

Wrapping interactions and a new source of corrections to the spin-chain/string duality

J. Ambjørn^{a,d}, *R. A. Janik*^b and *C. Kristjansen*^c

^a The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
email: ambjorn@nbi.dk

^b Institute of Physics, Jagellonian University,
Reymonta 4, PL 30-059 Krakow, Poland.
email: romuald@th.if.uj.edu.pl

^c NORDITA,
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
email kristjan@nbi.dk

^d Institute for Theoretical Physics, Utrecht University,
Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands.

Abstract

Assuming that the world-sheet sigma-model in the AdS/CFT correspondence is an integrable *quantum* field theory, we deduce that there might be new corrections to the spin-chain/string Bethe ansatz paradigm. These come from virtual particles propagating around the circumference of the cylinder and render Bethe ansatz quantization conditions only approximate. We determine the nature of these corrections both at weak and at strong coupling in the near BMN limit, and find that the first corrections behave qualitatively as wrapping interactions at weak coupling.

1 Introduction

The discovery of integrable structures underlying planar $\mathcal{N} = 4$ SYM [1, 2, 3, 4] as well as classical string theory on $AdS_5 \times S^5$ [5, 6, 7, 8, 9, 10, 11] has opened up new avenues for testing the AdS/CFT correspondence [12]. Moreover, the application of spin chain techniques has facilitated the analysis on the gauge theory side. Non-trivial comparisons of gauge theory anomalous dimensions and string state energies can now be carried out in three different regimes: The BMN limit [13], the near BMN limit [14] and the spinning string limit [15]. Whereas for the first of these three limits everything at the level of anomalous dimensions appears to work fine the latter two are plagued by vexing three-loop discrepancies [16, 17].

It has been suggested that the three-loop discrepancy for near BMN states and spinning strings is due to the non-commutativity of the two limits which are involved in performing the analysis and which are imposed in different order on respectively the string and gauge theory side [17]. To perform an analysis on the gauge theory side in which the order of limits would be the same as on the string theory side one would have to take into account so-called wrapping interactions [18]. In the spin chain language wrapping interactions are interactions whose range exceeds the length of the chain. It is a fact that wrapping interactions are important when it comes to the determination of anomalous dimensions of short operators. For instance, wrapping interactions contribute to the four-loop anomalous dimension of the Konishi operator [2, 19]. A systematic description of wrapping interactions in terms of Feynman diagrams can be found in [20]. An alternative explanation of the three-loop discrepancy was suggested in [21].

On the gauge theory side it is very hard to do any rigorous derivations beyond the two and three loop ones of [2, 22], see also [23]. However, assuming integrability and BMN scaling at higher loop orders a conjecture for an all loop Bethe ansatz has been put forward [18, 24, 25]. On the string theory side it is possible to derive integral equations encoding all classical string solutions [8, 26, 10, 11]. These integral equations can be viewed as classical, continuum Bethe equations. Inspired by these two achievements a suggestion for a Bethe ansatz for quantum strings was presented in [27, 24, 25]. The quantum string Bethe ansatz is identical to the gauge theory one up to two loop order but differs from it beyond two loops. The all loop gauge theory Bethe ansatz and the quantum string one are very similar in nature. They both express the condition for factorized scattering for a set of elementary

excitations with individual momenta p_k

$$\exp(iLp_k) = \prod_{j \neq k}^M S(p_k, p_j), \quad (1)$$

where S is the S -matrix for scattering of two of these excitations and L is the length of the spin chain. The elementary excitations have the same dispersion relation on the gauge and string theory side. What differs between the two sides is the form of the S -matrices. The difference can conveniently be encoded in a so-called dressing factor

$$S^{string} = \hat{S}^{dressing} S^{gauge}, \quad (2)$$

which can be expressed as a phase shift, i.e.

