left(\frac{1}{\mu(6B)}\int_B |T(f_1,f_2)|^{r}d\mu(z)\right)^{1/r}\\
\leq &C||b_1||_*||b_2||_*M_{r,(6)}(T(f_1,f_2))(x).
\end{aligned}\tag{2.18}$$

For E_2, let $s>1$ such that $\frac{1}{s}+\frac{1}{r}=\frac{1}{\delta}$, by Hölder's inequality and Lemma 2.2, we deduce

(2.19)
$$\begin{aligned}E_2 &\leq C\biggl(\frac{1}{\mu(6B)}\int_B |b_1(z)-m_{\tilde{B}}b_1|^s d\mu(z)\biggr)^{1/s}\\ &\quad\times\biggl(\frac{1}{\mu(6B)}\int_B |[b_2,T](f_1,f_2)|^r d\mu(z)\biggr)^{1/r}\\ &\leq C||b_1||_* M_{r,(6)}([b_2,T](f_1,f_2))(x).\end{aligned}$$

Similar to estimate E_2, we immediately get

$$E_3 \leq C||b_2||_* M_{r,(6)}([b_1,T](f_1,f_2))(x). \tag{2.20}$$

Let us turn to estimate E_4. Denote $f_j^1 = f_j\chi_{\frac{6}{5}B}$ and $f_j^2 = f_j - f_j^1$ for $j = 1, 2$. Then

(2.21)
$$\begin{aligned}E_4 &\leq C\biggl(\frac{1}{\mu(6B)}\int_B |T((b_1-m_{\tilde{B}}b_1)f_1^1(z),(b_2-m_{\tilde{B}}b_2)f_2^1)(z)|^\delta d\mu(z)\biggr)^{1/\delta}\\ &\quad+C\biggl(\frac{1}{\mu(6B)}\int_B |T((b_1-m_{\tilde{B}}b_1)f_1^1(z),(b_2-m_{\tilde{B}}b_2)f_2^2)(z)|^\delta d\mu(z)\biggr)^{1/\delta}\\ &\quad+C\biggl(\frac{1}{\mu(6B)}\int_B |T((b_1-m_{\tilde{B}}b_1)f_1^2(z),(b_2-m_{\tilde{B}}b_2)f_2^1)(z)|^\delta d\mu(z)\biggr)^{1/\delta}\\ &\quad+C\biggl(\frac{1}{\mu(6B)}\int_B |T((b_1-m_{\tilde{B}}b_1)f_1^2(z),(b_2-m_{\tilde{B}}b_2)f_2^2)(z)-h_B|^\delta d\mu(z)\biggr)^{1/\delta}\\ &=: E_{41}+E_{42}+E_{43}+E_{44}.\end{aligned}$$

To estimate E_{41}, we need the classical Kolmogorov's theorem: Let (X,μ) be a probability measure space and let $0 < p < q < \infty$, then there exists a constant $C > 0$, such that $||f||_{L^p(\mu)} \leq C||f||_{L^{q,\infty}(\mu)}$ for any measurable function f. Let $p = \delta$ and $q = 1/2$ such that $0 < \delta < 1/2$. Using Kolmogorov's theorem, Lemma 2.2 and Hölder's inequality, we obtain

(2.22)
$$\begin{aligned}E_{41} &\leq C||T((b_1-m_{\tilde{B}}b_1)f_1^1,(b_2-m_{\tilde{B}}b_2)f_2^1)||_{L^{1/2,\infty}(\frac{6}{5}B,\frac{d\mu(z)}{\mu(6B)})}\\ &\leq C\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|(b_1-m_{\tilde{B}}b_1)f_1(z)|d\mu(z)\\ &\quad\times\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|(b_2-m_{\tilde{B}}b_2)f_2(z)|d\mu(z)\\ &\leq C(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|(b_1-m_{\tilde{B}}b_1|^{p_1'}d\mu(z))^{1/p_1'}\\ &\quad\times(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|f_1(z)|^{p_1}d\mu(z))^{1/p_1}\\ &\quad\times(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|(b_2-m_{\tilde{Q}}b_2)|^{p_2'}d\mu(z))^{1/p_2'}\\ &\quad\times(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|f_2(z)|^{p_2}d\mu(z))^{1/p_2}\\ &\leq C||b_1||_*||b_2||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).\end{aligned}$$

