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Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes

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Abstract

Planar gluon amplitudes in $\mathcal{N} = 4$ SYM are remarkably similar to expectation values of Wilson loops made of light-like segments. We argue that the latter can be determined by making use of the conformal symmetry of the gauge theory, broken by cusp anomalies. We derive the corresponding anomalous conformal Ward identities valid to all loops and show that they uniquely fix the form of the finite part of a Wilson loop with n cusps (up to an additive constant) for $n = 4$ and 5 and reduce the freedom in it to a function of conformal invariants for $n \geq 6$. We also present an explicit two-loop calculation for $n = 5$. The result confirms the form predicted by the Ward identities and exactly matches the finite part of the two-loop five-gluon planar MHV amplitude. This constitutes another non-trivial test of the Wilson loop/gluon amplitude duality.

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1 Introduction

The recent surge of interest in gluon scattering amplitudes in $\mathcal{N} = 4$ SYM theory has been motivated by two findings. The four-gluon color-ordered planar amplitudes reveal an intriguing iterative structure at weak coupling – the Bern-Dixon-Smirnov (BDS) conjecture [1], which predicts the finite part of the amplitude in terms of the cusp anomalous dimensions [2, 3] and two other functions of the coupling [4]. Quite remarkably, the same structure also emerged at strong coupling within the Alday-Maldacena proposal [5] for a string description of the gluon scattering amplitude in the AdS/CFT correspondence. Their string construction makes contact with a Wilson loop on a specific contour with cusps and with like-like edges determined by the gluon momenta considered earlier in [6, 7].

Inspired by these recent developments, in [8] three of us have conjectured that a new type of duality between gluon amplitudes and Wilson loops in $\mathcal{N} = 4$ SYM may also exist at weak coupling. More precisely, we suggested that, for arbitrary values of the coupling constant, the four-gluon planar amplitude in $\mathcal{N} = 4$ SYM is related to the expectation value of a Wilson loop

$$\mathcal{M}_4 = \frac{1}{N}\langle 0|\mathrm{Tr\,P}\exp\left(i\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle + O(1/N^2) \ , \tag{1}$$

where the integration contour C_4 in Minkowski space-time consists of four light-like segments $[x_i, x_{i+1}]$ with the coordinates x_i^μ related to the on-shell gluon momenta, $x_i^\mu - x_{i+1}^\mu = p_i^\mu$. Alday and Maldacena [5] demonstrated the duality relation (1) at strong coupling. At weak coupling, we have shown by an explicit one-loop [8] and then two-loop calculation [9] that the relation (1) reproduces the known expression for the four-gluon amplitude [10].

The BDS conjecture [1] also applies to the case of $n-$gluon maximal helicity violating (MHV) planar amplitudes. So far the conjecture has been confirmed for $n = 4$ up to three loops in [1], while for $n > 4$ it has been tested only for $n = 5$ and at two loops in [11]. The Alday-Maldacena proposal also works for an arbitrary number of gluons, although the practical evaluation of the corresponding solution of the classical string equations is difficult even for $n = 5$ (for a recent discussion see Refs. [4]). However, very recently Alday and Maldacena [12] proposed a way to do this for n very large which gives a disagreement with the BDS conjecture at strong coupling. It is straightforward to generalize the weak-coupling duality relation (1) to MHV amplitudes with an arbitrary number $n \geq 5$ of external legs on the one hand, and the expectation value of a Wilson loop evaluated along a polygonal contour consisting of n light-like segments, on the other hand. This duality has been tested for arbitrary n at one loop in [13].

We argued in [9] that we can profit from the (broken) conformal symmetry of the light-like Wilson loops to make all-loops predictions about the form of the amplitude.[1] Due to the presence of a cusp anomaly in the Wilson loop, conformal invariance manifests itself in the form of anomalous Ward identities. We proposed a very simple anomalous conformal Ward identity for the finite part of the amplitude and conjectured it to be valid to all orders in the coupling. This identity uniquely fixes the form of the finite part (up to an additive constant) of the Wilson loop dual to the four- and five-gluon amplitudes, and gives partial restrictions on the functional dependence on the kinematical variables for $n \geq 6$. Quite remarkably, the BDS ansatz [1] for the $n-$gluon MHV amplitudes satisfies the conformal Ward identity for arbitrary n.

[1] It should be noted that conformal arguments were efficiently used by Alday and Maldacena to construct their strong-coupling dual in [5].

If the duality relation (1) holds for any n and to all loops, the conformal symmetry properties of the light-like Wilson loops impose strong constraints on the finite part of the gluon amplitudes in $\mathcal{N}=4$ SYM. This may provide the natural explanation of the BDS conjecture for $n=4$ and $n=5$, but cannot validate it for $n\geq 6$. One might also try to find a direct conformal symmetry argument on the MHV amplitude side. The encouraging fact is that the $n=4$ gluon amplitudes have been shown to possess a peculiar 'hidden' or 'dual' conformal symmetry [14] (different from the conformal symmetry of the underlying $\mathcal{N}=4$ theory) up to four loops [15] (and even at five loops [16]). It is natural to conjecture that the controlled breaking of this symmetry in the on-shell divergent amplitude will lead to the same type of Ward identities. In this paper we present a derivation of the all-order anomalous conformal Ward identities conjectured in [9]. In addition, we perform an explicit two-loop calculation of the Wilson loop made of five light-like segments (pentagon). We demonstrate that the two-loop expression for the pentagon Wilson loop indeed verifies the conformal Ward identities. Most importantly, it coincides with the known expression [11] for the two-loop correction to the five-gluon planar scattering amplitude, thus providing additional support for the gluon amplitude/Wilson loop correspondence (1).

The presentation is organized as follows. In Section 2 we review some properties of light-like Wilson loops which will be important in deriving the conformal Ward identities in section 3. Then, in section 4, we present an explicit two-loop calculation of the pentagon Wilson loop as an explicit check of our Ward identities.

2 General features of light-like Wilson loops

2.1 Definitions

The central object of our consideration is the light-like Wilson loop defined in $\mathcal{N}=4$ SYM theory with $SU(N)$ gauge group as

$$W(C_n) = \frac{1}{N}\langle 0|\,\mathrm{Tr}\,\mathrm{P}\exp\left(i\oint_{C_n} dx^\mu A_\mu(x)\right)|0\rangle\,, \tag{2}$$

where $A_\mu(x) = A^a_\mu(x)t^a$ is a gauge field, t^a are the $SU(N)$ generators in the fundamental representation normalized as $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ and P indicates the ordering of the $SU(N)$ indices along the integration contour C_n. This contour is made out of n light-like segments $C_n = \bigcup_{i=1}^n \ell_i$ joining the cusp points x_i^μ (with $i=1,2,\ldots,n$)

$$\ell_i = \{x^\mu(\tau_i) = \tau_i x_i^\mu + (1-\tau_i)x_{i+1}^\mu |\, \tau_i \in [0,1]\}\,, \tag{3}$$

such that the vectors

$$x^\mu_{i,i+1} \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu\,, \qquad p_i^2 = 0 \tag{4}$$

are identified with the external on-shell momenta of the n-gluon scattering amplitude. Thinking about $W(C_n)$ as a function of the cusp points we observe that it has the following symmetry

$$W(x_1,x_2,\ldots,x_n) = W(x_n,x_1,\ldots,x_{n-1}) = W(x_n,x_{n-1},\ldots,x_1)\,, \tag{5}$$

where the first relation is a consequence the cyclic property of the trace in the right-hand side of (2) while the second relation follows from reality condition $(W(C_n))^* = W(C_n)$.

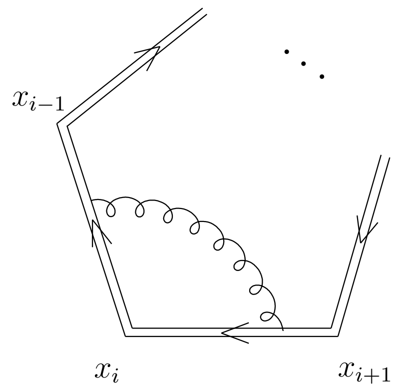

Figure 1: The Feynman diagram contributing to the one-loop divergence at the cusp point x_i. The double line depicts the integration contour C_n, the wiggly line the gluon propagator.

2.2 Cusp singularities

A distinctive feature of the contour C_n is the presence of n cusps located at the points x_i^μ. This causes specific ultraviolet divergences (UV) to appear in the Wilson loop [2]. The fact that the cusp edges are light-like makes them even more severe [6]. In order to get some insight into the structure of these divergences we start by giving a one-loop example. Later on in this subsection give some arguments which lead to the all-order form of the divergences.

2.2.1 One-loop example

Let us first discuss the origin of the cusp singularities in the n-gonal Wilson loop $W(C_n)$ to the lowest order in the coupling constant. According to the definition (2), it is given by a double contour integral,

$$W(C_n) = 1 + \frac{1}{2}(ig)^2 C_F \oint_{C_n} dx^\mu \oint_{C_n} dy^\nu \, G_{\mu\nu}(x-y) + O(g^4) \,. \tag{6}$$

Here $C_F = t^a t^a = (N^2-1)/(2N)$ is the quadratic Casimir of $SU(N)$ in the fundamental representation and $G_{\mu\nu}(x-y)$ is the gluon propagator in the coordinate representation

$$\langle A^a_\mu(x) \; A^b_\nu(y) \rangle = g^2 \delta^{ab} G_{\mu\nu}(x) \,. \tag{7}$$

To regularize the ultraviolet divergences of the integrals entering (6), we shall employ dimensional regularization, $D = 4 - 2\epsilon$ (with $\epsilon > 0$). Also, making use of the gauge invariance of the Wilson loop (2) and for the sake of simplicity, we perform the calculation in the Feynman gauge, where the gluon propagator is given by [2]

$$G_{\mu\nu}(x) = g_{\mu\nu} G(x) \,, \qquad G(x) = -\frac{\Gamma(1-\epsilon)}{4\pi^2}(-x^2 + i0)^{-1+\epsilon}\big(\mu^2 \mathrm{e}^{-\gamma}\big)^\epsilon \,. \tag{8}$$

The divergences in (6) originate from the integration of the position of the gluon in the vicinity of a light-like cusp in the diagram shown in Fig. 1. The calculation of this diagram is straightforward

[2] As in [9], we redefine the conventional dimensional regularization scale as $\mu^2 \pi \mathrm{e}^\gamma \mapsto \mu^2$ to avoid dealing with factors involving π and the Euler constant γ.

and the details can be found e.g. in [6]. Adding together the contributions of all cusps we obtain the following one-loop expression for the divergent part of $W(C_n)$:

$$\ln W(C_n) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \, \mu^2\right)^{\epsilon} + O(\epsilon^0) \right\} + O(g^4) \tag{9}$$

where the periodicity condition

$$x_{ij}^2 = x_{i+n,j}^2 = x_{i,j+n}^2 \equiv (x_i - x_j)^2 \tag{10}$$

is tacitly implied.