$$\hat{S}^{dressing}(p_k, p_j) = \exp(i\theta(p_k, p_j)), \quad (3)$$

where

$$\theta(p_k, p_j) = 2 \sum_{r=2}^{\infty} c_r(\lambda) \left(\frac{\lambda}{16\pi^2} \right)^r (q_r(p_k)q_{r+1}(p_j) - q_{r+1}(p_k)q_r(p_j)), \quad (4)$$

with λ being the 't Hooft coupling constant and the q_r 's certain conserved charges [18, 27]. The expansion coefficients $c_r(\lambda)$ must fulfill that $c_r(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$ in order that the classical string theory limit is correctly reproduced. Recently, it was found by a study of one-loop string sigma model corrections in the $sl(2)$ sector that it is not possible to have $c_r(\lambda) = 1$ for all values of λ [28]. The first string sigma model loop correction produces contributions to energies of strings spinning with total angular momentum L which contain half integer powers of $\lambda' = \frac{\lambda}{L^2}$ [29, 30] starting at order $(\lambda')^{5/2}$ [30] as well as non-perturbative contributions containing factors of the type $\exp(-\frac{1}{\sqrt{\lambda'}})$ [29]. The appearance of terms of the type $\exp(-\frac{1}{\sqrt{\lambda'}})$ as well as terms with half-integer powers of λ' was earlier observed in the BMN limit in [31] in connection with a study of the three string interaction vertex. The leading half integer power of λ' in string energies can be accounted for by the phase factor (4) if the coefficient $c_2(\lambda)$ has the expansion [30]

$$c_2(\lambda) = 1 - \frac{3}{4} \frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right). \quad (5)$$

This opens the interesting possibility that $c_2(\lambda)$ could tend to zero at weak coupling [30] and thus that the three loop discrepancy for near BMN states as well as for spinning strings could indeed be due to an order of limits problem. Recently, it was shown that it is possible to obtain the classical string equations of motion of the $su(2)$ sector as the classical limit of an integrable *quantum* field theory, namely the $osp(2m+2|2m)$ coset model [32]. This model is defined on the plane and not on the cylinder as needed for a quantum string theory on $AdS_5 \times S^5$. The current understanding is that the $osp(2m+2|2m)$ coset model is capable of capturing the quantum effects introducing half-integer powers of λ in the expansions of string energies but not the finite size corrections which appear when the theory is put on a cylinder [32].

In the present paper we will explore the consequences of putting an integrable quantum field theory (IQFT) on a cylinder. In particular we shall discuss what happens if the elementary excitations of the field theory instead of the standard relativistic dispersion relation obey the dispersion relation implied by the conjectured quantum string Bethe equations [27]. We shall work in the near BMN limit and shall show that wrapping interactions generically give rise to contributions of order λ^L at weak coupling and of order $\exp(-\frac{1}{\sqrt{\lambda}})$ at strong coupling.

One of the established properties of an integrable *quantum* field theory on a cylinder is the fact that the Bethe ansatz quantization conditions

$$e^{ip_k L} = \prod_{j \neq k} S(p_k, p_j), \quad (6)$$

are no longer exact (see e.g. [33] section 5). The effect is very generic and comes from virtual corrections – excitations going around the circumference of the cylinder (see fig. 1 below).

Intuitively such processes, involving a virtual particle going around the cylinder, when translated into Feynman graphs of the gauge theory, should correspond to wrapping interactions. These types of virtual effects can be described¹ through S-matrix information, where the S-matrix is defined in the infinite volume limit. Thus, since there are proposals for the asymptotic S-matrix for gauge theory/strings we propose to interpret them as the infinite volume scattering data and use the framework of the virtual corrections to IQFT to incorporate the effect of wrapping interactions. We show that at

¹At leading order in all relativistic IQFT's, and for some specific theories exactly.

weak coupling the order at which the virtual corrections set in is λ^L just as expected for wrapping interactions.

The outline of this paper is as follows. In section 2 we will review these corrections in ordinary relativistic integrable field theories. In section 3 we will motivate what changes need to be done in order to incorporate the non-standard dispersion relation characteristic of excitations in the conjectured long range Bethe ansatz, then in section 4 we estimate the size of the corrections both at weak coupling and in the near-BMN limit. We close the paper with a summary and outlook.