To compute E_{42}, using (i) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and the properties of λ, we know

$$\begin{aligned}
E_{42} &\leq C\frac{1}{\mu(6B)}\int_X\int_X\int_{X\setminus\frac{6}{5}B}\frac{|b_1(y_1)-m_{\tilde{B}}b_1||f_1^1(y_1)|}{[\lambda(z,d(z,y_1))+\lambda(z,d(z,y_2))]^2}\\
&\quad\times|b_2(y_2)-m_{\tilde{B}}b_2||f_1^2(y_2)|d\mu(y_1)d\mu(y_2)d\mu(z)\\
&\leq C\frac{1}{\mu(6B)}\int_B\int_{\frac{6}{5}B}|b_1(y_1)-m_{\tilde{B}}b_1||f_1(y_1)|d\mu(y_1)\\
&\quad\times\int_{X\setminus\frac{6}{5}B}\frac{|b_2(y_2)-m_{\tilde{B}}b_2||f_2(y_2)|d\mu(y_2)}{[\lambda(z,d(z,y_2))]^2}d\mu(z)\\
&\leq C(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|b_1(y_1)-m_{\tilde{B}}b_1|^{p_1'}d\mu(y_1))^{1/p_1'}\\
&\quad\times(\frac{1}{\mu(6B)}\int_{\frac{6}{5}B}|f_1(y_1)|^{p_1}d\mu(y_1))^{1/p_1}\\
&\quad\times\mu(B)\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}B}\frac{|b_2(y_2)-m_{\tilde{B}}b_2||f_2(y_2)|}{[\lambda(z,6^{k-1}\frac{6}{5}r_B)]^2}d\mu(y_2)\\
&\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\frac{\mu(B)}{\mu(\frac{6}{5}B)}\frac{\mu(\frac{6}{5}B)}{\lambda(z,\frac{6}{5}r_B)}\\
&\quad\times\frac{1}{\lambda(z,6^{k-1}\frac{6}{5}r_B)}\int_{6^k\frac{6}{5}B}|b_2(y_2)-m_{\tilde{B}}b_2||f_2(y_2)|d\mu(y_2)\\
&\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\\
&\quad\times\int_{6^k\frac{6}{5}B}|b_2(y_2)-m_{\widetilde{6^k\frac{6}{5}B}}(b_2)+m_{\widetilde{6^k\frac{6}{5}B}}(b_2)-m_{\tilde{B}}b_2||f_2(y_2)|d\mu(y_2)\\
&\leq C||b_1||_*M_{p_1,(5)}f_1(x)\sum_{k=1}^{\infty}6^{-km}\Bigg[\Bigg(\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\\
&\quad\times\int_{6^{k+1}\frac{6}{5}B}|b_2(y_2)-m_{\widetilde{6^k\frac{6}{5}B}}(b_2)|^{p_2'}d\mu(y_2)\Bigg)^{1/p_2'}\\
&\quad\times\Bigg(\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_2(y_2)|^{p_2}d\mu(y_2)\Bigg)^{1/p_2}\\
&\quad+Ck||b_2||_*\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_2(y_2)|d\mu(y_2)\Bigg]\\
&\leq C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned}\tag{2.23}$$

Similarly, we get

$$E_{43}\leq C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).\tag{2.24}$$

For E_{44}, by (ii) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and the properties of λ, we obtain

(2.25)