2.2.2 All-loop structure

Generalizing (9) to higher loops we find that the cusp singularities appear in $W(C_n)$ to the l-th loop-order as poles $W(C_n) \sim (a\mu^{2\epsilon})^l/\epsilon^m$ with $m \leq 2l$. Furthermore, $W(C_n)$ can be split into a divergent ('renormalization') factor Z_n and a finite ('renormalized') factor F_n as

$$W(C_n) = Z_n F_n \ . \tag{11}$$

From the studies of renormalization properties of light-like Wilson loops it is known [6] that cusp singularities exponentiate to all loops and, as a consequence, the factor Z_n has the special form [3]

$$\ln Z_n = -\frac{1}{4} \sum_{l\geq 1} a^l \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \mu^2\right)^{l\epsilon} \left(\frac{\Gamma_{\rm cusp}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right), \qquad a = \frac{g^2 N}{8\pi^2} \ . \tag{12}$$

Here $\Gamma_{\rm cusp}^{(l)}$ and $\Gamma^{(l)}$ are the expansion coefficients of the cusp anomalous dimension and the so-called collinear anomalous dimension, respectively, defined in the adjoint representation of $SU(N)$:

$$\Gamma_{\rm cusp}(a) = \sum_{l\geq 1} a^l \, \Gamma_{\rm cusp}^{(l)} = 2a - \frac{\pi^2}{3} a^2 + O(a^3) \ , \tag{13}$$

$$\Gamma(a) = \sum_{l\geq 1} a^l \, \Gamma^{(l)} = -7\zeta_3 a^2 + O(a^3) \, .$$

We have already seen at one-loop order that the divergences in $W(C_n)$ are due to the presence of the cusps on the integration contour C_n. They occur when the gluon propagator in Fig. 1 slides along the contour towards a given cusp point x_i. The divergences appear when the propagator becomes singular. This happens either when a gluon propagates at short distances in the vicinity of the cusp point (short distance divergences), or when a gluon propagates along the light-like segment adjacent to the cusp point (collinear divergences). In these two regimes we encounter, respectively, a double and single pole.

Going to higher orders in the coupling expansion of $W(C_n)$, we immediately realize that the structure of divergences coming from each individual diagram becomes much more complicated. Still, the divergences originate from the same part of the 'phase space' as at one-loop: from short distances in the vicinity of cusps and from propagation along light-like edges of the contour. The

[3] Formula (12) follows from the evolution equation (8) of the first reference in [6].

main difficulty in analyzing these divergences is due to the fact that in the diagrams with a few gluons attached to different segments of C_n various regimes could be realized simultaneously, thus enhancing the strength of poles in ϵ. This implies, in particular, that individual diagrams could generate higher order poles to $W(C_n)$.

The simplest way to understand the form of $\ln Z_n$ in (12) is to analyze the divergences of the Feynman diagrams not in the Feynman gauge but in the so-called axial gauge defined as

$$n \cdot A(x) = 0 \tag{14}$$

with n^μ being an arbitrary vector, $n^2 \neq 0$. The reason for this is that the same Feynman diagrams become less singular in the axial gauge and, most importantly, the potentially divergent graphs have a much simpler topology [31].[4] The axial gauge gluon propagator in the momentum representation is given by the following expression

$$\widetilde{G}_{\mu\nu}(k) = -i\frac{d_{\mu\nu}(k)}{k^2+i0}\,, \qquad d^{(A)}_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{(kn)} + k_\mu k_\nu \frac{n^2}{(kn)^2}\,. \tag{15}$$

The polarization tensor satisfies the relation

$$g^{\mu\nu} d^{(A)}_{\mu\nu}(k) = (D-2) + \frac{k^2 n^2}{(kn)^2} \tag{16}$$

(to be compared with the corresponding relation in the Feynman gauge, which is $g^{\mu\nu} d^{(F)}_{\mu\nu}(k) = D$) from which we deduce that for $k^2 = 0$, it describes only the physical polarizations of the on-shell gluon.[5] Let us consider a graph in which a gluon is attached to the i-th segment. In configuration space, the corresponding effective vertex is described by the contour integral $\int d\tau_i\, p_i^\mu\, A_\mu(x_i - p_i\tau_i)$. In the momentum representation, the same vertex reads $\int d\tau_i\, p_i^\mu\, \tilde{A}_\mu(k)\, \mathrm{e}^{ik(x_i-p_i\tau_i)}$ where the field $\tilde{A}_\mu(k)$ describes all possible polarizations (2 longitudinal and $D-2$ transverse) of the gluon with momentum k^μ. Let us examine the collinear regime, when the gluon propagates along the light-like direction p_i^μ. The fact that the gluon momentum is collinear, $k^\mu \sim p_i^\mu$, implies that it propagates close to the light-cone and, therefore, has a small virtuality k^2. Furthermore, since $p_i^\mu \tilde{A}_\mu(k) \sim k^\mu \tilde{A}_\mu(k)$ we conclude that the contribution of the transverse polarization of the gluon is suppressed as compared to the longitudinal ones. In other words, the most singular contribution in the collinear limit comes from the longitudinal components of the gauge field $\tilde{A}_\mu(k)$. The properties of the latter depend on the gauge, however. To see this, let us examine the form of the emission vertex in the Feynman and in the axial gauges. In the underlying Feynman integral, the gauge field $\tilde{A}_\mu(k)$ will be replaced by the propagator $\tilde{D}_{\mu\nu}(k)$ with ν the polarization index at the vertex to which the gluon is attached. In this way, we find that

$$p_i^\mu\, \tilde{G}_{\mu\nu}(k) \sim k^\mu \tilde{G}_{\mu\nu}(k) = -i\frac{k^\mu d_{\mu\nu}(k)}{k^2+i0} = -\frac{i}{k^2} \times \begin{cases} k^\nu\,, & \text{Feynman gauge} \\ k^2\left[\frac{k^\nu n^2}{(kn)^2} - \frac{n^\nu}{(kn)}\right], & \text{axial gauge} \end{cases} \tag{17}$$

Since $k^2 \to 0$ in the collinear limit, we conclude that the vertex is suppressed in the axial gauge, as compared to the Feynman gauge. This is in perfect agreement with our physical intuition

[4]The contribution of individual Feynman diagrams is gauge dependent and it is only their total sum that is gauge invariant.

[5]That is the reason why the axial gauge is called physical.

– the propagation of longitudinal polarizations of a gluon with momentum k^μ is suppressed for $k^2 \to 0$. This property is rather general and it holds not only for the vertices describing the attachment of gluons to the integration contour, but also for the 'genuine' interaction vertices of the $\mathcal{N} = 4$ SYM Lagrangian [31]. It should not be surprising now that the collinear divergences in the light-like Wilson loop come from graphs of very special topology that we shall explain in a moment.

Another piece of information that will be extensively used in our analysis comes from the so-called non-Abelian exponentiation property of Wilson loops [24]. It follows from the combinatorial properties of the path-ordered exponential and it is not sensitive to the particular form of the Lagrangian of the underlying gauge theory. For an arbitrary integration contour C it can be formulated as follows:

$$\langle W(C)\rangle = 1 + \sum_{k=1}^{\infty}\left(\frac{g^2}{4\pi^2}\right)^k W^{(k)} = \exp\left[\sum_{k=1}^{\infty}\left(\frac{g^2}{4\pi^2}\right)^k c^{(k)} w^{(k)}\right] . \tag{18}$$

Here $W^{(k)}$ denote the perturbative corrections to the Wilson loop, while $c^{(k)}w^{(k)}$ are given by the contribution to $W^{(k)}$ from 'webs' $w^{(k)}$ with the 'maximally non-Abelian' color factor $c^{(k)}$. To the first few orders, $k = 1, 2, 3$, the maximally non-Abelian color factor takes the form $c^{(k)} = C_F N^{k-1}$, but starting from $k = 4$ loops it is not expressible in terms of simple Casimir operators. Naively, one can think of the 'webs' $w^{(k)}$ as of Feynman diagrams with maximally interconnected gluon lines. For the precise definition of 'webs' we refer the interested reader to [24].

Let us now return to the analysis of the cusp divergences of the light-like Wilson loops and take advantage of both the axial gauge and the exponentiation (18). A characteristic feature of the 'webs' following from their maximal non-Abelian nature is that the corresponding Feynman integrals have 'maximally complicated' momentum loop flow, e.g. they cannot be factorized into a product of integrals. When applied to the light-like Wilson loop, this has the following remarkable consequences in the axial gauge:

- the 'webs' do not contain nested divergent subgraphs;
- each web produces a double pole in ϵ at most;
- the divergent contribution only comes from 'webs' localized at the cusp points as shown in Fig. 2(a) and (b)

These properties imply that the divergent part of $\ln W(C_n)$ is given by a sum over the cusp points x_i^μ with each cusp producing a double and single pole contribution. Moreover, the corresponding residue depend on $x_{i-1,i+1}^2$ – the only kinematical invariant that one can built out of three vectors x_{i-1}^μ, x_i^μ and x_{i+1}^μ satisfying $x_{i-1,i}^2 = x_{i,i+1}^2 = 0$. In this way, we arrive at the well-known relation (12) for the divergent part of the light-like Wilson loop.

Our consideration relied on the analysis of Feynman diagrams at weak coupling. It was recently shown in Refs. [7, 32, 43] that the structure of divergences of $\ln W(C_n)$ remains the same even at strong coupling.

3 Conformal symmetry of light-like Wilson loops

Were the Wilson loop $W(C_n)$ well defined in $D = 4$ dimensional Minkowski space-time, it would enjoy the (super)conformal invariance of the underlying $\mathcal{N} = 4$ SYM theory. More precisely, in

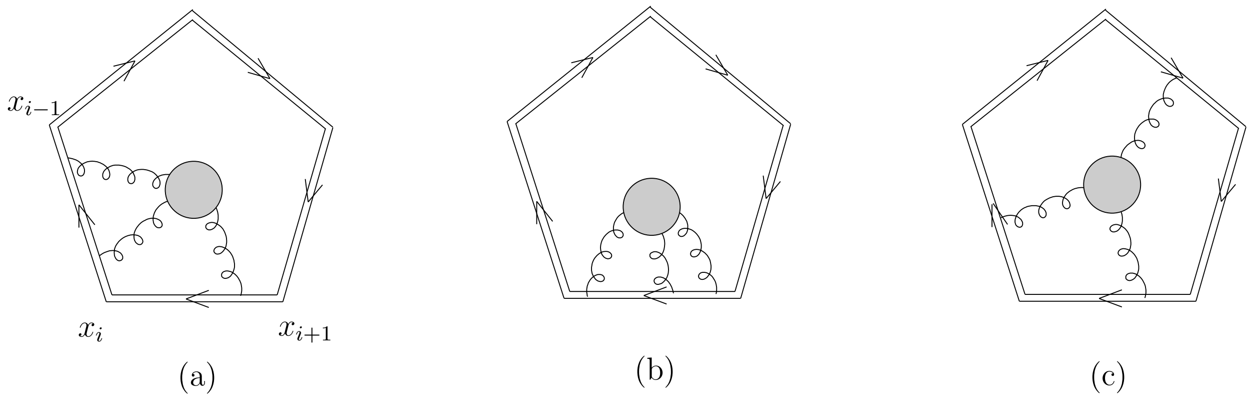

Figure 2: Maximally nonabelian Feynman diagrams of different topology ('webs') contributing to $\ln W(C_n)$. In the axial gauge, the vertex-type diagram (a) generates simple pole; the self-energy type diagram (b) generates double pole in ϵ; the diagram (c) with gluons attached to three and more segments is finite.

the absence of conformal anomalies we would conclude that

$$W(C'_n) = W(C_n)\,, \tag{19}$$

where C'_n is the image of the contour C_n upon an $SO(2,4)$ conformal transformations.

Our contour C_n is very special, however. Firstly, its shape is stable under conformal transformations, that is the contour C'_n is also made of n light-like segments with new cusp points $x_i'^{\mu}$ obtained as the images of the old ones x_i^μ. This property is rather obvious for translations, rotations and dilatations but we have to check it for special conformal transformations. Performing a conformal inversion [6], $x^{\mu\prime} = x^\mu/x^2$, of all points belonging to the segment (3), we obtain another segment of the same type, $x'^{\mu}(\tau'_i) = \tau'_i x_i'^{\mu} + (1-\tau'_i)x'^{\mu}_{i+1}$ with $(x'_{i+1}-x'_i)^2 = 0$ and $\tau'_i = \tau_i/[\tau_i + (1-\tau_i)(x'_i)^2/(x'_{i+1})^2]$.

The second distinctive feature of the contour C_n is the presence of cusps, which, as we already pointed out in section 2, cause specific ultraviolet divergences to appear in the Wilson loop (2). For this reason we employ dimensional regularization with $D = 4-2\epsilon$ (with $\epsilon > 0$).