2 Finite size mass shift in relativistic integrable field theories

In [34, 35] Lüscher calculated the leading order corrections to the (infinite volume) masses/energies of 1-particle states when put on a cylinder of circumference L :

$$m(L) = m(L = \infty) + \Delta m_\mu(L) + \Delta m_F(L). \quad (7)$$

These corrections arise from two different types of processes: the first is the so-called μ -term which arises when the particle splits into a pair of virtual on-shell particles which go around the cylinder and recombine later, while the second one, the F -term comes from a virtual particle loop where the virtual particle goes around the circumference of the cylinder (see fig. 1).

The expressions for these terms in the case of 1+1 dimensions and a single mass scale in the theory are [38]

$$\frac{\Delta m_\mu(L)}{m(\infty)} = -\frac{\sqrt{3}}{2} \sum_{b,c} M_{abc}(-i) \operatorname{res}_{\theta=2\pi i/3} S_{ab}^{ab}(\theta) \cdot e^{-\frac{\sqrt{3}}{2}mL}, \quad (8)$$

$$\frac{\Delta m_F(L)}{m(\infty)} = -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-mL \cosh \theta} \cosh \theta \sum_b \left(S_{ab}^{ab} \left(\theta + i\frac{\pi}{2} \right) - 1 \right). \quad (9)$$

where $S_{ab}^{ab}(\theta)$ is the (infinite volume) S -matrix, and $M_{abc} = 1$ if c is a bound state of a and b and zero otherwise. These formulas have been checked to agree with a wide variety of examples such as Ising field theory, the so-called scaling Lee-Yang model (SLYM), various perturbed minimal model CFT's etc. [37, 38].

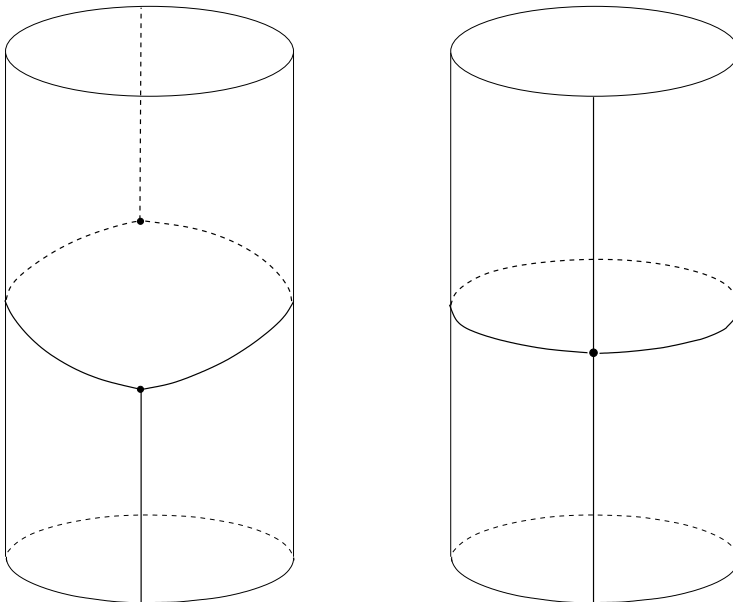


Figure 1: The diagram to the left (the μ -term) shows a particle splitting in two virtual, on-shell particles, traveling around the cylinder and recombining. The diagram to the right (the F-term) shows a virtual particle going around the circumference of the cylinder.

However, it is rather difficult to use these formulas directly in the case of the world sheet integrable field theories appearing in our present context. The proofs of Lüscher's formulas were a mixture of diagrammatic analysis and analytical continuation where Lorentz invariance was crucial (although clearly many aspects remain valid on the lattice [39]). In particular, one always had the dispersion relation $E^2 - p^2 = m^2$ and the parametrization of energy and momentum through rapidity

$$E = m \cosh \theta, \quad p = m \sinh \theta. \quad (10)$$

For the world sheet string theories in *curved* AdS space-time it was pointed out [40, 14, 16, 9] that the usual light cone gauge like $\tau = t$ (where t is e.g. the global AdS time) is inconsistent with putting a Minkowskian metric on the world sheet. Hence the second gauge condition is usually taken to be the uniform gauge where the density of a conserved charge (such as the R-charge J) is spread uniformly along the string². Then the gauge-fixed field theory