$$\begin{aligned}
&|T((b_1-m_{\tilde{B}}b_1)f_2^2,(b_2-m_{\tilde{B}}b_2)f_2^2)(z)-T((b_1-m_{\tilde{B}}b_1)f_2^2,(b_2-m_{\tilde{B}}b_2)f_2^2)(z_0)|\\
\leq& C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|\\
&\times|\prod_{i=1}^{2}(b_i(y_i)-m_{\tilde{B}}b_i)f_i(y_i)|d\mu(y_1)d\mu(y_2)\\
\leq& C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B}\frac{d(z,z_0)^{\delta}|\prod_{i=1}^{2}(b_i(y_i)-m_{\tilde{B}}b_i)f_i(y_i)|d\mu(y_1)d\mu(y_2)}{(d(z,y_1)+d(z,y_2))^{\delta}[\sum_{j=1}^{2}\lambda(z,d(z,y_j))]^2}\\
\leq& C\prod_{i=1}^{2}\int_{X\setminus\frac{6}{5}B}\frac{d(z,z_0)^{\delta_i}|b_i(y_i)-m_{\tilde{B}}b_i||f_i(y_i)d\mu(y_i)|}{d(z,y_i)^{\delta_i}\lambda(z,d(z,y_i))}\\
\leq& C\prod_{i=1}^{2}\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}B}6^{-k\delta_i}\frac{\mu(5\times6^k\frac{6}{5}B)}{\lambda(z,5\times6^k\frac{6}{5}r_B)}\frac{1}{\mu(5\times6^k\frac{6}{5}B)}|b_i(y_i)-m_{\tilde{B}}b_i||f_i|d\mu(y_i)\\
\leq& C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}(\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|b_i(y_i)-m_{\tilde{B}}b_i|^{p_i'}d\mu(y_i))^{1/p_i'}\\
&\times(\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|f_i|^{p_i})^{1/p_i}\\
\leq& C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}M_{p_i,(6)}f_i(x)(\frac{1}{\mu(5\times6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}|b_i(y_i)-m_{\widetilde{6^k\frac{6}{5}B}}\\
&+m_{\widetilde{6^k\frac{6}{5}B}}-m_{\tilde{B}}b_i|^{p_i'}d\mu(y_i))^{1/p_i'}\\
\leq& C\prod_{i=1}^{2}\sum_{k=1}^{\infty}6^{-k\delta_i}k||b_i||_*M_{p_i,(5)}f_i(x)\\
\leq& C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned}$$

where $\delta_1,\delta_2>0$ and $\delta_1+\delta_2=\delta$.

Taking the mean over $z_0\in B$, we deduce

$$E_{44}\leq C||b_1||_*||b_2||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.26}$$

So (2.14) can be obtain from (2.17) to (2.26).

Next we prove (2.15). Consider two balls $B\subset Q$ with $x\in B$, where B is an arbitrary ball and Q is a doubling ball. Let $N=N_{B,Q}+1$, then we obtain

$$
\begin{aligned}
&\Big||m_B T((b_1-m_{\tilde{B}}b_1)f_1^2,(b_2-m_{\tilde{B}}b_2)f_2^2)| \\
&\quad -|m_Q T((b_1-m_Q b_1)f_1^2,(b_2-m_Q b_2)f_2^2)|\Big| \\
\le&|m_B T((b_1-m_{\tilde{B}}b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{X\setminus 6^N B}) \\
&\quad - m_Q T((b_1-m_{\tilde{B}}b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{X\setminus 6^N B})| \\
&\quad + |m_Q T((b_1-m_Q b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_Q b_2)f_2\chi_{X\setminus 6^N B}) \\
&\quad - m_Q T((b_1-m_{\tilde{B}}b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{X\setminus 6^N B})| \\
&\quad + |m_B T((b_1-m_{\tilde{B}}b_1)f_1\chi_{6^N B\setminus \frac{6}{5}B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{X\setminus \frac{6}{5}B}) \\
&\quad + |m_B T((b_1-m_{\tilde{B}}b_1)f_1\chi_{X\setminus \frac{6}{5}B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{6^N B\setminus \frac{6}{5}B}) \\
&\quad + |m_Q T((b_1-m_Q b_1)f_1\chi_{6^N B\setminus \frac{6}{5}Q},(b_2-m_Q b_2)f_2\chi_{X\setminus 6^N B}) \\
&\quad + |m_Q T((b_1-m_Q b_1)f_1\chi_{X\setminus \frac{6}{5}Q},(b_2-m_Q b_2)f_2\chi_{6^N B\setminus \frac{6}{5}Q}) \\
=:&F_1+F_2+F_3+F_4+F_5+F_6.
\end{aligned}
\tag{2.27}
$$

Using the method to estimate E_4, we get

$$
F_1 \le CK_{B,Q}^2||b_1||_*||b_2||_* M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x). \tag{2.28}
$$