In the dimensionally regularized $\mathcal{N}=4$ SYM theory, the Wilson loop $W(C_n) \equiv \langle W_n\rangle$ is given by a functional integral

$$\langle W_n\rangle = \int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}\phi\;\mathrm{e}^{iS_\epsilon[A,\lambda,\phi]}\;\mathrm{Tr}\;\mathrm{P}\exp\left(i\oint_{C_n} dx\cdot A(x)\right)\,, \tag{20}$$

where integration goes over gauge fields, A, gaugino, λ, and scalars, ϕ, with the action

$$S_\epsilon = \frac{1}{g^2\mu^{2\epsilon}}\int d^Dx\;\mathcal{L}(x)\,,\qquad \mathcal{L} = \mathrm{Tr}\left[-\tfrac14 F_{\mu\nu}^2\right] + \text{gaugino} + \text{scalars} + \text{gauge fixing} + \text{ghosts}. \tag{21}$$

Here μ is the regularization scale and we redefined all fields in such a way that g does not appear inside the Lagrangian $\mathcal{L}(x)$. This allows us to keep the *canonical* dimension of all fields, in particular of the gauge field $A^\mu(x)$, and hence to preserve the conformal invariance of the path-ordered exponential entering the functional integral in (20). However, due to the change of dimension of the measure $\int d^Dx$ in (21) the action S_ϵ is not invariant under dilatations and conformal boosts, which yields an *anomalous* contribution to the Ward identities.

[6] A special conformal transformation (boost) is equivalent to an inversion followed by a translation and then another inversion.

3.1 Anomalous conformal Ward identities

The conformal Ward identities for the light-like Wilson loop $W(C_n)$ can be derived following the standard method [17, 18, 29, 28]. To begin with we recall the well-known expressions for the generators of the $SO(2,4)$ conformal transformations – rotations ($\mathbb{M}^{\mu\nu}$), dilatations ($\mathbb{D}$), translations ($\mathbb{P}^{\mu}$) and special conformal boosts ($\mathbb{K}^{\mu}$), acting on fundamental (gauge, gaugino, scalars) fields $\phi_I(x)$ with conformal weight d_ϕ and Lorentz indices I [7]

$$
\begin{aligned}
\mathbb{M}^{\mu\nu}\phi_I &= (x^\mu\partial^\nu - x^\nu\partial^\mu)\phi_I + (m^{\mu\nu})_I{}^J\phi_J \\
\mathbb{D}\,\phi_I &= x\cdot\partial\;\phi_I + d_\phi\,\phi_I \\
\mathbb{P}^\mu\phi_I &= \partial^\mu\;\phi_I \\
\mathbb{K}^\mu\phi_I &= \left(2x^\mu x\cdot\partial - x^2\partial^\mu\right)\phi_I + 2x^\mu\,d_\phi\,\phi_I + 2x_\nu(m^{\mu\nu})_I{}^J\phi_J\ ,
\end{aligned}
\tag{22}
$$

where $m^{\mu\nu}$ is the generator of spin rotations, e.g., $m^{\mu\nu}=0$ for a scalar field and $(m^{\mu\nu})_\lambda{}^\rho = g^{\nu\rho}\delta^\mu_\lambda - g^{\mu\rho}\delta^\nu_\lambda$ for a gauge field.

Let us start with the dilatations and perform a change of variables in the functional integral (20), $\phi'_I(x) = \phi_I(x) + \varepsilon\,\mathbb{D}\phi_I(x)$. This change of variables could be compensated by a coordinate transformation $x^{\mu\prime} = (1-\varepsilon)x^\mu$. We recall that the path-ordered exponential is invariant under dilatations, whereas the Lagrangian is covariant with canonical weight $d_{\mathcal{L}} = 4$. However, the measure $\int d^Dx$ with $D = 4-2\epsilon$ does not match the weight of the Lagrangian, which results in a non-vanishing variation of the action S_ϵ

$$
\delta_{\mathbb{D}} S_\epsilon = \frac{2\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ \mathcal{L}(x)\,. \tag{23}
$$

This variation generates an operator insertion into the expectation value, $\langle \delta_{\mathbb{D}} S_\epsilon\ W_n\rangle$, and yields an anomalous term in the action of the dilatation generator on $\langle W_n\rangle$

$$
\mathbb{D}\langle W_n\rangle = \sum_{i=1}^{n}(x_i\cdot\partial_i)\ \langle W_n\rangle = \frac{2i\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ \langle\mathcal{L}(x)W_n\rangle\ . \tag{24}
$$

In a similar manner, the anomalous special conformal (or conformal boost) Ward identity is derived by performing transformations generated by the operator $\mathbb{K}^\nu$, Eq. (22), on both sides of (20). In this case the nonvanishing variation of the action $\delta_{\mathbb{K}^\mu}S_\epsilon$ comes again from the mismatch of the conformal weights of the Lagrangian and of the measure $\int d^Dx$.[8] This amounts to considering just the d_ϕ term in (22) with $d_\phi = d_{\mathcal{L}} - D = 2\epsilon$, hence

$$
\mathbb{K}^\nu\langle W_n\rangle = \sum_{i=1}^{n}(2x_i^\nu x_i\cdot\partial_i - x_i^2\partial_i^\nu)\langle W_n\rangle = \frac{4i\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ x^\nu\ \langle\mathcal{L}(x)W_n\rangle\ . \tag{25}
$$

The relations (24) and (25) can be rewritten as

$$
\begin{aligned}
\mathbb{D}\ln\langle W_n\rangle &= -\frac{2i\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ \frac{\langle\mathcal{L}(x)W_n\rangle}{\langle W_n\rangle}\ , \\
\mathbb{K}^\nu\ln\langle W_n\rangle &= \frac{4i\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ x^\nu\ \frac{\langle\mathcal{L}(x)W_n\rangle}{\langle W_n\rangle}\ .
\end{aligned}
\tag{26}
$$

[7] The generators $\mathbb{G}$ determine the infinitesimal transformations with parameters ε: $\phi'(x) = \phi(x) + \varepsilon\cdot\mathbb{G}\phi(x)$.

[8] Another source of non-invariance of the action S_ϵ is the gauge-fixing term (and the associated ghost term of the non-Abelian theory) which is not conformally invariant even in four dimensions. However, due to gauge invariance of the Wilson loop, such anomalous terms do not appear on the right-hand side of (25).

To make use of these relations we have to evaluate the ratio $\langle\mathcal{L}(x)W_n\rangle/\langle W_n\rangle$ obtained by inserting the Lagrangian into the Wilson loop expectation value. Due to the presence of ϵ on the right-hand side of (26), it is sufficient to know only its divergent part.

3.2 Dilatation Ward identity

As we will now show, the dilatation Ward identity can be derived by dimensional arguments and this provides a consistency condition on the right-hand side of (26). By definition (20), the dimensionally regularized light-like Wilson loop $\langle W_n\rangle$ is a dimensionless scalar function of the cusp points x_i^ν and, as a consequence, it satisfies the relation

$$\left(\sum_{i=1}^{n}(x_i\cdot\partial_i)-\mu\frac{\partial}{\partial\mu}\right)\ln\langle W_n\rangle=0\,. \tag{27}$$

In addition, its perturbative expansion is expressed in powers of the coupling $g^2\mu^{2\epsilon}$ and, therefore,

$$\mu\frac{\partial}{\partial\mu}\langle W_n\rangle=2\epsilon g^2\frac{\partial}{\partial g^2}\langle W_n\rangle=-\frac{2i\epsilon}{g^2\mu^{2\epsilon}}\int d^Dx\ \langle\mathcal{L}(x)W_n\rangle\ , \tag{28}$$

where the last relation follows from (20).

The Wilson loop $\langle W_n\rangle$ can be split into the product of divergent and finite parts, Eq. (11). Notice that the definition of the divergent part is ambiguous as one can always add to $\ln Z_n$ a term finite for $\epsilon\to 0$. Our definition (12) is similar to the conventional MS scheme with the only difference that we choose the expansion parameter to be $a\mu^{2\epsilon}$ instead of a. The reason for this is that Z_n satisfies, in our scheme, the same relation $\mu\partial_\mu Z_n=2\epsilon g^2\partial_{g^2}Z_n$ as $\langle W_n\rangle$ (28). Together with (11) and (27), this implies that the finite part of the Wilson loop does not depend on the renormalization point, i.e.

$$\mu\partial_\mu F_n=O(\epsilon)\,. \tag{29}$$

Writing $\ln\langle W_n\rangle=\ln Z_n+\ln F_n$, and using the explicit form of Z_n in (12), the relation (28) leads to the following dilatation Ward identity[9]

$$\sum_{i=1}^{n}(x_i\cdot\partial_{x_i})\,F_n=0 \tag{30}$$

Adding to this the obvious requirement of Poincaré invariance, we conclude that the finite part F_n of the light-like Wilson loop can only depend on the dimensionless ratios x_{ij}^2/x_{kl}^2. In particular, for $n=4$ there is only one such independent ratio, i.e. $F_4=F_4\left(x_{13}^2/x_{24}^2\right)$.

Making use of (11), (12) and (30) we find that the all-loop dilatation Ward identity for $\langle W_n\rangle$ takes the form

$$\mathbb{D}\ln\langle W_n\rangle=\sum_{i=1}^{n}(x_i\cdot\partial_i)\ \ln\langle W_n\rangle=-\frac{1}{2}\sum_{l\geq 1}a^l\sum_{i=1}^{n}\left(-x_{i-1,i+1}^2\mu^2\right)^{l\epsilon}\left(\frac{\Gamma_{\rm cusp}^{(l)}}{l\epsilon}+\Gamma^{(l)}\right)\,. \tag{31}$$

This relation provides a constraint on the form of the Lagrangian insertion on the right-hand side of (26).

[9]In what follows, we shall systematically neglect corrections to F_n vanishing as $\epsilon\to 0$.

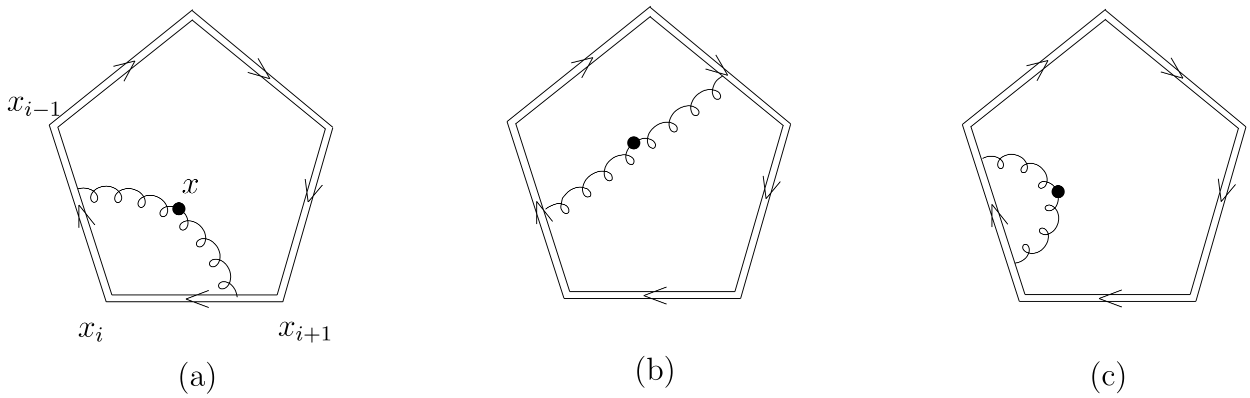

Figure 3: The Feynman diagrams contributing to $\langle \mathcal{L}(x) W_n \rangle$ to the lowest order in the coupling. The double line depicts the integration contour C_n, the wiggly line the gluon propagator and the blob the insertion point.