²Usually one makes the condition $p_\phi = J$ keeping the size of the cylinder to 2π [9]. But

is *not* explicitly Lorentz-invariant! A common ingredient of the asymptotic Bethe ansätze is the dispersion relation for all elementary excitations (here $E \equiv \Delta - J$):

$$E = \sqrt{1 + 8g^2 \sin^2 \frac{p}{2}}, \quad g^2 = \frac{\lambda}{8\pi^2}. \quad (11)$$

In the following we will *assume* that this is indeed the true dispersion relation for excitations in the integrable quantum field theories on the world sheet and explore the consequences of this assumption.

Another point which one should mention is that e.g. in the near BMN limit the zero momentum 1-particle states are protected by supersymmetry and what one is really interested in are corrections to 2-particle (or higher) states for which we lack similar expressions.

In order to attack the above problems it is convenient to look at a way in which the finite size virtual corrections can be calculated exactly, and which may shed more light on the origin of the formulas (8) and (9).

There are basically three (interrelated) ways of computing exactly the spectrum of an integrable QFT on a cylinder: the Thermodynamic Bethe Ansatz (TBA) [42], Nonlinear Integral equation (NLIE) or Destri-de-Vega (DDV) equation [43], and functional relations [33]. All this is still an area of active research [44]. We will here concentrate on the first of these approaches.

Before we start let us emphasize that the Thermodynamic Bethe Ansatz that we consider here is *not* the thermodynamic approximation which is used for approximately solving Algebraic Bethe Ansatz equations for a large number of roots in the spin chain/string literature. We use the term in the sense that it is used in the context of relativistic integrable field theories.

Thermodynamic Bethe Ansatz

The Thermodynamical Bethe Ansatz [42] was originally devised as a method for finding how the ground state of an integrable field theory depends on the circumference L . The main idea is to consider the theory on a very elongated torus with circumference L and length $R \rightarrow \infty$, and to compute the (euclidean) partition function. From the point of view of L being space and

this populates the action with explicit factors of J . It is more convenient for our purposes to set the density to be equal to 1 and then the total R-charge is encoded just in the size of the cylinder like in [41].

R being very large ‘time’ this gives the ground state energy being calculated:

$$E_0(L) = -\frac{1}{R} \log Z. \quad (12)$$

On the other hand if one looks at R being space and L being the (euclidean) time this is just the thermal free energy of the system on an *infinite* line, which can be calculated by minimising $E - TS$ with $T = 1/L$. The main point is that for the theory with $R \rightarrow \infty$, Bethe ansatz quantization remains exact. Then one introduces continuous densities for the occupied roots AND appropriate entropy factors. The result is the set of equations³:

$$\varepsilon(\theta) = LE_{TBA}(\theta) - \phi * L, \quad (13)$$

$$E_0 = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} p'_{TBA}(\theta) L(\theta), \quad (14)$$

where

$$L(\theta) = \log(1 + e^{-\varepsilon(\theta)}), \quad (15)$$

$$(\phi * L)(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \phi(\theta - \theta') L(\theta'), \quad (16)$$

$$\phi(\theta) = -i \frac{d}{d\theta} \log S(\theta), \quad (17)$$

and $E_{TBA}(\theta) = m \cosh \theta$, and $p_{TBA}(\theta) = m \sinh \theta$.

As it stands the Thermodynamic Bethe Ansatz seems to be confined to being a tool for finding just the ground state energy. However, in [45] Dorey and Tateo suggested that by analytical continuation one could obtain the energies of excited states. This program was carried out first in the scaling Lee-Yang model (SLYM) [45] and then in a series of perturbed minimal model CFT's [46].