Let us estimate F_2. At first, we compute

$$
\begin{aligned}
&T((b_1-m_Q b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_Q b_2)f_2\chi_{X\setminus 6^N B})(z) \\
&\quad - T((b_1-m_{\tilde{B}}b_1)f_1\chi_{X\setminus 6^N B},(b_2-m_{\tilde{B}}b_2)f_2\chi_{X\setminus 6^N B})(z) \\
=&(m_Q b_2-m_{\tilde{B}}b_2)T((b_1-m_Q b_1)f_1\chi_{X\setminus 6^N B},f_2\chi_{X\setminus 6^N B})(z) \\
&\quad + (m_Q b_1-m_{\tilde{B}}b_1)T(f_1\chi_{X\setminus 6^N B},(b_2-m_Q b_2)f_2\chi_{X\setminus 6^N B})(z) \\
&\quad + (m_Q b_1-m_{\tilde{B}}b_1)(m_Q b_2-m_{\tilde{B}}b_2)T(f_1\chi_{X\setminus 6^N B},f_2\chi_{X\setminus 6^N B})(z).
\end{aligned}
\tag{2.29}
$$

Hence

(2.30)

$$
\begin{aligned}
F_2 \le&|(m_Q b_2-m_{\tilde{B}}b_2)\frac{1}{\mu(Q)}\int_Q T((b_1-m_Q b_1)f_1\chi_{X\setminus 6^N B},f_2\chi_{X\setminus 6^N B})(z)d\mu(z)| \\
&+|(m_Q b_1-m_{\tilde{B}}b_1)\frac{1}{\mu(Q)}\int_Q T((f_1\chi_{X\setminus 6^N B},(b_2-m_Q b_2)f_2\chi_{X\setminus 6^N B})(z)d\mu(z)| \\
&+|(m_Q b_1-m_{\tilde{B}}b_1)(m_Q b_2-m_{\tilde{B}}b_2)\frac{1}{\mu(Q)}\int_Q T(f_1\chi_{X\setminus 6^N B},f_2\chi_{X\setminus 6^N B})(z)| \\
=:&F_{21}+F_{22}+F_{23}.
\end{aligned}
$$

To estimate F_{21}, we write

$$
\begin{aligned}
&T((b_1-m_Qb_1)f_1\chi_{X\setminus 6^NQ}, f_2\chi_{X\setminus 6^NQ})(z)\\
=&T((b_1-m_Qb_1)f_1,f_2)(z)-T((b_1-m_Qb_1)f_1\chi_{6^NB}\chi_{\frac{6}{5}Q}, f_2\chi_{\frac{6}{5}Q})(z))\\
&-T((b_1-m_Qb_1)f_1\chi_{\frac{6}{5}Q}, f_2\chi_{6^NB}\chi_{\frac{6}{5}Q})(z))\\
&+T((b_1-m_Qb_1)f_1\chi_{6^NB}\chi_{\frac{6}{5}Q}, f_2\chi_{6^NB}\chi_{\frac{6}{5}Q})(z)\\
&-T((b_1-m_Qb_1)f_1\chi_{X\setminus\frac{6}{5}Q}, f_2\chi_{6^NB})(z)\\
&-T((b_1-m_Qb_1)f_1\chi_{6^NB}, f_2\chi_{X\setminus\frac{6}{5}Q})(z)\\
&+T((b_1-m_Qb_1)f_1\chi_{6^NB\setminus\frac{6}{5}Q}, f_2\chi_{6^NB\setminus\frac{6}{5}Q})(z)\\
=:&H_1(z)+H_2(z)+H_3(z)+H_4(z)+H_5(z)+H_6(z)+H_7(z).
\end{aligned}
\tag{2.31}
$$

Let us estimate $H_1(z)$ firstly. Since

$$\frac{1}{\mu(Q)}\int_Q |T(b_1-b_1(z)f_1,f_2)(z)|d\mu(z)\le CM_{r,(6)}([b_1,T]f_1,f_2)(x)$$

and by Hölder's inequality, we have

$$\frac{1}{\mu(Q)}\int_Q |(b_1(z)-m_Q(b_1))T(f_1,f_2)(z)|d\mu(z)\le C||b_1||_*M_{r,(6)}(T(f_1,f_2))(x),$$

then we obtain

$$
\begin{aligned}
|m_Q(H_1)|&\le |m_Q(T(b_1-b_1(z)f_1,f_2))|+|m_Q((b_1(z)-m_Q(b_1))T(f_1,f_2))|\\
&\le CM_{r,(6)}([b_1,T]f_1,f_2)(x)+||b_1||_*M_{r,(6)}(T(f_1,f_2))(x).
\end{aligned}
\tag{2.32}
$$