3.3 One-loop calculation of the anomaly

The derivation of the dilatation Ward identities (30) relied on the known structure of cusp singularities (12) of the light-like Wilson loop and did not require a detailed knowledge of the properties of Lagrangian insertion $\langle \mathcal{L}(x) W_n \rangle / \langle W_n \rangle$. This is not the case anymore for the special conformal Ward identity.

To start with, let us perform an explicit one-loop computation of $\langle \mathcal{L}(x) W_n \rangle / \langle W_n \rangle$. To the lowest order in the coupling, we substitute $\langle W_n \rangle = 1 + O(g^2)$ and retain inside $\mathcal{L}(x)$ and W_n only terms quadratic in gauge field. The result is

$$\frac{\langle \mathcal{L}(x) W_n \rangle}{\langle W_n \rangle} = -\tfrac{1}{8} \left\langle \mathrm{Tr}\left[(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))^2\right] \mathrm{Tr}\left(\oint_{C_n} dy \cdot A(y) \right)^2 \right\rangle + O(g^6) \tag{32}$$

The Wick contractions between gauge fields coming from the Lagrangian and the path-ordered exponential yield a product of two gluon propagators (7), each connecting the point x with an arbitrary point y on the integration contour C_n. Gauge invariance allows us to choose, e.g., the Feynman gauge in which the gluon propagator is given by (8). To the lowest order in the coupling constant, the right-hand side of (32) receives non-vanishing contributions only from Feynman diagrams of three different topologies shown in Figs. 3(a) – (c).

In the Feynman gauge, only the vertex-like diagram shown in Fig. 3(a) develops poles in ϵ. Going through the calculation, we find after some algebra[10]

$$\frac{2i}{g^2 \mu^{2\epsilon}} \frac{\langle \mathcal{L}(x) W_n \rangle}{\langle W_n \rangle} = -a \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \mu^2\right)^\epsilon \left\{ \epsilon^{-2} \delta^{(D)}(x - x_i) + \epsilon^{-1} \Upsilon^{(1)}(x | x_{i-1}, x_i, x_{i+1}) + O(\epsilon^0) \right\} \tag{33}$$

where $a = g^2 N/(8\pi^2)$ and the notation was introduced for

$$\Upsilon^{(1)}(x|x_{i-1}, x_i, x_{i+1}) = \int_0^1 \frac{ds}{s} \left[\delta^{(D)}(x - x_i - s x_{i-1,i}) + \delta^{(D)}(x - x_i + s x_{i,i+1}) - 2\delta^{(D)}(x - x_i) \right] \tag{34}$$

[10] It is advantageous to perform the calculation by taking a Fourier transform with respect to x and later take the inverse transform of the final result.

We see that the leading double-pole singularities are localized at the cusp points $x = x_i$. The subleading single poles are still localized on the contour, but they are 'smeared' along the light-like edges adjacent to the cusp.

Substitution of (33) into (26) yields

$$\mathbb{D}\ln\langle W_n\rangle = -a\sum_{i=1}^{n}\left(-x_{i-1,i+1}^2\mu^2\right)^{\epsilon}\epsilon^{-1} + O(a^2),$$
$$\mathbb{K}^{\nu}\ln\langle W_n\rangle = -2a\sum_{i=1}^{n}x_i^{\nu}\left(-x_{i-1,i+1}^2\mu^2\right)^{\epsilon}\epsilon^{-1} + O(a^2)\ . \tag{35}$$

Notice that $\sum_{i=1}^{n}\Upsilon^{(1)}(x|x_{i-1},x_i,x_{i+1})$ does not contribute to the right-hand sides of these relations by virtue of

$$\int d^D x\,\Upsilon^{(1)}(x|x_{i-1},x_i,x_{i+1}) = 0\,,$$
$$\int d^D x\,x^{\nu}\Upsilon^{(1)}(x|x_{i-1},x_i,x_{i+1}) = (x_{i-1}+x_{i+1}-2x_i)^{\nu}\ . \tag{36}$$

As was already mentioned, the right-hand sides of the Ward identities (35) are different from zero due to the fact that the light-like Wilson loop has cusp singularities.

We verify with a help of (13) that to the lowest order in the coupling, the first relation in (35) is in agreement with (31).

3.4 Structure of the anomaly to all loops

To extend the analysis of the special conformal Ward identity to all loops we examine the all-loop structure of divergences in $\langle\mathcal{L}(x)W_n\rangle/\langle W_n\rangle$. They arise in a very similar way to those in the Wilson loop itself, which were discussed in section 2.2.2.

It is convenient to couple $\mathcal{L}(x)$ to an auxiliary 'source' $J(x)$ and rewrite the insertion of the Lagrangian into $\langle W_n\rangle$ as a functional derivative

$$\langle\mathcal{L}(x)W_n\rangle/\langle W_n\rangle = -i\frac{\delta}{\delta J(x)}\ln\langle W_n\rangle_J\Big|_{J=0} \tag{37}$$

where the subscript J indicates that the expectation value is taken in the $\mathcal{N}=4$ SYM theory with the additional term $\int d^D x J(x)\mathcal{L}(x)$ added to the action. This generates new interaction vertices inside Feynman diagrams for $\langle W_n\rangle_J$ but does not affect the non-Abelian exponentiation property (18).[11] The only difference compared with (18) is that the webs $w^{(k)}$ now depend on $J(x)$ through a new interaction vertex. Making use of the non-Abelian exponentiation we obtain

$$\langle\mathcal{L}(x)W_n\rangle/\langle W_n\rangle = \sum_{k=1}^{\infty}\left(\frac{g^2}{4\pi^2}\right)^k c^{(k)}\left[-i\frac{\delta w^{(k)}}{\delta J(x)}\right]\Big|_{J=0}\ . \tag{38}$$

[11] We recall that the non-Abelian exponentiation is not sensitive to the form of the action.

Similar to $\ln\langle W_n\rangle$, the webs produce at most double poles in ϵ localized at a given cusp. This allows us to write a general expression for divergent part of the Lagrangian insertion:

$$
\begin{aligned}
&\frac{2i\epsilon}{g^2\mu^{2\epsilon}}\,\frac{\langle\mathcal{L}(x)W_n\rangle}{\langle W_n\rangle} \\
&= -\sum_{l\geq 1} a^l \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2\mu^2\right)^{l\epsilon}\left\{\frac{1}{2}\left(\frac{\Gamma_{\rm cusp}^{(l)}}{l\epsilon}+\Gamma^{(l)}\right)\delta^{(D)}(x-x_i)+\Upsilon^{(l)}(x;x_{i-1},x_i,x_{i+1})\right\}+O(\epsilon)\ ,
\end{aligned}
\tag{39}
$$

which generalizes (33) to all loops. The following comments are in order.

In Eq. (39), the term proportional to $\delta^{(D)}(x-x_i)$ comes from the double pole contribution to the web $w^{(k)}$ which is indeed located at the short distances in the vicinity of the cusp x_i^μ. The residue of the simple pole in the right-hand side of (39) is given by the cusp anomalous dimension. This can be shown by substituting (39) into the dilatation Ward identity (24) and comparing with (31).

The contact nature of the leading singularity in (39) can also be understood in the following way. The correlator on the left-hand side can be viewed as a conformal $n+1$-point function with the Lagrangian at one point and the rest corresponding to the cusps. In the $\mathcal{N}=4$ SYM theory the Lagrangian belongs to the protected stress-tensor multiplet, therefore it has a fixed conformal dimension four. The Wilson loop itself, if it were not divergent, would be conformally invariant. This means that the n cusp points can be regarded as having vanishing conformal weights. Of course, the presence of divergences might make the conformal properties anomalous. However, the conformal behaviour of the leading singularity in (39) cannot be corrected by an anomaly.[12] Then we can argue that the only function of space-time points, which has conformal weight four at one point and zero at all other points, is the linear combination of delta functions appearing in (39).[13]

As was already emphasized, the residue of the simple pole of $\delta w^{(k)}/\delta J(x)$ at the cusp point x_i depends on its position and at most on its two nearest neighbors x_{i-1} and x_{i+1}, as well as on the insertion point. It gives rise to a function $\Upsilon^{(l)}(x;x_{i-1},x_i,x_{i+1})$, which is the same for all cusp points due to the cyclic symmetry of the Wilson loop (5).

Notice that in (39) we have chosen to separate the terms with the collinear anomalous dimension $\Gamma^{(l)}$ from the rest of the finite terms. This choice has implications for the function $\Upsilon^{(l)}(x;x_{i-1},x_i,x_{i+1})$. Substituting the known factor Z_n (12) and the particular form of (39) into the dilatation Ward identity (24), we derive

$$
\sum_{i=1}^{n}\int d^Dx\ \Upsilon^{(l)}(x;x_{i-1},x_i,x_{i+1})=0\ . \tag{40}
$$

We can argue that in fact each term in this sum vanishes. Indeed, each term in the sum is a Poincaré invariant dimensionless function of three points x_{i-1}, x_i and x_{i+1}. Given the light-like separation of the neighboring points, the only available invariant is $x_{i-1,i+1}^2$. Further, $\Upsilon^{(l)}(x;x_{i-1},x_i,x_{i+1})$ cannot depend on the regularization scale because μ always comes in the

[12] Such an anomalous contribution should come from a $1/\epsilon^3$ pole in the correlator with two insertions of the Lagrangian, but repeated use of the argument above shows that the order of the poles does not increase with the number of insertions.

[13] The mismatch of the conformal weight four of the Lagrangian and $D=4-2\epsilon$ of the delta functions does not affect the leading singularity in (39).

combination $a\mu^{2\epsilon}$ and thus contributes to the $O(\epsilon)$ terms in (39). The dimensionless Poincaré invariant $\int d^Dx\, \Upsilon^{(l)}(x; x_{i-1}, x_i, x_{i+1})$ depends on a single scale and, therefore, it must be a constant after which (40) implies

$$\int d^Dx\ \Upsilon^{(l)}(x; x_{i-1}, x_i, x_{i+1}) = 0\ . \tag{41}$$

For $l = 1$ this relation is in agreement with the one-loop result (36).

3.5 Special conformal Ward identity

We are now ready to investigate the special conformal Ward identity (26). Inserting (39) into the right-hand side and integrating over x we obtain

$$\begin{aligned} \mathbb{K}^\nu \ln W_n &= \sum_{i=1}^{n} (2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \ln W_n \\ &= -\sum_{l\geq 1} a^l \left(\frac{\Gamma^{(l)}_{\rm cusp}}{l\epsilon} + \Gamma^{(l)} \right) \sum_{i=1}^{n}\ \left(-x^2_{i-1,i+1}\mu^2\right)^{l\epsilon}\ x_i^\nu - 2\sum_{i=1}^{n} \Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) + O(\epsilon)\ , \end{aligned} \tag{42}$$

where

$$\Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) = \sum_{l\geq 1} a^l \int d^Dx\, x^\nu \Upsilon^{(l)}(x; x_{i-1}, x_i, x_{i+1})\ . \tag{43}$$

Next, we substitute $\ln W_n = \ln Z_n + \ln F_n$ into (42), replace $\ln Z_n$ by its explicit form (12) and expand the right-hand side in powers of ϵ to rewrite (42) as follows:

$$\mathbb{K}^\nu \ln F_n = \frac{1}{2}\Gamma_{\rm cusp}(a) \sum_{i=1}^{n} \ln \frac{x^2_{i,i+2}}{x^2_{i-1,i+1}} x^\nu_{i,i+1} - 2\sum_{i=1}^{n} \Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) + O(\epsilon)\ . \tag{44}$$