The underlying mechanism of the construction is that $L(\theta) = \log(1 + e^{-\varepsilon(\theta)})$ developed singularities which after a contour deformation added source terms to the right hand sides of equations (13) and (14). E.g. for a 1-particle state at zero momentum, the $mL \cosh \theta$ in (13) should be substituted by

$$mL \cosh \theta + \log \frac{S(\theta - \theta_0)}{S(\theta + \theta_0)}, \quad (18)$$

³For the simplest case of a single particle in the spectrum.

where θ_0 is a singularity of $L(\theta)$, which leads to an additional equation $\varepsilon(\theta_0) = i\pi$. This set of equations then gives *exact* results for the excited state energy $E_1(L)$ for any L . Moreover, Lüscher's formulas for (8) and (9) then immediately follow from solving that system of equations by iteration (see [45]).

For our purposes it is interesting to obtain the form of corrections for a 2-particle state for which we lacked an explicit expression.

From the results in [45] we obtain

$$E = 2 \cosh \theta_B + 2 \cosh \theta_B \cdot \delta - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta e^{-L \cosh \theta} S\left(\theta + i\frac{\pi}{2} - \theta_B\right) S\left(\theta + i\frac{\pi}{2} + \theta_B\right), \quad (19)$$

where θ_B is the Bethe root which follows from solving the Bethe quantization condition

$$e^{iL \sinh \theta_B} = S(2\theta_B). \quad (20)$$

We see that the first piece of (19) is the classical energy of the 2-particle state. The second piece is a μ -term of the form

$$\delta = 3e^{-\frac{\sqrt{3}}{2}L \cosh \theta_B} \coth \theta_B \sqrt{\frac{2 \cosh 2\theta_B + 1}{2 \cosh 2\theta_B - 1}}, \quad (21)$$

while the last term is the analogue of the F -term. The exponential factor remains the same, while the two S-matrices correspond to a virtual particle of rapidity $\theta + i\pi/2$ moving around the cylinder and scattering off the two particles with Bethe roots (rapidities) $\pm\theta_B$.

The point that we want to make here is that the structure of the F -term correction to the 2-particle state is a straightforward generalization of the Lüscher mass shift formula for a 1-particle state. In view of applications to the world sheet theory we would expect a first nontrivial correction to occur at the level of 2-particle states since the mass of the one-particle states are protected by supersymmetry. The absence of corrections for the ground state and 1-particle states should be a property of the full world sheet theory and cannot be obtained just in e.g. the $su(2)$ sector. The reason is that the $su(2)$ excitation scatters non-trivially from other excitations [25] which might therefore circulate in the loop.

Another motivation for discussing at length the TBA derivation of Lüscher's formulas is that the space-time interpretation of the derivation of TBA may

suggest its generalization to the case of the world sheet theory which, when gauge-fixed, is no longer Lorentz-invariant.

In the next section we will use the TBA intuition to propose what changes are needed when the dispersion relation is no longer exactly relativistic but is modified to (11). We will also try to motivate, using a solvable example, that such a procedure can be legitimate – which is of course not completely clear *a-priori*.

3 Nonstandard dispersion relations

As we saw above, the derivation of the TBA is based on a modular transformation on the torus i.e. an interchange of space and time. For a relativistic theory this does not really make a difference but it is far from clear if such a philosophy may be applied at all to the rather nonstandard world-sheet theories.

In order to motivate this we will show an example where the proposed procedure can be proved to give the exact result despite the fact that the dispersion relation is not exactly relativistic.

The Ising model off criticality

This example has been given by Lüscher in his Cargese lectures [34]. We need to be here more explicit in the calculation, as the intermediate steps will be crucial for our purposes. The Ising model on a finite $L \times L$ lattice can be completely solved in terms of transfer matrices which can be expressed through free fermion operators with the dispersion relation [36]:

$$\cosh E_q = \cosh m_0 + 1 - \cos q \equiv \cosh m_0 + 2 \sin^2 \frac{q}{2}. \quad (22)$$