For $H_2(z)$, let $r>1$ and $1<s_1<p_1$ such that $\frac{1}{r}=\frac{1}{s_1}+\frac{1}{p_2}$. Denote $\frac{1}{s_2}=\frac{1}{s_1}-\frac{1}{p_1}$, using the fact of Q is a doubling balls, Kolmogorov's inequality, Hölder's inequality and Lemma 2.4, we yield

$$
\begin{aligned}
|m_Q(H_2)|\le&C||H_2||_{L^{r,\infty}(Q,\frac{d\mu(z)}{\mu(Q)})}\\
\le&C(\frac{1}{\mu(Q)}\int_{\frac{6}{5}Q}|b_1-m_Qb_1|^{s_1}d\mu(z))^{1/s_1}(\frac{1}{\mu(Q)}\int_{\frac{6}{5}Q}|f_2|^{p_2}d\mu(z))^{1/p_2}\\
\le&C(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|b_1-m_Qb_1|^{s_2}d\mu(z))^{1/s_2}(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|f_2|^{p_1}d\mu(z))^{1/p_1}\\
&\times(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|f_2|^{p_2}d\mu(z))^{1/p_2}\\
\le&||b_1||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).
\end{aligned}
\tag{2.33}
$$

We also can obtain

$$|m_Q(H_3)|+|m_Q(H_4)|\le C||b_1||_*M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x).\tag{2.34}$$

For H_5, since $z \in Q$, by (i) of Definition 1.7, Lemma 2.1, Lemma 2.2, Hölder's inequality and Q is a doubling ball, we deduce

$$
\begin{aligned}
|H_5(z)| &\le C \int_{6^N B} \int_{X \setminus \frac{6}{5}Q} \frac{|b_1(y_1) - m_Q b_1||f_1(y_1)||f_2(y_2)| d\mu(y_1) d\mu(y_2)}{[\sum_{j=1}^{2} \lambda(z, d(x, y_j))]^2} \\
&\le C \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} \int_{6^k \frac{6}{5}Q} \frac{|b_1(y_1) - m_Q b_1||f_1(y_1)|}{(\lambda(z, 6^{k-1}\frac{6}{5} r_Q))^2} d\mu(y_1) \\
&\le C \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km} \\
&\quad \times \int_{6^k \frac{6}{5}Q} \frac{1}{\lambda(z, \frac{6}{5} r_Q)} \frac{|b_1(y_1) - m_Q b_1||f_1(y_1)| d\mu(y_1)}{\lambda(z, 6^{k-1}\frac{6}{5} r_Q)} \\
&\le C \frac{1}{\lambda(z, 6r_Q)} \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km} \frac{1}{\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)} \\
&\quad \times \Big[\int_{6^k \frac{6}{5}Q} |b_1(y_1) - m_{6^k \frac{6}{5}Q}(b_1)||f_1(y_1)| d\mu(y_1) \\
&\quad + \int_{6^k \frac{6}{5}Q} |m_{6^k \frac{6}{5}Q}(b_1) - m_Q b_1||f_1(y_1)| d\mu(y_1) \Big] \\
&\le C \frac{1}{\lambda(z, 6r_Q)} \int_{6^N B} |f_2(y_2)| d\mu(y_2) \sum_{k=1}^{\infty} 6^{-km} \\
&\quad \times \Big[\Big(\frac{1}{\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)} \int_{6^k \frac{6}{5}Q} |b_1(y_1) - m_{6^k \frac{6}{5}Q}(b_1)|^{p_1'} d\mu(y_1) \Big)^{1/p_1'} \\
&\quad \times \big(\frac{1}{\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)} \int_{6^k \frac{6}{5}Q} |f_1(y_1)|^{p_1} d\mu(y_1) \big)^{1/p_1} \\
&\quad + k ||b_1||_* \frac{1}{\lambda(z, 5 \times 6^k \frac{6}{5} r_Q)} \int_{6^k \frac{6}{5}Q} |f_1(y_1)| d\mu(y_1) \Big] \\
&\le C \sum_{k=1}^{N} \frac{1}{\lambda(z, 6r_Q)} \int_{6^k B} |f_2(y_2)| d\mu(y_2) ||b_1||_* M_{p_1,(5)} f_1(x) \\
&\le C \sum_{k=1}^{N} \frac{\mu(5 \times 6^k B)}{\lambda(z, 5 \times 6^k r_B)} \frac{\lambda(z, 5 \times 6^k r_B)}{\lambda(z, 6r_Q)} \\
&\quad \times \frac{1}{\mu(5 \times 6^k B)} \int_{6^k B} |f_2(y_2)| d\mu(y_2) ||b_1||_* M_{p_1,(5)} f_1(x) \\
&\le C K_{B,Q} ||b_1||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x).
\end{aligned} \tag{2.35}
$$