Note that the quantities $\Upsilon^\nu(x_{i-1}, x_i, x_{i+1})$ are translation invariant. Indeed, a translation under the integral in (43) only affects the factor x^ν (the functions $\Upsilon^{(l)}(x; x_{i-1}, x_i, x_{i+1})$ are translation invariant), but the result vanishes as a consequence of (41). Furthermore, $\Upsilon^\nu(x_{i-1}, x_i, x_{i+1})$ only depends on two neighboring light-like vectors $x^\mu_{i-1,i}$ and $x^\mu_{i,i+1}$, from which we can form only one non-vanishing Poincaré invariant, $x^2_{i-1,i+1}$. We have already argued that $\Upsilon^{(l)}$ are independent of μ, and so must be Υ^ν. Taking into account the scaling dimension one of Υ^ν, we conclude that

$$\Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) = \alpha x^\nu_{i-1,i} + \beta x^\nu_{i,i+1}\ , \tag{45}$$

where α, β only depend on the coupling. The symmetry of the Wilson loop W_n under mirror exchange of the cusp points, Eq. (5), translates into symmetry of (45) under exchange of the neighbors x_{i-1} and x_{i+1} of the cusp point x_i, which reduces (45) to

$$\Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) = \alpha\, (x^\nu_{i-1} + x^\nu_{i+1} - 2x^\nu_i)\ , \tag{46}$$

with $\alpha = a + O(a^2)$ according to (36). Substituting this relation into (44) we find

$$\sum_{i=1}^{n} \Upsilon^\nu(x_{i-1}, x_i, x_{i+1}) = 0\ . \tag{47}$$

This concludes the derivation of the special conformal Ward identity. In the limit $\epsilon \to 0$ it takes the form (cf. [9])

$$\sum_{i=1}^{n}(2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \ln F_n = \frac{1}{2}\Gamma_{\rm cusp}(a) \sum_{i=1}^{n} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu \,. \tag{48}$$

3.6 Solution and implications for W_n

Let us now examine the consequences of the conformal Ward identity (48) for the finite part of the Wilson loop W_n. We find that the cases of $n = 4$ and $n = 5$ are special because here the Ward identity (48) has a unique solution up to an additive constant. The solutions are, respectively,

$$\begin{aligned} \ln F_4 &= \frac{1}{4}\Gamma_{\rm cusp}(a) \ln^2\Big(\frac{x_{13}^2}{x_{24}^2}\Big) + \text{ const} \\ \ln F_5 &= -\frac{1}{8}\Gamma_{\rm cusp}(a) \sum_{i=1}^{5} \ln\Big(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\Big) \ln\Big(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\Big) + \text{ const} \quad . \end{aligned} \tag{49}$$

as can be easily verified by making use of identity $\mathbb{K}^\mu x_{ij}^2 = 2(x_i^\mu + x_j^\mu)x_{ij}^2$. We find that, upon identification of the kinematical invariants

$$x_{k,k+r}^2 := (p_k + \ldots + p_{k+r-1})^2 \,, \tag{50}$$

the relations (49) are exactly the functional forms of the ansatz of [1] for the finite parts of the four- and five-point MHV amplitudes (or rather the ratio of the amplitude to the corresponding tree amplitude).

The reason why the functional form of F_4 and F_5 is fixed up to an additive constant is that there are no conformal invariants one can build from four or five points x_i with light-like separations $x_{i,i+1}^2 = 0$. Such invariants take the form of cross-ratios[14]

$$\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} \,. \tag{51}$$

It is obvious that with four or five points they cannot be constructed. This becomes possible starting with six points, where there are three such cross-ratios, e.g.,

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} \,. \tag{52}$$

Hence the general solution of the Ward identity at six cusp points and higher will contain an arbitrary function of the conformal cross-ratios.

4 Light-like pentagon Wilson loop

In this section we perform an explicit two-loop calculation of the light-like pentagon Wilson loop $W(C_5)$ in $\mathcal{N} = 4$ SYM theory. It goes along the same line as the analysis of the rectangular Wilson loop $W(C_4)$ performed in Refs. [9, 6], where the interested reader could find the details of the technique employed.

[14]For n generic points x_i^μ, $i = 1 \ldots n$ in a D-dimensional space-time the number of invariants is $(n-1)(n-2)/2-1$ if $n \le D+1$ and $nD-(D+1)(D+2)/2$ if $n > D+1$. The additional conditions of light-like separations, $x_{i,i+1}^2 = 0$, remove n of them.

4.1 One loop result

The one-loop calculation of $\ln W(C_5)$ was done in [13]. The relevant Feynman diagrams are shown in Fig. 4(a) and Fig. 4(b).

Let us split $\ln W(C_5)$ into divergent, D, and finite, F, parts

$$\ln W(C_5) = \frac{g^2}{4\pi^2} C_F \left[D^{(1)} + F^{(1)}\right] + O(g^4) \tag{53}$$

where by definition

$$\begin{aligned} D^{(1)} &= -\frac{1}{2\epsilon^2}\sum_{i=1}^{5}\left(-x^2_{i,i+2}\,\mu^2\right)^{\epsilon} \\ F^{(1)} &= -\frac{1}{4}\sum_{i=1}^{5}\ln\Big(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\Big)\ln\Big(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\Big) + \frac{5\pi^2}{24} + O(\epsilon) \end{aligned} \tag{54}$$

We verify that (53) also fulfills the duality relation (1). To this end, we substitute (53) into the duality relation (1) and apply (4) to identify the coordinates $x^\mu_{i,i+1}$ with the on-shell gluon momenta p^μ_i. This leads to

$$x^2_{i,i+2} := (p_i + p_{i+1})^2 \equiv s_{i,i+1}\,, \qquad x^2_{i,i+3} = x^2_{i,i-2} := (p_{i-2}+p_{i-1})^2 \equiv s_{i-1,i-2}\,, \tag{55}$$

where $s_{jk} = (p_j + p_k)^2$ are the Mandelstam invariants corresponding to the five-gluon amplitude. A specific feature of the $n = 5$ on-shell gluon amplitude as compared with $n \geq 6$ is that it only depends on two-particle invariants $s_{i,i+1}$. Similarly, for the light-like Wilson loop one finds that $W(C_5)$ only depends on the distances $x^2_{i,i+2}$ between next-to-neighboring vertices on the contour C_5. Then we observe that, firstly, upon identification of the dimensional regularization parameters

$$x^2_{i,i+2}\,\mu^2 := s_{i,i+1}/\mu^2_{\rm IR}\,, \qquad x^2_{i,i+2}/x^2_{k,k+2} := s_{i,i+1}/s_{k,k+1} \tag{56}$$

the UV divergences of the light-like Wilson loop match the IR divergent part of the five-gluon scattering amplitude and, secondly, the finite corrections to these two objects indeed coincide to one loop. In the next section, we extend the analysis beyond one loop and demonstrate that the planar $n = 5$ gluon amplitude/pentagon Wilson loop duality also holds to two loops.

4.2 Two-loop calculation

As was explained in detail in [9], the two-loop calculation of $W(C_5)$ can be significantly simplified by making use of the non-Abelian exponentiation property of Wilson loops [24]. In application to $W(C_5)$, it can be formulated as follows:

$$\ln W(C_5) = \frac{g^2}{4\pi^2} C_F\, w^{(1)} + \left(\frac{g^2}{4\pi^2}\right)^2 C_F N\, w^{(2)} + O(g^6)\,, \tag{57}$$

where $w^{(1)}$ and $w^{(2)}$ are functions of the distances $x_{i,i+2}$, independent of the Casimirs of the gauge group $SU(N)$. Matching (57) into (53) we find that

$$w^{(1)} = D^{(1)} + F^{(1)} \tag{58}$$

with $D^{(1)}$ and $F^{(1)}$ defined in (54). The relation (57) implies that the coefficient in front of $g^4C_F^2/(4\pi^2)^2$ in the two-loop expression for the pentagon Wilson loop $W(C_5)$ is given by $\frac{1}{2}\big(w^{(1)}\big)^2$ and, therefore, it is uniquely determined by the one-loop correction to $W(C_5)$. Thus, in order to determine the function $w^{(2)}$ it sufficient to calculate the contribution to $W(C_5)$ only from two-loop diagrams containing 'maximally non-Abelian' color factor C_FN. This property allows us to reduce significantly the number of relevant two-loop diagrams. In addition, as yet another advantage of using the Feynman gauge, we observe [9] that some of the 'maximally non-Abelian' diagrams like those where both ends of a gluon are attached to the same light-like segment vanish by virtue of $x_{j,j+1}^2 = 0$. The corresponding Feynman diagrams have the same topology as for the rectangular Wilson loop $W(C_4)$ and they can be easily identified by applying the selection rules formulated in Ref. [9].

To summarize, in Figs. 4 (c) – (o) we list all non-vanishing two-loop diagrams of different topologies contributing to $w^{(2)}$. The diagrams in Figs. 4 (c) – (l) have the same topology as for the rectangular Wilson loop (see Ref. [9]), while the diagrams in Figs. 4 (m) – (o) are specific to the pentagon Wilson loop $W(C_5)$. The diagrams in Figs. 4 (f), (g), (l) and (o) involve the three-gluon interaction vertex of the $\mathcal{N}=4$ SYM Lagrangian. Their color factors equal C_FN, and therefore they contribute to the function $w^{(2)}$ in (57). The diagrams shown in Figs. 4 (d), (e), (h), (i), (j), (m) and (n) are non-planar and their color factors equal $C_F(C_F - N/2)$. To identify their contribution to $w^{(2)}$, we have to retain its maximally non-Abelian part only, that is, to replace their color factors by $C_F(C_F - N/2) \to -C_FN/2$. Finally, the diagrams in Figs. 4(c) and (k) involve the one-loop correction to the gluon propagator with the blob denoting gauge fields/gauginos/scalars/ghosts propagating along the loop. Their color factors equal C_FN and they directly contribute to $w^{(2)}$. To preserve supersymmetry, we evaluate these diagrams within the dimensional reduction (DRED) scheme. The two-loop correction $w^{(2)}$ is given by the sum over the individual diagrams shown in Fig. 4 plus crossing symmetric diagrams. Computing these diagrams, we employ the technique developed in Refs. [25, 6]. The result of our calculation can be summarized as follows. It is convenient to expand the contribution of each diagram in powers of $1/\epsilon$ and separate the UV divergent and finite parts as

$$w^{(2)} = \sum_\alpha \left\{ \frac{1}{2}\left(\frac{1}{\epsilon^4}A_{-4}^{(\alpha)} + \frac{1}{\epsilon^3}A_{-3}^{(\alpha)} + \frac{1}{\epsilon^2}A_{-2}^{(\alpha)} + \frac{1}{\epsilon}A_{-1}^{(\alpha)}\right)\sum_{i=1}^{5}\left(-x_{i,i+2}^2\,\mu^2\right)^{2\epsilon} + A_0^{(\alpha)}\right\} + O(\epsilon)\,, \quad (59)$$

where the sum goes over the two-loop Feynman diagrams of different topologies shown in Fig. 4(c)–(o) and the factor 1/2 has been inserted for later convenience. Here $A_{-n}^{(\alpha)}$ (with $0 \le n \le 4$) are dimensionless functions of the ratio of distances $x_{k,k+2}^2$ (with $k = 1, \dots, 5$). Making use of (59), we parameterize the contribution of each individual diagram to $w^{(2)}$ by the set of coefficient functions $A_{-n}^{(\alpha)}$. We would like to stress that the contribution of each individual diagram to $w^{(2)}$, or equivalently the functions $A_{-n}^{(\alpha)}$, are gauge dependent and it is only their sum in the right-hand side of (59) that remains gauge invariant.