The mass gap (i.e. the mass/energy of the lowest lying excitation) can be expressed through an exact formula:

$$m(L) = m_0 + \frac{1}{2} \sum_{\nu=0}^{L-1} E_{\frac{\pi}{L}(2\nu+1)} - \frac{1}{2} \sum_{\nu=0}^{L-1} E_{\frac{\pi}{L} \cdot 2\nu} = m_0 - \frac{1}{2} \sum_{\nu=0}^{2L-1} e^{i\pi\nu} E_{\frac{\pi}{L} \cdot \nu}. \quad (23)$$

The discrete sum can be rewritten using the (finite) Poisson resummation formula [47] as

$$m(L) = m_0 - L \sum_{\nu=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqL(2\nu+1)} E_q. \quad (24)$$

Now after an integration by parts one changes variables to

$$Q = -iE_q, \quad (25)$$

Q will play the role of the new momentum in what follows. For *integer* L one then gets

$$m(L) = m_0 + \sum_{\nu=-\infty}^{\infty} \frac{1}{2\nu+1} \int_{-iE_{-\pi}}^{-iE_{\pi}} \frac{dQ}{2\pi} e^{iq(Q)L(2\nu+1)}. \quad (26)$$

We note that due to the form of (22) $q(Q) = iE_Q$ so the exponent simplifies to $-E_Q L(2\nu+1)$. After a slightly nontrivial contour argument it turns out that one can change the contour of integration to go from $-\pi$ to π just as if Q was a physical momentum. One then gets

$$m(L) = m_0 + \sum_{\nu=0}^{\infty} \frac{2}{2\nu+1} \int_{-\pi}^{\pi} \frac{dQ}{2\pi} e^{-L(2\nu+1)E_Q} \sim m_0 + 2 \int_{-\pi}^{\pi} \frac{dQ}{2\pi} e^{-LE_Q} \quad (27)$$

The leading correction is then the term with $\nu = 0$. The exponent is just as in the formula for relativistic F -term correction (with the S matrix equal to -1 , which is the value for the massive deformation of the Ising model).

Space-time interchange

Let us now go back and analyze what kind of analytical continuations were made here and adopt a notation more reminiscent of the TBA expressions. What plays the role of momentum in the final formula (27) is

$$p_{TBA} \equiv Q = -iE_q, \quad (28)$$

while the energy appearing in the exponent of (27) is in fact $-i$ times the original momentum expressed in terms of $Q \equiv p_{TBA}$:

$$E_{TBA} = -iq(Q). \quad (29)$$

The above substitutions are indeed quite natural from the point of view of a modular transformation exchanging space and time. Since time should be interchanged with space one should Wick rotate both coordinates (hence the $-i$'s in (28) and (29)), and also exchange E and q with each other. In the final integral (27) $p_{TBA} \equiv Q$ is taken to be real.

To add some more plausibility to the interpretation of the above formulas, one sees that in the Lorentz invariant case the dispersion relation in the double Wick rotated space remains the same and the shift of the rapidity of the virtual particle by $i\pi/2$ in the F -term exactly corresponds to the above transformations (28)–(29).

$$m \cosh(\theta + i\pi/2) = im \sinh \theta, \quad m \sinh(\theta + i\pi/2) = im \cosh \theta. \quad (30)$$

In the following section we will apply these ideas to estimate the size of the correction terms for the world sheet theory with the dispersion relation (11) but before we do that, we will show that the above procedure works for a modified Ising model with (11).

Modified Ising model

Since we lack any explicit example of a completely defined integrable theory with the dispersion relation (11), we will construct a somewhat artificial but exactly solvable example. Let us modify the Ising model defined by the transfer matrices involving free fermions by changing their dispersion relation from (22) to (11).

The exact mass shift can then be obtained in exactly the same fashion. It is given by formulae analogous to (23)–(24), i.e.

$$m(L) = 1 - L \sum_{\nu=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqL(2\nu+1)} \sqrt{1 + 8g^2 \sin^2 \frac{q}{2}}. \quad (31)$$

Let us now evaluate $E_{TBA}(p_{TBA})$. Performing the substitutions (28)–(29) we obtain

$$E_{TBA}(p_{TBA}) = 2 \operatorname{arcsinh} \left(\frac{1}{2\sqrt{2}g} \cdot \sqrt{1 + p_{TBA}^2} \right). \quad (32)$$