Then it yields

$$
|m_Q(H_5)| \le C K_{B,Q} ||b_1||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \tag{2.36}
$$

In the similar way to estimate $m_Q(H_5)$, we also obtain

$$
|m_Q(H_6)| + |m_Q(H_7)| \le C K_{B,Q} ||b_1||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \tag{2.37}
$$

From (2.8) in Lemma 2.2, we deduce

$$\begin{aligned} F_{21} \leq & CK^2_{B,Q}\{||b_1||_*||b_2||_* M_{r,(6)}(T(f_1,f_2))(x) \\ & + ||b_1||_* M_{r,(6)}([b_2,T](f_1,f_2))(x) \\ & + ||b_2||_* M_{r,(6)}([b_1,T](f_1,f_2))(x) \\ & + ||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x)\}. \end{aligned} \tag{2.38}$$

F_{22} and F_{23} also have similar estimate of F_{21}, therefore,

$$\begin{aligned} F_2 \leq & CK^2_{B,Q}\Big\{||b_1||_*||b_2||_* M_{r,(6)}(T(f_1,f_2))(x) \\ & + ||b_1||_* M_{r,(6)}([b_2,T](f_1,f_2))(x) \\ & + ||b_2||_* M_{r,(6)}([b_1,T](f_1,f_2))(x) \\ & + ||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x)\Big\}. \end{aligned} \tag{2.39}$$

From F_3 to F_6, using the similar method to estimate I_4, we conclude

$$F_3 + F_4 + F_5 + F_6 \leq C||b_1||_*||b_2||_* M_{p_1,(5)} f_1(x) M_{p_2,(5)} f_2(x). \tag{2.40}$$

Thus (2.15) holds by from (2.27) to (2.40) and hence (2.11) is proved. With the same method to prove (2.11), we can obtain that (2.12) and (2.13) are also hold. Here we omit the details. Thus the Lemma 2.5 is proved. □

Proof of Theorem 1.11. Let $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < r < q$, $f_1 \in L^{p_1}(\mu)$, $f_2 \in L^{p_2}(\mu)$, $b_1 \in RBMO(\mu)$ and $b_2 \in RBMO(\mu)$. By $|f(x)| \leq N_\delta f(x)$, Lemma 2.1, Lemma 2.3, Lemma 2.4, Hörder's inequality and the boundedness of $M_{(\rho)}$ and $M_{r,(\rho)}$ for $\rho \geq 5$ and $q > r$, we obtain

$$\begin{aligned}
&||[b_1, b_2, T](f_1, f_2)||_{L^q(\mu)} \leq ||N_\delta([b_1, b_2, T](f_1, f_2))||_{L^q(\mu)} \\
\leq & C||M^\sharp_\delta([b_1, b_2, T](f_1, f_2))||_{L^q(\mu)} \\
\leq & C||b_1||_*||b_2||_*||M_{r,(6)}(T(f_1, f_2))||_{L^q(\mu)} \\
& + C||b_1||_*||M_{r,(6)}([b_2, T](f_1, f_2))||_{L^q(\mu)} \\
& + C||b_2||_*||M_{r,(6)}([b_1, T](f_1, f_2))||_{L^q(\mu)} \\
& + C||b_1||_*||b_2||_*||M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)||_{L^q(\mu)} \\
\leq & C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)} \\
& + C||b_1||_*||([b_2, T](f_1, f_2))||_{L^q(\mu)} \\
& + C||b_2||_*||([b_1, T](f_1, f_2))||_{L^q(\mu)} \\
\leq & C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)} \\
& + C||b_1||_*||M^\sharp_\delta([b_2, T](f_1, f_2))||_{L^q(\mu)} \\
& + C||b_2||_*||M^\sharp_\delta([b_1, T](f_1, f_2))||_{L^q(\mu)} \\
\leq & ||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)} \\
& + C||b_1||_*||M_{r,(6)}(T(f_1, f_2))(x)||_{L^q(\mu)} \\
& + C||b_1||_*||M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)||_{L^q(\mu)} \\
& + C||b_2||_*||M_{r,(6)}(T(f_1, f_2))(x)||_{L^q(\mu)} \\
& + C||b_2||_*||M_{p_1,(5)}f_1(x)M_{p_2,(5)}f_2(x)||_{L^q(\mu)} \\
\leq & C||b_1||_*||b_2||_*||f_1(x)||_{L^{p_1}(\mu)}||f_2(x)||_{L^{p_2}(\mu)}.
\end{aligned} \tag{2.41}$$

Thus the proof of Theorem 1.11 is completed. □

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 10371087) and Excellent Young Talent Foundation of Anhui Province (Grant No.2013SQRL080ZD).