In our analysis, we have heavily used the results of the two-loop calculation of the rectangular light-like Wilson loop $W(C_4)$ [6, 9]. The expression for $W(C_4)$ has the same general form as $W(C_5)$, Eqs. (57) and (59), with the corresponding coefficient functions $A_{-n}^{(\alpha)}$ known explicitly. There are no reasons to expect *a priori* that $W(C_4)$ and $W(C_5)$ should be related to each other. Still, as we will see in a moment, there exist remarkable relations between the contributions of various diagrams to the two Wilson loops. As an example, let us consider the vertex-like diagrams shown in Fig. 4(c), 4(d) and 4(f). The corresponding Feynman integrals only depend

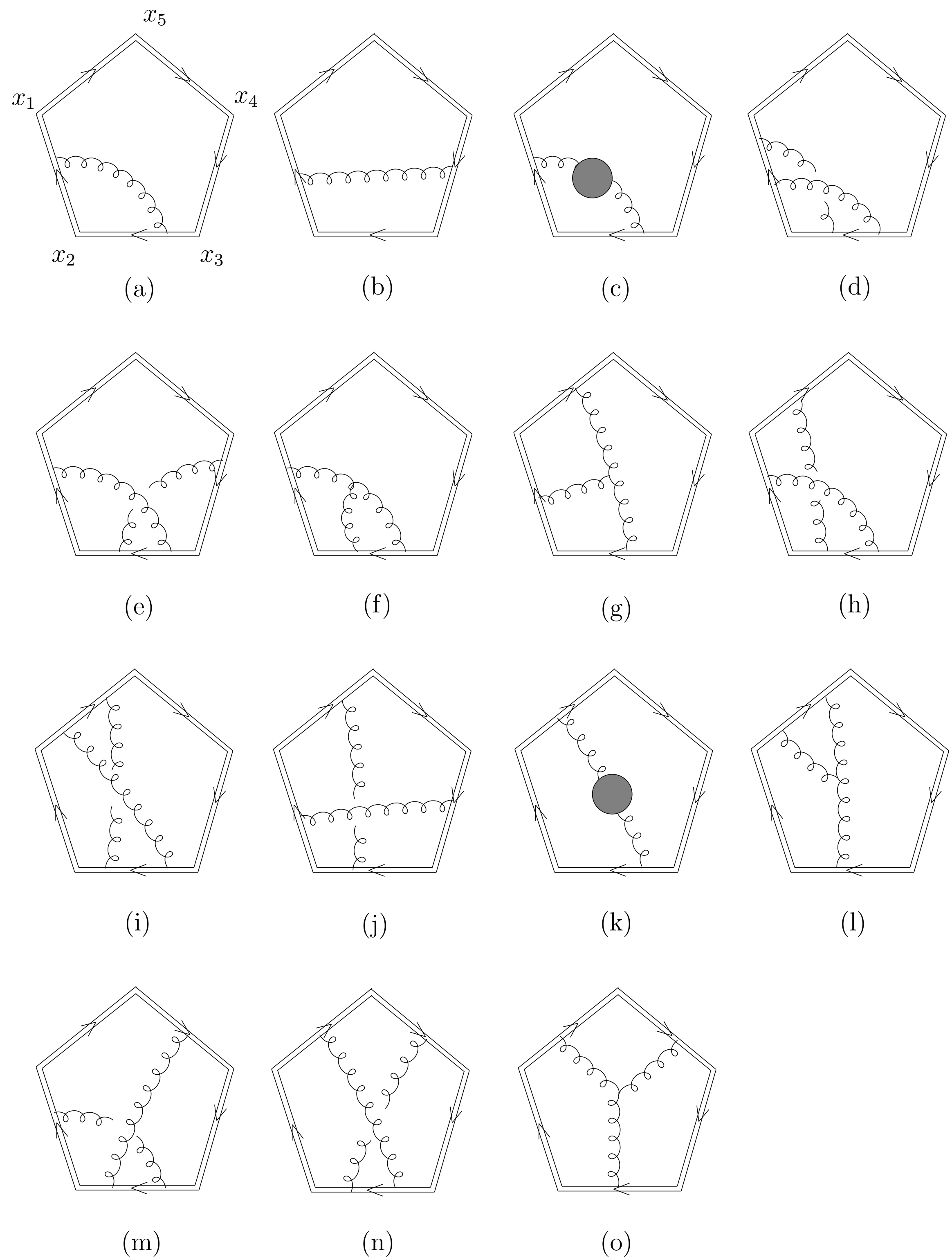

Figure 4: The Feynman diagrams contributing to $\ln W(C_5)$ to two loops. The double line depicts the integration contour C_5, the wiggly line the gluon propagator and the blob the one-loop polarization operator.

on the single distance x_{13}^2, and therefore they give the same contributions to $W(C_4)$ and $W(C_5)$. In a similar manner, the contribution of the diagram shown in Fig. 4(e) only depends on two distances, x_{13}^2 and x_{24}^2, and as a consequence it is the same for $W(C_4)$ and $W(C_5)$. At the same time, examining the diagram shown in Fig. 4(g) we found that it gives the same contribution to $W(C_4)$ and $W(C_5)$ for the leading, $O(1/\epsilon^2)$ term but this is not true for the $O(1/\epsilon)$ term, as well as for the finite $O(\epsilon^0)$ term.

Summarizing our results for $W(C_5)$, we find that the coefficient functions entering the right-hand side of (59) are given by

- UV divergent $O(1/\epsilon^4)$ terms in (59) only come from the two Feynman diagrams shown in Figs. 4(d) and 4(f). They coincide with those for the rectangular Wilson loop $W(C_4)$

$$A_{-4}^{(\mathrm{d})} = -\frac{1}{16}\,, \qquad A_{-4}^{(\mathrm{f})} = \frac{1}{16} \tag{60}$$

- UV divergent $O(1/\epsilon^3)$ terms in (59) only come from the two Feynman diagrams shown in Figs. 4(c) and 4(f). They coincide with those for the rectangular Wilson loop $W(C_4)$

$$A_{-3}^{(\mathrm{c})} = \frac{1}{8}\,, \qquad A_{-3}^{(\mathrm{f})} = -\frac{1}{8} \tag{61}$$

- UV divergent $O(1/\epsilon^2)$ terms only come from the Feynman diagrams shown in Figs. 4(c)–4(g). They coincide with those for the rectangular Wilson loop $W(C_4)$

$$A_{-2}^{(\mathrm{c})} = \frac{1}{4}\,, \qquad A_{-2}^{(\mathrm{d})} = -\frac{\pi^2}{96}\,, \qquad A_{-2}^{(\mathrm{e})} = -\frac{\pi^2}{24}\,, \qquad A_{-2}^{(\mathrm{f})} = -\frac{1}{4}+\frac{5}{96}\pi^2\,, \qquad A_{-2}^{(\mathrm{g})} = \frac{\pi^2}{48} \tag{62}$$

- UV divergent $O(1/\epsilon^1)$ terms come from the Feynman diagrams shown in Figs. 4(c)–4(h),4(k) – 4(m) and 4(o).

- Finite $O(\epsilon^0)$ terms come from all Feynman diagrams shown in Figs. 4(c)–4(o).

  The expressions for $A_{-1,0}^{(\mathrm{c})}$, $A_{-1,0}^{(\mathrm{d})}$, $A_{-1,0}^{(\mathrm{e})}$ and $A_{-1,0}^{(\mathrm{f})}$ are the same as for the rectangular Wilson loop [9], while the remaining coefficients $A_{-1}-$ and A_0- are given by complicated functions of the distances $x_{i,i+2}^2$. Instead of calculating each of them separately, below we determine their total sum.

We would like to stress that the relations (60), (61) and (62) are not specific to the pentagon Wilson loop. The same relations hold true for an arbitrary $n-$polygon light-like Wilson loop $W(C_n)$ (with $n \geq 4$). They ensure that, independently on the number of light-like segments, the $O(1/\epsilon^4)$ and $O(1/\epsilon^3)$ terms cancel inside $w^{(2)}$ in the sum of all diagrams leading to

$$w^{(2)} = \left\{ \epsilon^{-2}\frac{\pi^2}{96} + \epsilon^{-1}\frac{1}{2}\sum_\alpha A_{-1}^{(\alpha)} \right\} \sum_{i=1}^{n} (-x_{i,i+2}^2\,\mu^2)^{2\epsilon} + O(\epsilon^0)\,. \tag{63}$$

Substituting this relation into (57), we find that it agrees with the expected structure of UV divergences of a light-like Wilson loop in planar $\mathcal{N}=4$ SYM theory,

$$\begin{aligned} \ln W(C_n) &= a\, w^{(1)} + 2a^2\, w^{(2)} + O(a^3) \\ &= -\frac{1}{4}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\mathrm{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right) \sum_{i=1}^{n} (-x_{i,i+2}^2\,\mu^2)^{l\epsilon} + O(\epsilon^0)\,. \end{aligned} \tag{64}$$

Here $a = g^2N/(8\pi^2)$ is the 't Hooft coupling, $\Gamma_{\rm cusp}(a) = \sum_{l=1}^{\infty} a^l \Gamma_{\rm cusp}^{(l)}$ is the cusp anomalous dimension and $\Gamma(a) = \sum_{l=1}^{\infty} a^l \Gamma^{(l)}$ is the collinear anomalous dimension given by (13). coefficient in front of $1/\epsilon$ in the right-hand side of (63) should be equal to

$$\sum_{\alpha} A_{-1}^{(\alpha)} = \frac{7}{8}\zeta_3 \,. \tag{65}$$

This relation is extremely non-trivial, given the fact that each individual term in the sum $A_{-1}^{(\alpha)}$ in general depends both on the number of segments n and on the distances x_{jk}^2 between the cusp points on the contour C_n. For the rectangular Wilson loop, the relation (65) has been verified in Ref. [9]. In the next section we show that it also holds true for arbitrary $n \geq 5$.

4.2.1 Structure of the simple poles

As was already mentioned, the coefficients $A_{-1}^{(\alpha)}$ of the simple poles in the right-hand side of (59) are functions of the distances x_{jk}^2. In the simplest case of the rectangular Wilson loop $W(C_4)$ they can be expressed in terms of polylogarithm functions. Going over to the pentagon and, in general, to $n-$polygon light-like Wilson loops, we should expect to encounter even more complicated expressions for the coefficients $A_{-1}^{(\alpha)}$. However, having the relation (65) in mind, we should also expect to find a dramatic simplification in the sum over all diagrams. This suggests to consider the total sum $\sum_\alpha A_{-1}^{(\alpha)}$ rather than analyze each individual term. The next step would be to uncover the mechanism responsible for the above mentioned simplification which could eventually lead to (65). Going through the analysis of the Feynman diagrams shown in Figs. 4(c)–4(h),4(k) – 4(m) and 4(o), we have indeed found such a mechanism.

To start with, let us introduce an auxiliary Feynman integral that will play a crucial rôle in our analysis (see footnote [2]),

$$J(z_1, z_2, z_3) = i(2\pi)^4 \left(\frac{\mu^2}{\pi \mathrm{e}^\gamma}\right)^\epsilon \int d^{4-2\epsilon} z \, G(z-z_1)G(z-z_2)G(z-z_3) \,, \tag{66}$$

with the gluon propagator $G(x)$ defined in (8). Obviously, $J(z_1,z_2,z_3)$ is a symmetric function of the three points z_i^μ (with $i=1,2,3$) in Minkowski space-time. In what follows, we will need its value in the special limit when two of the points are separated by a light-like interval. Assuming that $z_{23}^2 \equiv (z_2 - z_3)^2 = 0$, we find (see second reference in [25])

$$J(z_1, z_2, z_3) \stackrel{z_{23}^2=0}{=} (\mu^2 \mathrm{e}^{-\gamma})^{2\varepsilon} \frac{\Gamma(1-2\varepsilon)}{4\varepsilon} \int_0^1 \frac{d\tau\, (\tau\bar\tau)^{-\varepsilon}}{[-(\tau z_{21} + \bar\tau z_{31})^2]^{1-2\varepsilon}} \,, \tag{67}$$

where the notation was introduced for $\bar\tau = 1-\tau$ and $z_{jk} \equiv z_j - z_k$. Notice that the integral (67) develops a single pole which has a simple physical interpretation [25]. It originates from the integration in (66) over z^μ approaching the light-like direction defined by the vector $(z_2 - z_3)^\mu$, so that the distances $(z-z_2)^2$ and $(z-z_3)^2$ vanish simultaneously and the two propagators in the right-hand side of (66) become singular.