Now again the quite remarkable identity holds for integer L :

$$\begin{aligned} -L \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqL(2\nu+1)} \sqrt{1 + 8g^2 \sin^2 \frac{q}{2}} \\ = \frac{1}{2\nu + 1} \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-L(2\nu+1) \cdot 2 \operatorname{arcsinh} \left(\frac{\sqrt{1+p_{TBA}^2}}{2\sqrt{2}g} \right)}, \end{aligned} \quad (33)$$

which for $\nu = 0$ reproduces the expected correction

$$m(L) = 1 + 2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-L \cdot 2 \operatorname{arcsinh} \left(\frac{\sqrt{1+p_{TBA}^2}}{2\sqrt{2}g} \right)} \quad (34)$$

but with the somewhat odd-looking expression (32) for the energy.

4 Wrapping interactions and near-BMN limit

Let us now apply the above considerations to the case of the world sheet theory with the dispersion relation (11). We will evaluate at what order does the virtual correction enter both at weak and at strong gauge theory coupling (in the near BMN limit). We will not try here to obtain the exact form of the correction since this would have to involve the full world sheet theory whose S-matrix is so far not known completely. Moreover the scattering is non-diagonal making the procedure even more complicated. The conjectured Bethe ansätze [25] may be used in this respect but carrying out this program still remains a very nontrivial task which we leave for future investigation.

In the following we will discuss first the F -term and then the μ -term.

The F -term

The size of the F term at large L is governed by the exponential factor

$$e^{-LE_{TBA}} = e^{-L \cdot 2 \operatorname{arcsinh} \left(\frac{\sqrt{1+p_{TBA}^2}}{2\sqrt{2}g} \right)}. \quad (35)$$

Let us see how this expression behaves at small coupling. Then the argument of the $\operatorname{arcsinh}$ is very large and we may substitute it by a logarithm $E_{TBA} \sim -2 \log g + \dots$. Consequently the correction term is of the order

$$g^{2L}, \quad (36)$$

which is exactly the expected order when wrapping interactions should set in. This further supports the intuition that virtual corrections in the world sheet theory corresponding to particles moving in loops around the cylinder should correspond to wrapping interactions on the gauge theory side.

Let us now proceed to take the near BMN limit. We take⁴ $L \sim J \rightarrow \infty$ but at the same time keep $\lambda' = \lambda/J^2$. Therefore

$$8g^2 = \frac{\lambda' J^2}{\pi^2}. \quad (37)$$

⁴For a two impurity state in the $su(2)$ sector $L = J + 2$.

In this limit the argument of the arcsinh is small and the exponent (35) governing the size of the corrections takes the form⁵

$$e^{-\frac{2\pi}{\sqrt{\lambda'}}\sqrt{1+p_{TBA}^2}}. \quad (38)$$

μ -term

We have much less control over the precise form of the μ -term, however its physical origin is quite clear. It occurs when there is a possible process when a particle can decay into a pair of virtual particles which are *on-shell* (but yet will have in general complex momenta).

For Lorentz-invariant theories, the exponents of the μ -terms (8) and (21) are seen to follow from the imaginary parts of the momenta of the two virtual particles. Explicitly, suppose that an on-shell particle of momentum p_0 disintegrates into a pair of particles with momenta p_1 and p_2 which are on-shell. From the kinematics we obtain for $p_{1,2}$:

$$p_{1,2} = \frac{p_0}{2} \pm i\frac{\sqrt{3}}{2}\sqrt{1+p_0^2}. \quad (39)$$

We do not have a precise expression for the μ -term in the case of multi-particle states. We will here just use an ad-hoc formula $\exp(-L \cdot \text{Im } p_i)$, and we find that it reproduces the exponents in (8) and (21).