References

[1] T. A. Bui and X. T. Duong, *Hardy Spaces, regularized BMO spaces and the Boundedness of Calderón-Zygmund operators on non-homogeneous spaces*, J. Geom. Anal., **23**, (2013), 895-932.

[2] W. Chen and E. Sawyer, *A note on commutators of fractional integrals with RBMO(μ) functions*, Illinois J. Math., **46**, (2002), 1287-1298.

[3] R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc., **212**, (1975), 315-331.

[4] X. Fu, D. C.Yang and W. Yuan, *Boundedness of multilinear commutators of Calderón-Zygmund operators on orlicz spaces over non-homogeneous spaces*, Taiwan. J. Math., **16**, (2012), 2203-2238.

[5] J. García-Cuerva and J. María-Martell, *Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces*, Indiana Univer. Math. J., **50**, (2001), 1241-1280.

[6] L. Grafakos and R. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math., **165**, (2002), 124-164.

[7] L. Grafakos and R.Torres, *On multilinear singular integrals of Calderón-Zygmund type*, Publ. Mat., (2002), 57-91. (extra)

[8] G. Hu, Y. Meng and D. Yang, *Multilinear commutators of singular integrals with non doubling measures*, Integr. Equat. Oper. Th., **51**, (2005), 235-255.

[9] G. Hu, Y. Meng and D. Yang, *New atomic characterization of H^1 space with nondoubling measures and its applications*, Math. Proc. Cambridge Philos. Soc., **138**, (2005), 151-171.

[10] G. Hu, Y. Meng and D. Yang, *Weighted norm inequalities for multilinear Calderón-Zygmund operators on non-homogeneous metric measure spaces,* Forum Math., (2012), doi: 10. 1515/forum-2011-0042.

[11] T. Hytönen, *A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa*, Publ. Mat., **54**, (2010), 485-504.

[12] T. Hytönen and H. Martikainen, *Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces*, J. Geom. Anal., **22**, (2012), 1071-1107.

[13] T. Hytönen, S. Liu, Da. Yang and Do. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces*, Canad. J. Math., DOI 10.4153/CJM-2011-065-2 (arXiv: 1011.2937).

[14] T. Hytönen, Da. Yang and Do. Yang, *The Hardy space H^1 on non-homogeneous metric spaces*, Math. Proc. Cambridge, **153**, (2012), 9-31.

[15] H. Lin and D. Yang, *Spaces of type BLO on non-homogeneous metric measure spaces*, Front. Math. China, **6**, (2011), 271-292.

[16] S. Liu, Da. Yang and Do. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: equivalent characterizations*, J. Math. Anal. Appl., **386**, (2012), 258-272.

[17] F. Nazarov, S. Treil and A. Volberg, *The Tb-theorem on non-homogeneous spaces*, Acta Math., **190**, (2003), 151-239.

[18] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math., **190**, (2003), 105-149.

[19] X. Tolsa, *BMO, H^1 and Calderón-Zygmund operators for non-doubling measures*, Math. Ann., **319**, (2001), 89-101.

[20] R. Xie and L. Shu, Θ*-type Calderón-Zygmund operators with non-doubling measures*, Acta Math. Appl. Sinica, English Series, **29**, (2013), 263-280.

[21] J. Xu, *Boundedness of multilinear singular integrals for non-doubling measures*, J. Math. Anal. Appl., **327**, (2007), 471-480.

[22] J. Xu, *Boundedness in Lebesgue spaces for commutators of multilinear singular integrals and RBMO functions with non-doubling measures*, Sci. China, Series A, **50**, (2007), 361-376.

School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China

E-mail address: xierl@mail.ustc.edu.cn;huajun84@hotmail.com;zhouxiaoyaodeyouxian@126.com