Let us now interpret the integral $J(z_1,z_2,z_3)$ as defining a new fake 'interaction vertex' for three gluons and examine the auxiliary Feynman diagrams involving this vertex with all three points z_i^μ attached to the integration contour C_4 as shown in Fig. 5. The rational for introducing these diagrams is that, as we will see later in this section, they describe the simple pole contribution to the 'genuine' two-loop Feynman diagrams shown in Fig. 4. Notice that one of the gluons in Fig. 5 is attached to the vertex of the pentagon, $z_2 = x_2$, while the positions of the two remaining gluons are integrated over the adjacent and the remaining non-adjacent light-like segments, $z_3 = x_3+p_2\tau_2$ and $z_1 = x_{i+1}+p_i\tau_i$ (with $i = 4, 5$). By definition, the Feynman diagrams associated with the two diagrams shown in Fig. 5 are

$$\begin{aligned} I_{\rm aux}^{\rm (a)} &= (p_2 \cdot p_5) \int_0^1 d\tau_2 \int_0^1 d\tau_5 \, J(x_1 + p_5\tau_5, x_2, x_3 + p_2\tau_2) \\ I_{\rm aux}^{\rm (b)} &= (p_2 \cdot p_4) \int_0^1 d\tau_2 \int_0^1 d\tau_4 \, J(x_5 + p_4\tau_4, x_2, x_3 + p_2\tau_2) \end{aligned} \tag{68}$$

with $p_i = x_i - x_{i+1}$ being light-like vector, $p_i^2 = 0$. Substituting (67) into (68), we obtain expressions for $I_{\rm aux}^{\rm (a,b)}$ in the form of a three-folded integral. According to (67), the integrals $I_{\rm aux}^{\rm (a,b)}$ have a single pole in $1/\epsilon$. Then, we add together the crossing symmetric diagrams of the same topology as in Fig. 5 and expand their contributions in ϵ similar to (59)

$$I_{\rm aux}^{\rm (a,b)} + (\text{cross-symmetry}) = \frac{1}{2\epsilon} M_{-1}^{\rm (a,b)} \sum_{i=1}^{5} \left(-x_{i,i+2}^2\, \mu^2\right)^{2\epsilon} + M_0^{\rm (a,b)} + O(\epsilon)\,. \tag{69}$$

where $M_{-1}^{\rm (a,b)}$ and $M_0^{\rm (a,b)}$ are complicated functions of the distances x_{jk}^2 defined by (68) and (67). We will not need their explicit form for our purposes.

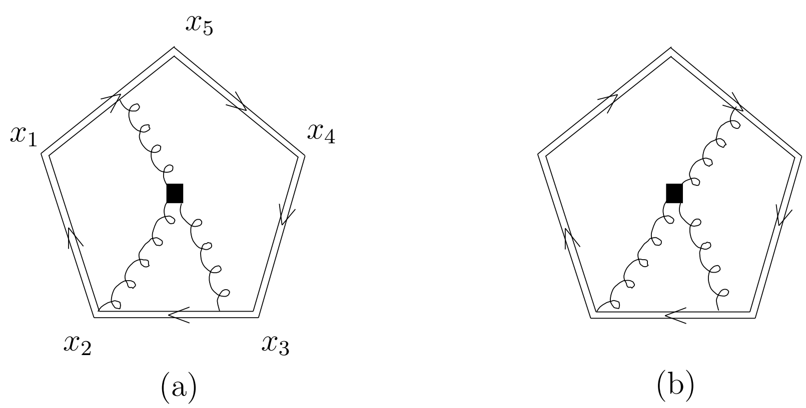

Figure 5: The auxiliary Feynman diagrams defined in (68). The double line depicts the integration contour C_5, the wiggly line the gluon propagator and the box the fake three-gluon vertex (66).

The crucial observation is that the coefficient functions $A_{-1}^{(\alpha)}$ corresponding to the various diagrams shown in Fig. 4 can be expressed in terms of the two functions $M_{-1}^{\rm (a)}$ and $M_{-1}^{\rm (b)}$. More precisely, our analysis leads to the following expressions for the coefficient functions accompanying the simple poles in the pentagon Wilson loop $W(C_5)$:

$$\begin{aligned} &A_{-1}^{\rm (d)} = -\frac{1}{24}\zeta_3\,, && A_{-1}^{\rm (e)} = \frac{1}{2}\zeta_3\,, && A_{-1}^{\rm (c)} + A_{-1}^{\rm (f)} = \frac{7}{24}\zeta_3\,, \\ &A_{-1}^{\rm (g)} = -M_{-1}^{\rm (a)} + \frac{1}{8}\zeta_3\,, && A_{-1}^{\rm (h)} = 2M_{-1}^{\rm (a)}\,, && A_{-1}^{\rm (k)} + A_{-1}^{\rm (l)} = -M_{-1}^{\rm (a)} - M_{-1}^{\rm (b)}\,, \\ &A_{-1}^{\rm (m)} = 2M_{-1}^{\rm (b)}\,, && A_{-1}^{\rm (o)} = -M_{-1}^{\rm (b)}\,. \end{aligned} \tag{70}$$

Putting these functions together we find that the functions $M_{-1}^{\rm (a)}$ and $M_{-1}^{\rm (b)}$ cancel in the sum of all diagrams! Most importantly, their sum equals $\sum_\alpha A_{-1}^{(\alpha)} = \frac{7}{8}\zeta_3$, in perfect agreement with (65).

The following comments are in order.

The relations (70) generalize similar ones for the rectangular Wilson loop (see Eq. (22) in Ref. [9]). The former relations can be recovered by discarding the contribution of $M_{-1}^{(\rm b)}$ specific to the pentagon contour and by identifying the function $M_{-1}^{(\rm a)}$ in the limit $x_5 = x_4$ with the corresponding function $\frac{1}{8}M_2$ (see Eq. (24) in Ref. [9]).

We notice that the expressions entering the right-hand side of (70) have the same transcendentality 3. This is however not the case for $A_{-1}^{(\rm c)}$ and $A_{-1}^{(\rm f)}$ separately (as well as for $A_{-1}^{(\rm k)}$ and $A_{-1}^{(\rm l)}$). Each of these functions also involves terms of lower transcendentality which cancel in the sum.[15]

It is straightforward to extend the analysis to arbitrary $n-$polygon Wilson loops $W(C_n)$. As was already explained, the two-loop corrections to $\ln W(C_n)$, Eq. (64), are described (63) by the sum $\sum_\alpha A_{-1}^{(\alpha)}$ receiving additional contribution from Feynman diagrams of new topologies specific to the $n-$polygon. Remarkably enough, the coefficient functions $A_{-1}^{(\alpha)}$ can again be expressed in terms of the auxiliary Feynman diagrams shown in Fig. 5 in which the gluon coming out of the box is attached to all possible light-like segments not adjacent to the segments $[x_1, x_2]$ and $[x_2, x_3]$. Combining these diagrams together we find that the single poles cancel in their sum leading to (65) for any n.

To summarize, we demonstrated by our explicit two-loop calculation that the divergent part of an $n-$polygon light-like Wilson loop has the expected form (64) with the cusp and collinear anomalous dimensions given by (13).

4.2.2 Finite part

We now turn to the most complicated part of our calculation. The finite part of the pentagon Wilson loop receives contributions from the Feynman diagrams shown in Figs. 4(c)–4(o). As in the previous section, instead of analyzing each of them separately, we will consider the total sum. Furthermore, based on the prediction of the conformal Ward identities we expect that

$$\sum_{\alpha=\rm c,\dots,o} A_0^{(\alpha)} = \frac{\pi^2}{48}\sum_{i=1}^{5} \ln\Big(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\Big)\ln\Big(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\Big) - r\frac{\pi^4}{144}\,. \tag{71}$$

Notice that the constant term in this relation is not determined by the Ward identities, so r is an arbitrary factor. The reason why we wrote it in the form $\sim \pi^4$ is that the first term in the right-hand side of (71) has transcendentality 4. Based on our analysis of the rectangular Wilson loop [9], we expect that the constant term should have the same transcendentality which implies that r should be a rational number.

We recall that the contribution of the diagrams shown in Figs. 4(c)–4(f) to the finite part of $\ln W(C_5)$ can be obtained from the similar expressions for the rectangular Wilson loop [9]

$$A_0^{(\rm d)} = -\frac{5}{4}\cdot\frac{7}{2880}\pi^4\,,\quad A_0^{(\rm c)}+A_0^{(\rm f)} = \frac{5}{4}\cdot\frac{119}{2880}\pi^4\,,\quad A_0^{(\rm e)} = \frac{\pi^2}{96}\sum_{i=1}^{5}\ln^2\left(\frac{x_{i+1,i-1}^2}{x_{i,i-2}^2}\right) - \frac{5}{4}\cdot\frac{19}{720}\pi^4 \tag{72}$$

where the combinatorial factor $\frac{5}{4}$ accounts for the different number of diagrams of the same topology contributing to the pentagon and rectangular Wilson loops. The finite contribution to

[15] We would like to thank Jan Plefka for pointing out this property to us.

$w^{(2)}$ from the remaining diagrams shown in Figs. 4(g)–4(o) has to be calculated anew (with the only exception of the diagram shown in Fig. 4(j) which factorizes into a product of two one-loop integrals corresponding to the diagram shown in Fig. 4(b)). Let us separate them into two groups according to their behavior as $\epsilon \to 0$:

- The diagrams shown in Figs. 4(i), (j), (n) are finite;
- The diagrams shown in Figs. 4(h), (k)-(m),(o) and in Fig. 4(g) have, respectively, simple and double pole in ϵ.

For the second group of diagrams we have to carefully separate the divergent and finite parts following the definition (59). This can easily be done by employing the following 'subtraction procedure'. As was shown in the previous section, the simple poles in these diagrams are described by the functions $M_{-1}^{(a,b)}$ corresponding to the auxiliary diagrams shown in Fig. 5. Therefore, in order to compensate the simple poles in the above mentioned diagrams it is sufficient to subtract from them the auxiliary diagrams with the appropriate weights defined in (70). Since the auxiliary diagrams also generate a finite part $M_0^{(a,b)}$, Eq. (69), the subtractions will modify the finite parts of the individual diagrams, $A_0^{(\alpha)} \to \widehat{A}_0^{(\alpha)}$ with

$$
\begin{aligned}
&\widehat{A}_0^{(\mathrm{g})} = A_0^{(\mathrm{g})} + M_0^{(\mathrm{a})}\,, && \widehat{A}_0^{(\mathrm{h})} = A_0^{(\mathrm{h})} - 2M_0^{(\mathrm{a})}\,, \\
&\widehat{A}_0^{(\mathrm{m})} = A_0^{(\mathrm{m})} - 2M_0^{(\mathrm{b})}\,, && \widehat{A}_0^{(\mathrm{o})} = A_0^{(\mathrm{o})} + M_0^{(\mathrm{b})}\,, \\
&\widehat{A}_0^{(\mathrm{k})+(\mathrm{l})} = A_0^{(\mathrm{k})} + A_0^{(\mathrm{l})} + M_0^{(\mathrm{a})} + M_0^{(\mathrm{b})}\,. &&
\end{aligned} \tag{73}
$$

Still, it is easy to see that the total sum of diagrams remains unchanged,

$$
\sum_\alpha A_0^{(\alpha)} = \sum_\alpha \widehat{A}_0^{(\alpha)}\,. \tag{74}
$$

The main advantage in dealing with the subtracted Feynman diagrams is that, by construction, they are free from UV divergences[16] and, therefore, can be directly evaluated in $D = 4$ dimensions. In this way, we found that, remarkably enough,

$$
\widehat{A}_0^{(\mathrm{k})+(\mathrm{l})} = 0 \tag{75}
$$

and obtained the functions $\widehat{A}_0^{(\alpha)}$ (with $\alpha =$ g,h,m,o) in the form of convergent multiple integrals. Their explicit expressions are lengthy and we do not present them here to save space.