Repeating the same kind of calculation with the dispersion relation (11) gives in general quite complicated expressions. We may however take p_0 to be the momentum of a particle in a 2-particle state in the near BMN limit:

$$p_0 = \frac{2\pi n}{J+2} + \frac{2\pi n}{\sqrt{1+\lambda'n^2}(J+2)^2} + \dots \quad (40)$$

Assuming p_i to be small we obtain

$$p_{1,2} = \frac{1}{J} \left(n\pi \pm i\frac{\sqrt{3}}{\sqrt{\lambda'}}\pi\sqrt{1+\lambda'n^2} \right). \quad (41)$$

This suggests that the μ -term, if present, would also give $1/\sqrt{\lambda'}$ corrections in the near BMN limit.

⁵We note that in this limit the same result would also arise from an unmodified BMN dispersion relation $E = \sqrt{1+2g^2p^2}$.

5 Conclusions

Lacking a direct proof of the AdS/CFT correspondence, tests which are truly reflecting non-trivial interactions are of utmost importance. The integrable structures discovered both at the string side and the $\mathcal{N} = 4$ SYM side provide a good testing ground and there has been extensive research comparing the predictions coming from the string world-sheet sigma model with the predictions from the spin-chain model of $\mathcal{N} = 4$ SYM.

Numerous examples of very detailed agreement between gauge and string theory have been found, nevertheless there remains some subtle but significant discrepancies. One notable example is the notorious three loop discrepancy occurring both in the near-BMN limit and for spinning strings. Another is the appearance half-integer powers of λ in string sigma model loop corrections, cf. the introduction. However, it is known that at least some ingredients have not been included in the asymptotic Bethe ansätze such as e.g. gauge theory wrapping interactions.

While integrability on the string theory side is fairly developed for the classical sigma model not much is known for the complete quantum theory. A very recent attempt to study the quantum effects via S-matrix (Bethe Ansatz) techniques can be found in the paper by Mann and Polchinski [32], who considered directly a relativistic quantum field theory on R^2 (a $OSp(2m+2|2m)$ supercoset model) with a known S-matrix. They found an embedding of the ‘classical’ string $su(2)$ sector into the Bethe ansatz of their full quantum theory, and found generic corrections of the type $1/\sqrt{\lambda}$, but did not consider finite size effects.

In our paper we wanted to concentrate on the specific features of an integrable *quantum* field theory when put on a cylinder, as is always ultimately required for the closed string worldsheet theory. The generic feature is then that the Bethe ansatz quantization condition is no longer *exact* and receives virtual corrections from particles moving in loops around the circumference of the cylinder.

The energy shifts due to these effects in a finite volume are most conveniently calculated using the Thermodynamic Bethe Ansatz.⁶ We provided arguments that despite the non-Lorentz-invariant nature of the dispersion relation (11) conjectured to hold for the excitations of the *AdS* string, one

⁶Although no general proof of this method for excited states exists, extensive tests have been performed in various relativistic integrable field theories [45, 46, 48].

can still use TBA. It does not seem possible at this stage to perform a complete calculation for the world-sheet theory. Therefore, in this paper we have confined ourselves to understand just the general nature of the corrections, using only the dispersion relation (11) as an assumption.

At weak coupling we find that the first corrections are of the order λ^L . It is natural to conjecture that the analogous term on the gauge-theory side is due to wrapping interactions in the spin-chain picture. This is also very intuitive as the propagation of a virtual excitation around the cylinder would presumably, when translated somehow into gauge theory Feynman graphs, be described by graphs ‘wrapping’ the cylinder.

At strong coupling the contribution is of the order $e^{-1/\sqrt{\lambda'}}$ which cannot be seen directly in perturbative gauge theory. However, as mentioned earlier, in string sigma model loop corrections one has found contributions to string energies involving half integer powers of λ' as well as contributions of the form $e^{-1/\sqrt{\lambda'}}$ [28, 29, 30], see also [31].

It would be very interesting to perform a more complete calculation, perhaps using all or a part of the asymptotic Bethe ansätze of [25]. The same could be also considered in the supercoset model of [32], which has the additional simplification that it is a true relativistic integrable field theory – so it does not suffer from the conceptual difficulties associated with the non-standard dispersion relation. Last but not least, it would be very interesting to try to extend the formalism of such finite size virtual corrections to many particle states dual to macroscopic spinning strings.

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