Having an integral representation for the sum of the functions $\sum_{\alpha=\mathrm{g,h,m,o}} A_0^{(\alpha)}$ and explicit expressions for the remaining functions (72), we are now in a position to test the relation (71) and to determine the factor r. Instead of trying to simplify the sum of complicated multiple integrals, we performed thorough numerical tests. Namely, both sides of (71) depend on the ratios of the distances between the vertices of the pentagon C_5

$$
X = \{x_{13}^2, x_{14}^2, x_{24}^2, x_{25}^2, x_{35}^2\} \tag{76}
$$

To test (71) it is sufficient to evaluate both sides of (71) for certain numerical values of X and then to compare the resulting numerical values. To be insensitive to the value of the constant

[16] For the diagram shown in Fig. 4(g) we have to perform the additional subtraction of the double pole defined in (62).

term $r\pi^4$, we evaluated the sum $\sum_{\alpha=\mathrm{c},\dots,\mathrm{o}} A_0^{(\alpha)}$ for several sets of distances X, and examined their difference. The results of our numerical tests are summarized in Table 1. We found that our results for the functions $A_0^{(\alpha)}$ are in perfect agreement with the first term in the right-hand side of (71). Then we determined the numerical value of the factor r parameterizing the constant term in (71) leading to

$$r = 0.99996\dots \,. \tag{77}$$

Assuming maximal transcendentality of the constant term, we expect r to be rational. Our calculation suggests that $r = 1$.

We conclude that our result for the two-loop pentagon Wilson loop is in agreement with the prediction (49) based on the conformal Ward identities.

$\{x_{13}^2, x_{14}^2, x_{24}^2, x_{25}^2, x_{35}^2\}$	$\widehat{A}_0^{(\mathrm{g})}$	$\widehat{A}_0^{(\mathrm{h})}$	$\widehat{A}_0^{(\mathrm{m})}$	$\widehat{A}_0^{(\mathrm{o})}$	$\sum_{\alpha=\mathrm{c},\dots,\mathrm{o}} A_0^{(\alpha)}$
$\{0.23, 1.32, 0.28, 0.72, 1.57\}$	$+1.6415$	-1.6209	-5.4799	$+6.1485$	$+0.5440$
$\{2.23, 1.32, 0.28, 0.72, 1.57\}$	$+5.8793$	-7.8712	-4.4920	$+6.2236$	-1.6328
$\{1.34, 0.04, 0.98, 3.21, 1.43\}$	$+5.4853$	-8.3748	-5.2143	$+5.8084$	-2.4030
$\{2.43, 1.30, 0.03, 1.41, 1.49\}$	$+4.1626$	-7.8060	-5.1487	$+5.8841$	-1.7502
$\{5.32, 0.42, 1.23, 7.76, 2.53\}$	$+3.5254$	-3.9424	-5.9461	$+5.1958$	-1.0333

Table 1: The coefficient functions (73) and their total sum (71) evaluated for different sets of the distances (76).

5 Conclusions

In this paper we have provided further evidence for the weak-coupling duality between MHV planar gluon amplitudes and light-like Wilson loops. Let us summarize the main arguments in favor of this conjecture.

The light-like Wilson loops with cusps have an intrinsic conformal symmetry which is broken, in a controlled way, by the cusp anomalies. This gives rise to anomalous conformal Ward identities which unambiguously fix the form of the finite part of the $n-$gonal Wilson loop for $n = 4$ and 5, and reduce the freedom in the dependence on the kinematical variables to a single function of conformal invariants for $n \geq 6$. The main open question is whether we can further restrict or even fully determine this function by evoking some additional properties. A promising scenario is to impose the consistency condition that the Wilson loop with n cusps reduce to the one with $n-1$ cusp when one of the cusps is 'flattened'. This requirement is analogous to the so-called 'collinear limit' for MHV planar gluon amplitudes, as discussed in [30]. It consists in taking two adjacent gluons to be near collinear, thus relating the $n-$gluon to the $n-1-$gluon amplitude. The combination of conformal symmetry with these additional consistency restrictions may turn out to be powerful enough to completely fix the functional form of the $n-$gonal Wilson loop to all orders. Even so, it will still contain some dynamically determined parameters like the cusp and collinear anomalous dimensions. We know that the former can be determined from a Bethe Ansatz [44, 45, 46]. It is not unlikely that the remaining parameters are also determined from integrability. If all of this is true, the light-like Wilson loop could be a very interesting example of a soluble theory.

Let us now turn to the MHV planar gluon amplitudes. What reasons do we have to believe that they are equivalent to the perturbative light-like Wilson loops? By now we have a lot of 'experimental' evidence to this effect.

First of all, the finite part of the

- four-gluon amplitude has been calculated up to three loops [1]
- five-gluon amplitude has been calculated up to two loops [11]
- n-gluon amplitude is known at one loop [30]

In all these cases the results match those for the Wilson loops, found either by explicit calculations ([8], [13], [9] and the present paper) or predicted by conformal invariance, as we have explained in [9] and in the present paper.

Secondly, the gluon amplitudes have the intriguing property of 'dual' conformal symmetry [14]. All the scalar Feynman integrals appearing in the calculations of Bern et al up to four loops[17] are dual to conformal integrals, after we take them off shell and perform the change of variables (4) from momenta to 'dual coordinates'.[18] At present, we do not have a good understanding of the origin of this hidden symmetry of the gluon amplitudes, nor we know if it persists to all loops. However, it is certainly true that such 'dual conformal' integrals give rise to an amplitude which satisfies differential equations coinciding with the Ward identities we discuss in this paper. This makes us believe that the dual conformal symmetry is not an accidental property of the gluon amplitudes, and that its breaking is controlled by the same type of anomalies as for the Wilson loop. In our opinion, investigating the origin of this symmetry is one of the important tasks at the moment.

On the other hand, we know that dual conformal symmetry cannot uniquely fix the gluon amplitudes with six or more gluons, just as in the case of a Wilson loop with more than five cusps. As mentioned earlier, the additional input may come from the collinear limit behavior studied in [30] and whose relevance was reiterated in [15]. We believe that the chances for this scenario to work out improve if one adds the requirement that the unknown function in the case $n \geq 6$ depend only on conformal invariants.

However, we should also be prepared to face a few less optimistic possibilities. Even if dual conformal invariance is a true symmetry of the gluon amplitudes and it restricts them in the way the intrinsic conformal invariance of the Wilson loop does, it may be that the additional requirements (collinear limits for gluon amplitudes and their equivalent for Wilson loops) are not enough to fully determine the amplitude. This may happen at a relatively high loop order, where the variety of Feynman integral and of associated functions is rich enough and one might be able to construct more than one function meeting all requirements. In this case gluon amplitudes and Wilson loops will fully agree for $n = 4, 5$, but they will start deviating from each other for $n \geq 6$ at two or higher loops.

Another possibility is that dual conformal symmetry is an accidental property of the low-loop gluon amplitudes. In this case, starting at some higher perturbative order, even the four- or five-gluon amplitudes might loose their simplicity and become similar to the very complicated generic QCD amplitudes.

[17] Taking dual conformal symmetry as an assumption, in [16] a five-loop four-gluon amplitude was constructed, which meets all the available unitarity consistency requirements.

[18] It should be pointed out that exactly the same change of variables appears in the 'T-duality' [20] transformation employed by Alday and Maldacena [5] in their strong coupling treatment of the gluon amplitudes.

All of these scenarios should be confronted with the discrepancy between the large n limit of the BDS conjecture and the strong coupling results recently found by Alday and Maldacena [12]. It may be that at strong coupling one sees the entire perturbative expansion, with all possible breakdowns of the above mechanisms.

What should be done at present to shed more light on these interesting issues? In our opinion, two parallel calculations should be carried out in the near future: the six-gluon two-loop MHV amplitude and the matching Wilson loop with six cusps. When the results become available, we may have the answers to some of the above questions.

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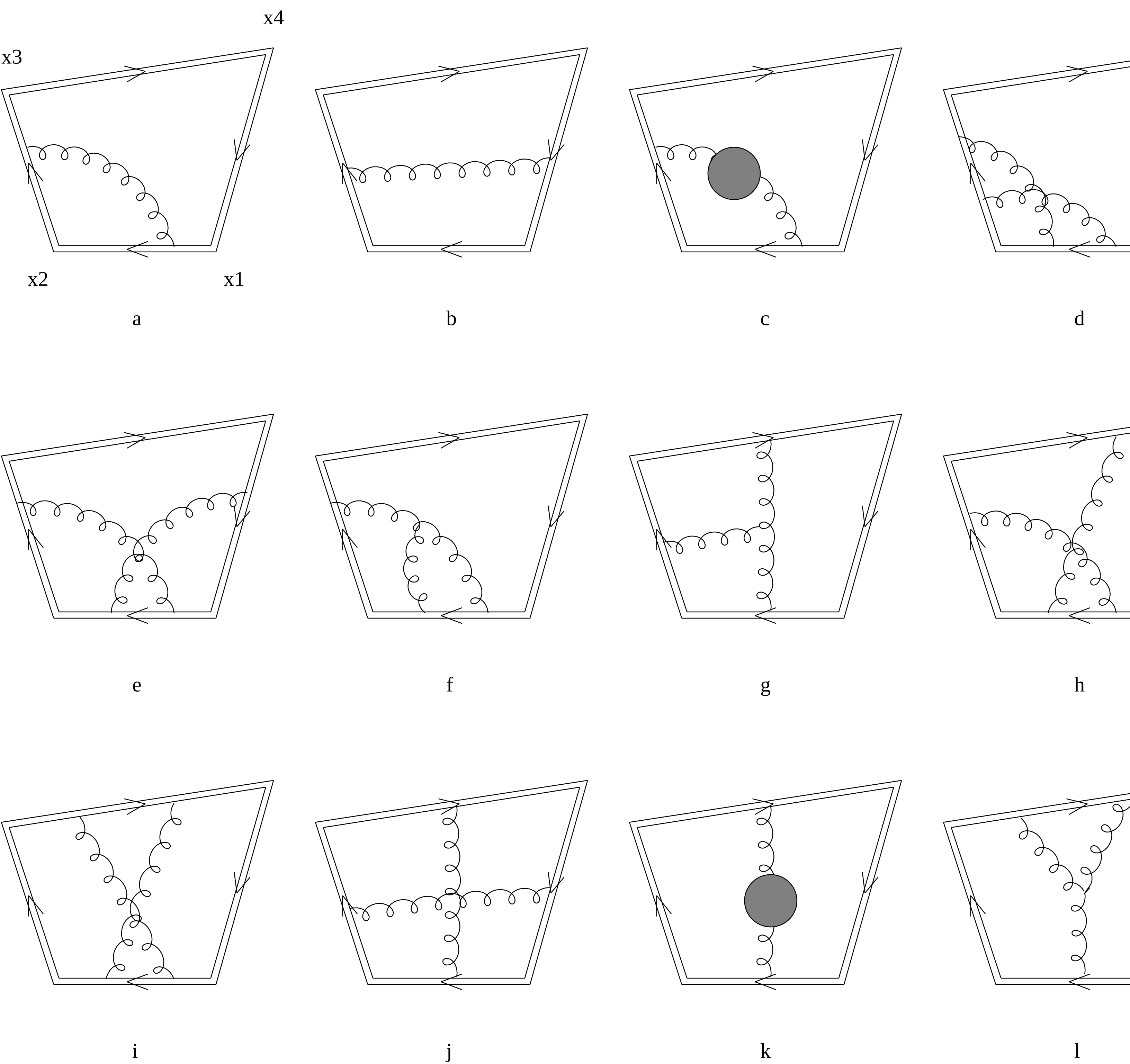
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